Problem sheet 3

Due Monday, May 7th at 17:30.

Question 1

Prove the following statements. Assume that in any ϵ-regular pair (X, Y), $X \cap Y = \emptyset$.

(a) An ϵ-regular partition of a graph G is an ϵ-regular partition the complement of G.

(b) If (X, Y) is an ϵ-regular pair with density d, then at most $\epsilon|X|$ vertices in X have more than $(d+\epsilon)|Y|$ neighbors in Y.

(c) For each $\epsilon > 0$ each graph G with $|E(G)| \leq \epsilon^3 \lfloor \frac{n^2}{4} \rfloor$ has an ϵ-regular partition with two parts.

Solution

(a) Let $V(G) = X_1 \cup \cdots \cup X_k$ be an ϵ-regular partition of G. Let $\overline{G}$ be the complement of G. Consider two disjoint sets of vertices X, Y. Let $e_G(X, Y)$ be the number of edges between X and Y in G. Then the density in G is $d_G(X, Y) = \frac{e_G(X,Y)}{|X||Y|}$. Since there are $|X||Y| - e_G(X, Y)$ edges between X and Y in $\overline{G}$ the density in $\overline{G}$ is

$$d_{\overline{G}}(X, Y) = \frac{|X||Y| - e_G(X, Y)}{|X||Y|} = 1 - d_G(X, Y).$$

If (X, Y) is an ϵ-regular pair in G, then for $X' \subseteq X$, $Y' \subseteq Y$ with $|X'| \geq \epsilon|X|$, $|Y'| \geq \epsilon|Y|$ we have

$$|d_{\overline{G}}(X, Y) - d_{\overline{G}}(X', Y')| = |1 - d_G(X, Y) - (1 - d_G(X', Y'))| = |d_G(X, Y) - d_G(X', Y')| \leq \epsilon.$$

Thus (X, Y) is an ϵ-regular pair in $\overline{G}$. Moreover the following summation over all not ϵ-regular pairs (X_i, X_j) does not dependent on the graph between X_i and X_j and thus

$$\sum \frac{|X_i||X_j|}{n^2} \leq \epsilon.$$

Hence $V(G)$=$V(\overline{G}) = X_1 \cup \cdots \cup X_k$ is an ϵ-regular partition of $\overline{G}$.

(b) Let X' denote the set of vertices in X that have more than $(d+\epsilon)|Y|$ neighbors in Y. Then

$$d(X',Y) > \frac{|X'|(d+\epsilon)|Y|}{|X'||Y|} = d+\epsilon.$$

Therefore $|d - d(X',Y)| > \epsilon$ and thus $|X'| < \epsilon|X|$.

(c) Observe that for any two disjoint sets $X,\ Y \subseteq V(G)$, with $|X| \geq \epsilon\lfloor\frac{n}{2}\rfloor$, $|Y| \geq \epsilon\lceil\frac{n}{2}\rceil$ we have

$$d(X,Y) = \frac{|E(X,Y)|}{|X||Y|} \leq \frac{\epsilon^3\lfloor\frac{n^2}{4}\rfloor}{\epsilon^2\lfloor\frac{n^2}{4}\rfloor} = \epsilon.$$

Therefore any partition of the vertex set into two sets of size $\lfloor\frac{n}{2}\rfloor$ and $\lceil\frac{n}{2}\rceil$ is ϵ-regular.

Question 2

A partition $V(G) = X_0 \cup X_1 \cup \cdots \cup X_k$ with exceptional set X_0 is called ϵ-regular if $\sum |X_i||X_j| \leq \epsilon n^2$, where the sum is taken over all not ϵ-regular pairs (X_i, X_j), $\underline{1 \leq i,j \leq k}$. Consider the following function for a partition P that is similar to the mean square density

$$q(P) = \sum_{1\leq i,j\leq k} \tfrac{|X_i||X_j|}{n^2} d(X_i,X_j)^2 + \sum_{v\in X_0}\sum_{i=1}^{k} \tfrac{|X_i|}{n^2} d(X_i,\{v\})^2 + \sum_{v\in X_0}\sum_{u\in X_0} \tfrac{1}{n^2} d(\{u\},\{v\})^2.$$

Prove the following statements.

(a) Let P be a partition $V(G) = X_0 \cup X_1 \cup \cdots \cup X_k$ with $|X_1| = \ldots = |X_k|$, that is not ϵ-regular. Then there is a partition $V(G) = Y_0 \cup Y_1 \cup \cdots \cup Y_\ell$, with $k \leq \ell \leq k2^{3k}$, $|Y_0| \leq |X_0| + \frac{n}{2^k}$, $|Y_1| = \ldots = |Y_k|$, that has q-value least $q(P) + \epsilon^5$.

(b) For each integer $k_0 \geq 2$ and each $\epsilon > 0$ there is an integer K such that for any graph G on $n \geq k_0$ vertices there is an ϵ-regular partition $V(G) = X_0 \cup X_1 \cup \cdots \cup X_k$, with $k_0 \leq k \leq K$, exceptional set X_0 of size at most ϵn, and $|X_1| = \ldots = |X_k|$.

(**Hint:** Follow the proof in Conlon's lecture notes while after each application of Lemma 1 in Lecture 5 cut the sets into pieces of size $\lfloor 2^{-3k}|X_i|\rfloor$.)

Solution

(a) Observe that the q-value of a partition $V(G) = X_0 \cup X_1 \cup \cdots \cup X_k$ is the mean square density of the partition $V(G) = \left(\bigcup_{v\in X_0}\{v\}\right) \cup X_1 \cup \cdots \cup X_k$. Therefore the q-value does not decrease under refinement by Lemma 2 in Lecture 4.

Let $I = \{(i,j) \mid (X_i, X_j) \text{ is not } \epsilon\text{-regular}, 1 \leq i,j \leq k\}$. Like in the proof in the lecture (Lecture 5, Lemma 1 in the lecture notes) for each i, there are refinements $X_i = X_i^1, \ldots, X_i^{a_i}$, with $a_i \leq 2^{2k}$ such that for each pair $(i,j) \in I$

$$\sum_{p=1}^{a_i} \sum_{q=1}^{a_j} \frac{|X_i^p||X_j^q|}{|X_i||X_j|} d(X_i^p, X_j^q)^2 \geq d(X_i, X_j)^2 + \epsilon^4.$$

We apply Lemma 2 from Lecture 4 to show that the partition $V(G) = \bigcup_{i=0}^{k} \bigcup_{p=1}^{a_i} X_i^p$ has q-value

$$\begin{aligned}
&\sum_{1\leq i,j\leq k} \sum_{p=1}^{a_i} \sum_{q=1}^{a_j} \tfrac{|X_i^p||X_j^q|}{n^2} d(X_i^p, X_j^q)^2 + \sum_{v\in X_0} \sum_{i=1}^{k} \sum_{p=1}^{a_i} \tfrac{|X_i^p|}{n^2} d(X_i^p, \{v\})^2 + \sum_{v\in X_0} \sum_{u\in X_0} \tfrac{1}{n^2} d(\{u\},\{v\})^2 \\
\geq &\sum_{1\leq i,j\leq k} \sum_{p=1}^{a_i} \sum_{q=1}^{a_j} \tfrac{|X_i^p||X_j^q|}{n^2} d(X_i^p, X_j^q)^2 + \underbrace{\sum_{v\in X_0} \sum_{i=1}^{k} \tfrac{|X_i|}{n^2} d(X_i, \{v\})^2 + \sum_{v\in X_0} \sum_{u\in X_0} \tfrac{1}{n^2} d(\{u\},\{v\})^2}_{=A} \\
= &\sum_{(i,j)\notin I} \sum_{p=1}^{a_i} \sum_{q=1}^{a_j} \tfrac{|X_i^p||X_j^q|}{n^2} d(X_i^p, X_j^q)^2 + \sum_{(i,j)\in I} \sum_{p=1}^{a_i} \sum_{q=1}^{a_j} \tfrac{|X_i^p||X_j^q|}{n^2} d(X_i^p, X_j^q)^2 + A \\
\geq &\sum_{(i,j)\notin I} \tfrac{|X_i||X_j|}{n^2} d(X_i, X_j)^2 + \sum_{(i,j)\in I} \tfrac{|X_i||X_j|}{n^2} \sum_{p=1}^{a_i} \sum_{q=1}^{a_j} \tfrac{|X_i^p||X_j^q|}{|X_i||X_j|} d(X_i^p, X_j^q)^2 + A \\
\geq &\sum_{(i,j)\notin I} \tfrac{|X_i||X_j|}{n^2} d(X_i, X_j)^2 + \sum_{(i,j)\in I} \tfrac{|X_i||X_j|}{n^2} \left(d(X_i, X_j)^2 + \epsilon^4\right) + A \\
= &\sum_{1\leq i,j\leq k} \tfrac{|X_i||X_j|}{n^2} d(X_i, X_j)^2 + \epsilon^4 \cdot \underbrace{\sum_{(i,j)\in I} \tfrac{|X_i||X_j|}{n^2}}_{>\epsilon \text{ since } P \text{ not } \epsilon\text{-regular}} + A \\
\geq &\sum_{1\leq i,j\leq k} \tfrac{|X_i||X_j|}{n^2} d(X_i, X_j)^2 + A + \epsilon^5 \\
= &q(P) + \epsilon^5.
\end{aligned}$$

This shows that the partition $V(G) = \bigcup_{i=0}^{k} \bigcup_{p=1}^{a_i} X_i^p$ has q-value at least ϵ^5 larger than P. Next we will cut the sets X_i^p, $1 \leq i \leq k$, $1 \leq p \leq a_i$, into not too many pieces of the same size, collecting the leftover in the exceptional set.

Let $c = |X_1| = \cdots = |X_k|$ and $d = \lfloor \frac{c}{2^{3k}} \rfloor$. Choose a maximal collection $Y_1 \ldots, Y_\ell$ of sets of size d such that each Y_t, $1 \leq t \leq \ell$, is contained in some X_j^p. Further let $Y_0 = V(G) \setminus \cup_{i=1}^{\ell} Y_i$. Then all parts $Y_1, \ldots, Y_\ell$ have same size and the q-value did not decrease by Lemma 2 from Lecture 4, so is still at least $q(P) + \epsilon^5$. It remains to show that $\ell \leq k2^{3k}$ and $|Y_0| \leq |X_0| + \frac{n}{2^k}$.

Observe that each X_j contains at most $|X_j|/\lfloor \frac{c}{2^{3k}} \rfloor \leq 2^{3k}$ sets Y_i. Hence $k \leq \ell \leq k2^{3k}$.

Moreover for each $j \in [k]$, $p \in [a_i]$ there are at most $\lfloor \frac{c}{2^{3k}} \rfloor$ vertices in X_j^p not contained in any set Y_i, $1 \leq i \leq \ell$. Since $\sum_{j=1}^{k} a_i \leq k2^{2k}$ and $kc \leq n$ we have

$$|Y_0| \leq |X_0| + k2^{2k} \lfloor \tfrac{c}{2^{3k}} \rfloor \leq |X_0| + k2^{2k} \tfrac{c}{2^{3k}} \leq |X_0| + \tfrac{n}{2^k}.$$

(b) We want to apply part (a) repeatedly to obtain an ϵ-regular partition with parts of the same size. We need to apply (a) at most ϵ^{-5} times, since the q-value increases by ϵ^5 each time and cannot be larger than 1.

Let $k \geq k_0$ be large enough such that $\frac{1}{\epsilon^5 2^k} \leq \frac{\epsilon}{2}$. Further let n_0 be large enough such that $\frac{k}{n_0} \leq \frac{\epsilon}{2}$. This ensures that for any $n \geq n_0$ we have

$$k + \frac{n}{\epsilon^5 2^k} \leq \epsilon n. \tag{1}$$

If G has at most n_0 vertices then a partition where each vertex forms its own part satisfies the conditions. Otherwise we start with an arbitrary partition $X_0 \cup X_1 \cup \cdots \cup X_k$ with $|X_0| \leq k$ and $|X_1| = \cdots = |X_k|$. When applying the refinement the size of the exceptional set X_0 increases by at most $\frac{n}{2^k}$. Therefore inequality (1) ensures that the exceptional set after ϵ^{-5} steps has size at most ϵn. Moreover the number K of parts in the final partition is at most a tower of 2's of height $3\epsilon^{-5}$ like in Conlon's notes.

Question 3

Let $0 < \nu < \frac{2}{3}$ and consider a triangle-free graph G with n vertices and minimum degree at least $(\frac{1}{3} + \nu)n$.

(a) Apply the regularity lemma as in question 2 part (b) to obtain an ϵ-regular partition $V(G) = X_0 \cup X_1 \cup \cdots \cup X_k$ with exceptional set X_0. Consider the partition

$$V(G) = \bigcup_{I \subseteq \{1,\ldots,k\}} V_I \qquad \text{with} \qquad V_I = \{v \in V(G) \mid |N_G(v) \cap X_i| \geq \tfrac{\nu}{9}|X_i| \Leftrightarrow i \in I\}.$$

Prove the following statements for each $I \subseteq [k]$.

(1) If $|\cup_{i \in I} X_i| \leq \frac{2}{3}n$, then any two vertices in V_I have a common neighbor.

(2) If $|\cup_{i \in I} X_i| \geq \frac{2}{3}n$, then there is an ϵ-regular pair (X_i, X_j), $i, j \in I$, of density at least $\frac{\nu}{9}$.

(3) Each set V_I is either independent or empty.

(b) Prove that there is a constant C, depending on ν only, such that $\chi(G) \leq C$.

(*Remark:* One can show that four colors are sufficient using more sophisticated arguments. Moreover there are graphs on n vertices with minimum degree at most $\frac{1}{3}n$ and arbitrarily large chromatic number.)

Solution

(a) We consider some fixed $I \subseteq [k]$. Let $U_I = \cup_{i \in I} X_i$.

Case 1: $|U_I| \leq \frac{2}{3}n$. Each vertex in V_I has at most $|X_0| \leq \epsilon n$ neighbors in X_0 and at most $\frac{\nu}{9}|X_i|$ neighbors in X_i, $i \notin I$. Therefore each vertex in V_I has at least

$$(\tfrac{1}{3} + \nu)n - \epsilon n - \tfrac{\nu}{9}n = (\tfrac{1}{3} + \nu - \tfrac{\nu}{12} - \tfrac{\nu}{9})n > \tfrac{1}{3}n = \tfrac{1}{2}|U_I|$$

neighbors in U_I. Hence each pair of vertices in V_I has a common neighbor in U_I. Therefore $\underline{V_I \text{ is independent}}$, since G is triangle-free.

Case 2: $|U_I| \geq \frac{2}{3}n$. Then $|I| \geq \frac{2}{3}k$. Consider the graph H obtained from $G[U_I]$ by removing all edges having both endpoints in some X_i, $i \in I$. Observe that each vertex in G has at least $(\frac{1}{3}+\nu)n - |V(G) \setminus U_I| \geq (\frac{1}{3}+\nu)n - \frac{1}{3}n = \nu n$ neighbors in U_I. Moreover, since $k \geq k_0 = \epsilon^{-1}$, for each $v \in X_i$ at most $|X_i| \leq \frac{n}{k} \leq \epsilon n$ of its neighbors are contained in X_i, $i \in I$. Hence each vertex in H has degree at least $(\nu - \epsilon)n$ in H. By the Handshake-Lemma there are at least $\frac{1}{2}|U_I| \cdot (\nu - \epsilon)n \geq \frac{\nu - \epsilon}{3}n^2$ edges in H.

Note that $|X_i| \leq \frac{n}{k}$ for each $i \in [k]$. The total number of edges in G, and thus in H, in not ϵ-regular pairs (X_i, X_j), $1 \leq i, j \leq k$, is at most ϵn^2. The total number of edges in H in pairs (X_i, X_j) of density at most $\frac{\nu}{9}$ is at most $k^2 \frac{\nu}{9} (\frac{n}{k})^2 = \frac{\nu}{9} n^2$. Hence there is a pair (X_p, X_q), p, $q \in I$, that is ϵ-regular and has density at least $\frac{\nu}{9}$ since

$$\left(\tfrac{\nu - \epsilon}{3} - \epsilon - \tfrac{\nu}{9}\right) n^2 = \tfrac{1}{3}\left(\nu - \tfrac{1}{3}\nu - \tfrac{1}{3}\nu\right) n^2 = \tfrac{1}{9}\nu\, n^2 > 0.$$

Consider a vertex $v \in V_I$. Then v has at least $\frac{\nu}{9}|X_p| \geq \epsilon |X_p|$ neighbors in X_p and at least $\frac{\nu}{9}|X_q| \geq \epsilon |X_q|$ neighbors in X_q. Hence the graph between these neighborhoods has density at least $d(X_p, Y_p) - \epsilon \geq \frac{\nu}{9} - \frac{\nu}{12} > 0$. Thus there is an edge in this graph connecting two neighbors of v and hence forming a triangle in G, a contradiction. Therefore $\underline{V_I = \emptyset}$.

(b) Consider a partition $V(G) = \cup_{I \subseteq [k]} V_I$ from part (a). Since each V_I is independent we obtain a proper coloring of G by coloring all vertices in V_I with the same color. This coloring uses at most $2^k \leq 2^K$ colors, where K is the constant given by the regularity lemma that depends on ν only (note that ϵ and k_0 depend on ν only).