Problem sheet 3
Due Monday, May 7th at 17:30.

Question 1

Prove the following statements. Assume that in any \( \epsilon \)-regular pair \((X, Y)\), \( X \cap Y = \emptyset \).

(a) An \( \epsilon \)-regular partition of a graph \( G \) is an \( \epsilon \)-regular partition the complement of \( G \).

(b) If \((X, Y)\) is an \( \epsilon \)-regular pair with density \( d \), then at most \( \epsilon |X| \) vertices in \( X \) have more than \((d + \epsilon)|Y|\) neighbors in \( Y \).

(c) For each \( \epsilon > 0 \) each graph \( G \) with \(|E(G)| \leq \epsilon^3 \lfloor \frac{n^2}{4} \rfloor \) has an \( \epsilon \)-regular partition with two parts.

Solution

(a) Let \( V(G) = X_1 \cup \cdots \cup X_k \) be an \( \epsilon \)-regular partition of \( G \). Let \( \overline{G} \) be the complement of \( G \). Consider two disjoint sets of vertices \( X, Y \). Let \( e_G(X, Y) \) be the number of edges between \( X \) and \( Y \) in \( G \). Then the density in \( G \) is \( d_G(X, Y) = \frac{e_G(X, Y)}{|X||Y|} \). Since there are \(|X||Y| - e_G(X, Y)\) edges between \( X \) and \( Y \) in \( \overline{G} \) the density in \( \overline{G} \) is

\[
d_{\overline{G}}(X, Y) = \frac{|X||Y| - e_G(X, Y)}{|X||Y|} = 1 - d_G(X, Y).
\]

If \((X, Y)\) is an \( \epsilon \)-regular pair in \( G \), then for \( X' \subseteq X, Y' \subseteq Y \) with \(|X'| \geq \epsilon |X|, |Y'| \geq \epsilon |Y|\) we have

\[
|d_{\overline{G}}(X, Y) - d_{\overline{G}}(X', Y')| = |1 - d_G(X, Y) - (1 - d_G(X', Y'))| = |d_G(X, Y) - d_G(X', Y')| \leq \epsilon.
\]

Thus \((X, Y)\) is an \( \epsilon \)-regular pair in \( \overline{G} \). Moreover the following summation over all not \( \epsilon \)-regular pairs \((X_i, X_j)\) does not dependent on the graph between \( X_i \) and \( X_j \) and thus

\[
\sum \frac{|X_i||X_j|}{n^2} \leq \epsilon.
\]

Hence \( V(G) = V(\overline{G}) = X_1 \cup \cdots \cup X_k \) is an \( \epsilon \)-regular partition of \( \overline{G} \).
(b) Let \( X' \) denote the set of vertices in \( X \) that have more than \((d + \epsilon)|Y|\) neighbors in \( Y \). Then
\[
d(X', Y) > \frac{|X'|(d + \epsilon)|Y|}{|X'||Y|} = d + \epsilon.
\]
Therefore \(|d - d(X', Y)| > \epsilon \) and thus \(|X'| < \epsilon |X|\).

(c) Observe that for any two disjoint sets \( X, Y \subseteq V(G) \), with \(|X| \geq \epsilon \left\lceil \frac{n}{2} \right\rceil \), \(|Y| \geq \epsilon \left\lfloor \frac{n}{2} \right\rfloor \) we have
\[
d(X, Y) = \frac{|E(X, Y)|}{|X||Y|} \leq \frac{\epsilon^3 \left\lfloor \frac{n^2}{4} \right\rfloor}{\epsilon^2 \left\lceil \frac{n^2}{4} \right\rceil} = \epsilon.
\]
Therefore any partition of the vertex set into two sets of size \( \left\lceil \frac{n}{2} \right\rceil \) and \( \left\lfloor \frac{n}{2} \right\rfloor \) is \( \epsilon \)-regular.

**Question 2**

A partition \( V(G) = X_0 \cup X_1 \cup \cdots \cup X_k \) with exceptional set \( X_0 \) is called \( \epsilon \)-regular if \( \sum |X_i||X_j| \leq \epsilon n^2 \), where the sum is taken over all not \( \epsilon \)-regular pairs \((X_i, X_j), 1 \leq i, j \leq k\). Consider the following function for a partition \( P \) that is similar to the mean square density
\[
q(P) = \sum_{1 \leq i, j \leq k} \frac{|X_i||X_j|}{n^2} d(X_i, X_j)^2 + \sum_{v \in X_0} \sum_{i=1}^k \frac{|X_i|}{n^2} d(X_i, \{v\})^2 + \sum_{v \in X_0} \sum_{u \in X_0} \frac{1}{n^2} d(\{u\}, \{v\})^2.
\]

Prove the following statements.

(a) Let \( P \) be a partition \( V(G) = X_0 \cup X_1 \cup \cdots \cup X_k \) with \( |X_1| = \ldots = |X_k| \), that is not \( \epsilon \)-regular. Then there is a partition \( V(G) = Y_0 \cup Y_1 \cup \cdots \cup Y_\ell \), with \( k \leq \ell \leq k^{2^k} \), \(|Y_0| \leq |X_0| + \frac{n}{2^k}, |Y_1| = \ldots = |Y_\ell| \), that has \( q \)-value least \( q(P) + \epsilon^5 \).

(b) For each integer \( k_0 \geq 2 \) and each \( \epsilon > 0 \) there is an integer \( K \) such that for any graph \( G \) on \( n \geq k_0 \) vertices there is an \( \epsilon \)-regular partition \( V(G) = X_0 \cup X_1 \cup \cdots \cup X_k \), with \( k_0 \leq k \leq K \), exceptional set \( X_0 \) of size at most \( \epsilon n \), and \( |X_1| = \ldots = |X_k| \).

( Hint: Follow the proof in Conlon’s lecture notes while after each application of Lemma 1 in Lecture 5 cut the sets into pieces of size \( \left\lceil 2^{-3k} |X_i| \right\rceil \).)

**Solution**

(a) Observe that the \( q \)-value of a partition \( V(G) = X_0 \cup X_1 \cup \cdots \cup X_k \) is the mean square density of the partition \( V(G) = \bigcup_{v \in X_0} \{v\} \cup X_1 \cup \cdots \cup X_k \). Therefore the \( q \)-value does not decrease under refinement by Lemma 2 in Lecture 4.
Let \( I = \{(i, j) \mid (X_i, X_j) \) is not \( \epsilon \)-regular, \( 1 \leq i, j \leq k \}\). Like in the proof in the lecture (Lecture 5, Lemma 1 in the lecture notes) for each \( i \), there are refinements \( X_i = X_{i1}, \ldots, X_{i a_i} \), with \( a_i \leq 2^{2k} \) such that for each pair \( (i, j) \in I \)

\[
\sum_{p=1}^{a_i} \sum_{q=1}^{a_j} \frac{|X_i^p||X_j^q|}{|X_i||X_j|} d(X_i^p, X_j^q)^2 \geq d(X_i, X_j)^2 + \epsilon^4.
\]

We apply Lemma 2 from Lecture 4 to show that the partition \( V(G) = \bigcup_{i=0}^{k} \bigcup_{p=1}^{a_i} X_i^p \) has \( q \)-value

\[
\sum_{1 \leq i, j \leq k} \sum_{p=1}^{a_i} \sum_{q=1}^{a_j} \frac{|X_i^p||X_j^q|}{n^2} d(X_i^p, X_j^q)^2 + \sum_{v \in X_0} \sum_{i=1}^{k} \sum_{p=1}^{a_i} \frac{|X_i^p|}{n^2} d(X_i^p, \{v\})^2 + \sum_{v \in X_0} \sum_{u \in X_0} \frac{1}{n^2} d(\{u\}, \{v\})^2
\]

\[
\geq \sum_{1 \leq i, j \leq k} \sum_{p=1}^{a_i} \sum_{q=1}^{a_j} \frac{|X_i^p||X_j^q|}{n^2} d(X_i^p, X_j^q)^2 + \sum_{v \in X_0} \sum_{i=1}^{k} \sum_{p=1}^{a_i} \frac{|X_i^p|}{n^2} d(X_i^p, \{v\})^2 + \sum_{v \in X_0} \sum_{u \in X_0} \frac{1}{n^2} d(\{u\}, \{v\})^2
\]

\[
= \sum_{(i, j) \notin I} \sum_{p=1}^{a_i} \sum_{q=1}^{a_j} \frac{|X_i^p||X_j^q|}{n^2} d(X_i^p, X_j^q)^2 + \sum_{(i, j) \in I} \sum_{p=1}^{a_i} \sum_{q=1}^{a_j} \frac{|X_i^p||X_j^q|}{n^2} d(X_i^p, X_j^q)^2 + A
\]

\[
\geq \sum_{(i, j) \notin I} \frac{|X_i||X_j|}{n^2} d(X_i, X_j)^2 + \sum_{(i, j) \in I} \frac{|X_i||X_j|}{n^2} d(X_i, X_j)^2 + \epsilon^4 + A
\]

\[
= \sum_{1 \leq i, j \leq k} \frac{|X_i||X_j|}{n^2} d(X_i, X_j)^2 + \epsilon^4 \cdot \sum_{(i, j) \in I} \frac{|X_i||X_j|}{n^2} + A
\]

\[
> \epsilon \text{ since } P \text{ not } \epsilon \text{-regular}
\]

\[
\geq \sum_{1 \leq i, j \leq k} \frac{|X_i||X_j|}{n^2} d(X_i, X_j)^2 + A + \epsilon^5
\]

\[
=q(P) + \epsilon^5.
\]

This shows that the partition \( V(G) = \bigcup_{i=0}^{k} \bigcup_{p=1}^{a_i} X_i^p \) has \( q \)-value at least \( \epsilon^5 \) larger than \( P \).

Next we will cut the sets \( X_i^p \), \( 1 \leq i \leq k \), \( 1 \leq p \leq a_i \), into not too many pieces of the same size, collecting the leftover in the exceptional set.

Let \( c = |X_1| = \cdots = |X_k| \) and \( d = \lfloor \frac{c}{2 \pi} \rfloor \). Choose a maximal collection \( Y_1, \ldots, Y_\ell \) of sets of size \( d \) such that each \( Y_t \), \( 1 \leq t \leq \ell \), is contained in some \( X_i^p \). Further let \( Y_0 = V(G) \setminus \bigcup_{i=1}^{\ell} Y_i \). Then all parts \( Y_1, \ldots, Y_\ell \) have same size and the \( q \)-value did not
decrease by Lemma 2 from Lecture 4, so is still at least $q(P) + \epsilon^5$. It remains to show that $\ell \leq k2^{3k}$ and $|Y_0| \leq |X_0| + \frac{n}{2 \pi}$.

Observe that each $X_j$ contains at most $|X_j|/\left\lfloor \frac{3}{2 \pi} \right\rfloor \leq 2^{3k}$ sets $Y_i$. Hence $k \leq \ell \leq k2^{3k}$.

Moreover for each $j \in [k], p \in [a_i]$ there are at most $\left\lfloor \frac{3}{2 \pi} \right\rfloor$ vertices in $X^p_j$ not contained in any set $Y_i$, $1 \leq i \leq \ell$. Since $\sum_{j=1}^k a_i \leq k2^{2k}$ and $kc \leq n$ we have

$$|Y_0| \leq |X_0| + k2^{2k} \left\lfloor \frac{3}{2 \pi} \right\rfloor \leq |X_0| + k2^{2k} \frac{3}{2 \pi} \leq |X_0| + \frac{n}{2 \pi}.$$

(b) We want to apply part (a) repeatedly to obtain an $\epsilon$-regular partition with parts of the same size. We need to apply (a) at most $\epsilon^{-5}$ times, since the $q$-value increases by $\epsilon^5$ each time and cannot be larger than 1.

Let $k \geq k_0$ be large enough such that $\frac{k}{\epsilon^{52k}} \leq \frac{\epsilon}{2}$. Further let $n_0$ be large enough such that $\frac{k}{n_0} \leq \frac{\epsilon}{2}$. This ensures that for any $n \geq n_0$ we have

$$k + \frac{n}{\epsilon^{52k}} \leq \epsilon n. \tag{1}$$

If $G$ has at most $n_0$ vertices then a partition where each vertex forms its own part satisfies the conditions. Otherwise we start with an arbitrary partition $X_0 \cup X_1 \cup \cdots \cup X_k$ with $|X_0| \leq k$ and $|X_1| = \cdots = |X_k|$. When applying the refinement the size of the exceptional set $X_0$ increases by at most $\frac{n}{2 \pi}$. Therefore inequality (1) ensures that the exceptional set after $\epsilon^{-5}$ steps has size at most $\epsilon n$. Moreover the number $K$ of parts in the final partition is at most a tower of 2’s of height $3\epsilon^{-5}$ like in Conlon’s notes.

**Question 3**

Let $0 < \nu < \frac{2}{3}$ and consider a triangle-free graph $G$ with $n$ vertices and minimum degree at least $(\frac{1}{3} + \nu)n$.

(a) Apply the regularity lemma as in question 2 part (b) to obtain an $\epsilon$-regular partition $V(G) = X_0 \cup X_1 \cup \cdots \cup X_k$ with exceptional set $X_0$. Consider the partition

$$V(G) = \bigcup_{I \subseteq \{1, \ldots, k\}} V_I \quad \text{with} \quad V_I = \{v \in V(G) \mid |N_G(v) \cap X_i| \geq \frac{\epsilon}{6} |X_i| \Leftrightarrow i \in I\}.$$

Prove the following statements for each $I \subseteq [k]$.

1. If $\bigcup_{i \in I} X_i \leq \frac{2}{3} n$, then any two vertices in $V_I$ have a common neighbor.
2. If $\bigcup_{i \in I} X_i \geq \frac{2}{3} n$, then there is an $\epsilon$-regular pair $(X_i, X_j), i, j \in I$, of density at least $\frac{\nu}{5}$.
(3) Each set \( V_I \) is either independent or empty.

(b) Prove that there is a constant \( C \), depending on \( \nu \) only, such that \( \chi(G) \leq C \).

(Remark: One can show that four colors are sufficient using more sophisticated arguments. Moreover there are graphs on \( n \) vertices with minimum degree at most \( \frac{1}{3} n \) and arbitrarily large chromatic number.)

Solution

(a) We consider some fixed \( I \subseteq [k] \). Let \( U_I = \cup_{i \in I} X_i \).

**Case 1:** \( |U_I| \leq \frac{2}{3} n \). Each vertex in \( V_I \) has at most \( |X_0| \leq \epsilon n \) neighbors in \( X_0 \) and at most \( \frac{\nu}{6} |X_i| \) neighbors in \( X_i, i \not\in I \). Therefore each vertex in \( V_I \) has at least

\[
(\frac{1}{3} + \nu)n - \epsilon n - \frac{\nu}{6} n = (\frac{1}{3} + \nu - \frac{\nu}{12} - \frac{\nu}{6})n > \frac{1}{3} n = \frac{1}{2}|U_I|
\]

neighbors in \( U_I \). Hence each pair of vertices in \( V_I \) has a common neighbor in \( U_I \). Therefore \( V_I \) is independent, since \( G \) is triangle-free.

**Case 2:** \( |U_I| \geq \frac{2}{3} n \). Then \( |I| \geq \frac{2}{3} k \). Consider the graph \( H \) obtained from \( G[U_I] \) by removing all edges having both endpoints in some \( X_i, i \in I \). Observe that each vertex in \( G \) has at least \( (\frac{1}{3} + \nu)n - |V(G)\setminus U_I| \geq (\frac{1}{3} + \nu)n - \frac{1}{3} n = \nu n \) neighbors in \( U_I \). Moreover, since \( k \geq k_0 = \epsilon^{-1} \), for each \( v \in X_i \) at most \( |X_i| \leq \frac{\nu}{6} \leq \epsilon n \) of its neighbors are contained in \( X_i, i \in I \). Hence each vertex in \( H \) has degree at least \( (\nu - \epsilon)n \) in \( H \). By the Handshake-Lemma there are at least \( \frac{1}{2}|U_I| \cdot (\nu - \epsilon)n \geq \frac{\nu - \epsilon}{3} n^2 \) edges in \( H \).

Note that \( |X_i| \leq \frac{\nu}{6} \) for each \( i \in [k] \). The total number of edges in \( G \), and thus in \( H \), in not \( \epsilon \)-regular pairs \( (X_i, X_j), 1 \leq i, j \leq k \), is at most \( \epsilon n^2 \). The total number of edges in \( H \) in pairs \( (X_i, X_j) \) of density at most \( \frac{\nu}{6} \) is at most \( k^2 \nu \frac{(\nu)}{6}^2 = \frac{\nu}{5}^2 n^2 \). Hence there is a pair \( (X_p, X_q), p, q \in I \), that is \( \epsilon \)-regular and has density at least \( \frac{\nu}{6} \) since

\[
(\frac{\nu - \epsilon}{3} - \epsilon - \frac{\nu}{6}) n^2 = \frac{1}{3} (\nu - \frac{1}{3} \nu - \frac{1}{3} \nu) n^2 = \frac{1}{9} \nu n^2 > 0.
\]

Consider a vertex \( v \in V_I \). Then \( v \) has at least \( \frac{\nu}{6} |X_p| \geq \epsilon |X_p| \) neighbors in \( X_p \) and at least \( \frac{\nu}{6} |X_q| \geq \epsilon |X_q| \) neighbors in \( X_q \). Hence the graph between these neighborhoods has density at least \( d(X_p, Y_p) - \epsilon \geq \frac{\nu}{12} - \frac{\nu}{12} > 0 \). Thus there is an edge in this graph connecting two neighbors of \( v \) and hence forming a triangle in \( G \), a contradiction. Therefore \( V_I = \emptyset \).

(b) Consider a partition \( V(G) = \cup_{I \subseteq [k]} V_I \) from part (a). Since each \( V_I \) is independent we obtain a proper coloring of \( G \) by coloring all vertices in \( V_I \) with the same color. This coloring uses at most \( 2^k \leq 2^K \) colors, where \( K \) is the constant given by the regularity lemma that depends on \( \nu \) only (note that \( \epsilon \) and \( k_0 \) depend on \( \nu \) only).