Problem sheet 4

Due Monday, May 14th at 17:30.

Question 1

A graph is *triangle-formed* if each edge is contained in exactly one triangle. Prove that for every $\lambda > 0$ there is n_0 such that each triangle-formed graph on n vertices, with $n \geq n_0$, has at most λn^2 edges.

Question 2

(a) Prove that if G is a a graph on n vertices which consists of the union of n induced matchings, then $|E(G)| = o(n^2)$.

(**Hint:** Start the proof by removing edges as we did in the lecture for the triangle removal lemma and the alternative proof of Erdős-Stone-Simonovits. Moreover remove the following additional set of edges. For each of the given induced matchings M, remove the edges which are incident to V_i and satisfying $|V_i \cap V(M)| \leq \epsilon |V_i|$ for each i.)

(b) Use part (a) to give an alternative proof of Roth's theorem.

(c) Use part (a) to show that if H is a 3-uniform hypergraph with no 6 vertices spanning at least 3 edges then $|E(H)| = o(n^2)$.

(**Definition:** A hypergraph is a pair (V, E) where V is a set of elements and E is a collection of non-empty subsets of V. A k-uniform hypergraph is a hypergraph where all the elements in E are of size k. Note that a graph is a 2-uniform hypergraph.)

Question 3

Suppose that $\Delta \in \mathbb{N}$. Prove that there exists a constant c such that

$$R(H) \leq c|V(H)|$$

for every graph H with maximum degree $\Delta(H) \leq \Delta$.

(**Definition**: We define $R(H)$, the Ramsey number of H, to be the smallest number n such that in any two colouring of K_n there is a monochromatic copy of H.)

(**Hint:** For a solution to this question you would need Szemerédi's regularity lemma, Turán's theorem, the fact that $R(K_{\Delta+1}) \leq 4^{\Delta}$ and the counting lemma.)