Problem sheet 3

Due Monday, May 7th at 17:30.

Question 1

Prove the following statements.

(a) An ϵ-regular partition of a graph G is an ϵ-regular partition the complement of G.

(b) If (X, Y) is an ϵ-regular pair with density d, then at most $\epsilon|X|$ vertices in X have more than $(d+\epsilon)|Y|$ neighbors in Y.

(c) For each $\epsilon > 0$ each graph G with $|E(G)| \leq \epsilon^3 \lfloor \frac{n^2}{4} \rfloor$ has an ϵ-regular partition with two parts.

Question 2

A partition $V(G) = X_0 \cup X_1 \cup \cdots \cup X_k$ with exceptional set X_0 is called ϵ-regular if $\sum |X_i||X_j| \leq \epsilon n^2$, where the sum is taken over all not ϵ-regular pairs (X_i, X_j), $\underline{1 \leq i, j \leq k}$. Consider the following function for a partition P that is similar to the mean square density

$$q(P) = \sum_{1 \leq i,j \leq k} \tfrac{|X_i||X_j|}{n^2} d(X_i, X_j)^2 + \sum_{v \in X_0} \sum_{i=1}^{k} \tfrac{|X_i|}{n^2} d(X_i, \{v\})^2 + \sum_{v \in X_0} \sum_{u \in X_0} \tfrac{1}{n^2} d(\{u\}, \{v\})^2.$$

Prove the following statements.

(a) Let P be a partition $V(G) = X_0 \cup X_1 \cup \cdots \cup X_k$ with $|X_1| = \ldots = |X_k|$, that is not ϵ-regular. Then there is a partition $V(G) = Y_0 \cup Y_1 \cup \cdots \cup Y_\ell$, with $k \leq \ell \leq k2^{3k}$, $|Y_0| \leq |X_0| + \frac{n}{2^k}$, $|Y_1| = \ldots = |Y_k|$, that has q-value least $q(P) + \epsilon^5$.

(b) For each integer $k_0 \geq 2$ and each $\epsilon > 0$ there is an integer K such that for any graph G on $n \geq k_0$ vertices there is an ϵ-regular partition $V(G) = X_0 \cup X_1 \cup \cdots \cup X_k$, with $k_0 \leq k \leq K$, exceptional set X_0 of size at most ϵn, and $|X_1| = \ldots = |X_k|$.

(**Hint:** Follow the proof in Conlon's lecture notes while after each application of Lemma 1 in Lecture 5 cut the sets into pieces of size $\lfloor 2^{-3k}|X_i| \rfloor$.)

Question 3

Let $0 < \nu < \frac{2}{3}$ and consider a triangle-free graph G with n vertices and minimum degree at least $(\frac{1}{3} + \nu)n$.

(a) Apply the regularity lemma as in question 2 part (b) to obtain an ϵ-regular partition $V(G) = X_0 \cup X_1 \cup \cdots \cup X_k$ with exceptional set X_0. Consider the partition

$$V(G) = \bigcup_{I \subseteq \{1,\ldots,k\}} V_I \qquad \text{with} \qquad V_I = \{v \in V(G) \mid |N_G(v) \cap X_i| \geq \tfrac{\nu}{9}|X_i| \Leftrightarrow i \in I\}.$$

Prove the following statements for each $I \subseteq [k]$.

(1) If $|\cup_{i \in I} X_i| \leq \frac{2}{3}n$, then any two vertices in V_I have a common neighbour.

(2) If $|\cup_{i \in I} X_i| \geq \frac{2}{3}n$, then there is an ϵ-regular pair (X_i, X_j), $i, j \in I$, of density at least $\frac{\nu}{9}$.

(3) Each set V_I is either independent or empty.

(b) Prove that there is a constant C, depending on ν only, such that $\chi(G) \leq C$.

(*Remark:* One can show that four colours are sufficient using more sophisticated arguments. Moreover there are graphs on n vertices with minimum degree at most $\frac{1}{3}n$ and arbitrarily large chromatic number.)