Problem sheet 4

Due Monday, May 14th at 17:30.

Question 1

A graph is *triangle-formed* if each edge is contained in exactly one triangle. Prove that for every $\lambda > 0$ there is n_0 such that each triangle-formed graph on n vertices, with $n \geq n_0$, has at most λn^2 edges.

Solution Let t denote the number of triangles in a graph G. If G is triangles-formed, then $|E(G)| = 3\,t$, since each edge is contained in exactly one triangle. Hence one needs to remove $t = \frac{|E(G)|}{3}$ edges from G to make G triangle-free. Let $\epsilon = \frac{\lambda}{3}$. Due to the triangle removal lemma, there is $\delta > 0$ such that every graph on n vertices with at most δn^3 triangles can be made triangle-free by removing at most ϵn^2 edges.

Let $n_0 = \frac{1}{6\delta}$. Assume that G is triangle-formed with $n \geq n_0$ vertices and more than λn^2 edges. Then more than $\frac{\lambda n^2}{3} = \epsilon n^2$ edges are needed to make G triangle-free, as argued above. Due to the triangle removal lemma, G has more than

$$\delta n^3 \geq \delta n_0 n^2 = \tfrac{1}{3}\tfrac{1}{2}n^2 \geq \tfrac{1}{3}\binom{n}{2} \geq \tfrac{1}{3}|E(G)| = t$$

triangles, a contradiction.

Question 2

(a) Prove that if G is a a graph on n vertices which consists of the union of n induced matchings. Then $|E(G)| = o(n^2)$.

(**Hint:** Start the proof by removing edges as we did in the lecture for the triangle removal lemma and the alternative proof of Erdős-Stone-Simonovits. Moreover remove the following additional set of edges. For each of the given induced matchings M, remove the edges which are incident to V_i and satisfying $|V_i \cap V(M)| \leq \epsilon |V_i|$ for each i.)

(b) Use part (a) to give an alternative proof of Roth's theorem.

(c) Use part (a) to show that if H is a 3-uniform hypergraph with no 6 vertices spanning at least 3 edges then $|E(H)| = o(n^2)$.

Solution The result in the question was first proved by Ruzsa and Szemerédi.

(a) Let $\delta > 0$, we show that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, any graph on n vertices as above has $|E(G)| \leq \delta n^2$. Assume to the contrary that $|E(G)| \geq \delta n$. Apply Szemerédi's regularity lemma as in Problem sheet 3 with $0 < \epsilon < \frac{\delta}{8}$ and $k_0 > \frac{1}{\varepsilon}$. Let K be the resulting constant and let $V_0, V_1, ..., V_k$ be an ϵ-regular partition such that $|V_0| \leq \epsilon n$ and $|V_1| = |V_2| = ... = |V_k|$. We remove from G the following sets of edges,

(1) Edges incident to vertices in V_0. There are at most $|V_0| \cdot n = \epsilon n^2$ such edges.

(2) Edges contained in the sets V_i, $i \in [k]$. There are at most $k \cdot \left(\frac{n}{k}\right)^2 \leq \epsilon n^2$ such edges.

(3) Edges between the sets V_i, V_j so (V_i, V_j) is not an ϵ-regular pair. There are at most ϵn^2 such edges.

(4) Edges between the sets V_i, V_j so $d(V_i, V_j) \leq 2\epsilon$. There are at most $2\epsilon \left(\frac{n}{k}\right)^2 \cdot k^2 \leq 2\epsilon n^2$ such edges.

(5) For each matching M, and $i \in [k]$. the edges which are incident to V_i and satisfying $|V_i \cap M| \leq \epsilon |V_i|$. There are at most $n \cdot \epsilon n = \epsilon n^2$ such edges.

Let G' be the resulting subgraph of G. Then $|E(G')| \geq (\delta - 6\epsilon)n^2$. By the pigeonhole principle there must be a matching M so $|M \cap E(G')| \geq (\delta - 6\epsilon)n \geq 1$. Let $i \neq j \in [k]$ be such that there is an edge $e \in M$ such that $V_i \cap e \neq \emptyset$ and $V_j \cap e \neq \emptyset$. By (5) we know that $|V(M) \cap V_i| \geq \epsilon n$ and $|V(M) \cap V_j| \geq \epsilon n$. The pair (V_i, V_j) is ϵ-regular and $d(V_i, V_j) \geq 2\epsilon$, therefore $|E\left(V(M) \cap V_i, V(M) \cap V_j\right)| \geq \epsilon |V(M) \cap V_i| \cdot |V(M) \cap V_j| > \min\{|V(M) \cap V_i|, |V(M) \cap V_j|\}$ for n large enough, in contradiction to the fact that M is an induced matching.

(b) Let $A \subseteq [n]$ such that there is $\delta > 0$ such that $|A| \geq \delta n$. Let G be a bipartite graph with a bipartition (X, Y) so $X = [2n]$ and $Y = [3n]$. For each $x \in X$ define a matching $M_x = \{\{x+a, x+2a\} | a \in A\}$ where the computation is modulo $3n$. Let $E(G) = \cup_{x \in X} M_x$. Then $|V(G)| = 5n$ and $|E(G)| \geq \frac{\delta}{5}|V(G)|$ edges. By part (a) this means that not all the matchings are induced. Let $x \in X$ so M_x is not an induced matching. Let $a \neq b \in A$ so we have an edge e between the edges $\{x+a, x+2a\}, \{x+b, x+2b\}$. Assume without loss of generality that $e = \{x+a, x+2b\}$, then there are x' and c such that $x+a = x'+c$ and

$x + 2b = x' + 2c$. Hence $c = 2b - a$ which can rewritten as $c - b = b - a$. The sequence a, b, c is an arithmetic progression. Indeed, it can be rewritten as $a, a + (b - a), b + (c - b)$ and $b + (c - b) = b + (b - a) = a + 2(b - a)$ as required.

(c) We can assume that every vertex is contained in at least 2 hyperedges. If there are vertices which are contained in at most 1 hyperedge then we remove them from the hypergraph. For each vertex $v \in V(H)$, let $M_v := \{e \setminus \{v\} | e \in E(H), e \cap \{v\} \neq \emptyset\}$. If there is $v \in V(H)$ such that M_v is not a matching then there are hyperedges $e_1 = \{v, x, y\}$ and $e_2 = \{v, x, y'\}$ in $E(H)$, choosing e_1, e_2 together with some edge e_3 that contains y which is different from e_1 gives us the required set of vertices and hyperedges. A hyperedge e_3 exists due to our assumption that every vertex is contained in at least 2 hyperedges.

Let G be a graph with $V(G) = V(H)$ and $E(G) = \cup_{v \in V(H)} M_v$. By part (a) there must be a vertex $v \in V(H)$ so M_v is not an induced matching. Let $\{x, y\}, \{x', y'\} \in M_v$ and assume without loss of generality that $\{x', y\} \in E(G)$. Then the hyperedges $\{v, x, y\}, \{v, x', y'\}, \{v', x, y\}$ gives us the required set of vertices and hyperedges.

Question 3

Suppose that $\Delta \in \mathbb{N}$. Prove that there exists a constant c such that

$$R(H) \leq c|V(H)|$$

for every graph H with maximum degree $\Delta(H) \leq \Delta$. We define $R(H)$, the Ramsey number of H, to be the smallest number n such that in any two colouring of K_n we get a monochromatic copy of H.

(**Hint:** For a solution to this question you would need Szemerédi's regularity lemma, Turán's theorem, the fact that $R(K_{\Delta+1}) \leq 4^{\Delta}$ and the counting lemma.)

Solution The result in the question was first proved by Chvátal, Rődl, Szemerédi and Trotter.

Let $|V(H)| = h$, let $0 < \epsilon < \frac{1}{4}$, let $k_0 = 4^{\Delta}$ and let $0 < \epsilon' < \min\{\frac{1}{k_0^2}, \frac{\epsilon^{\Delta}}{2\Delta}\}$. Let $K = K(\epsilon')$ be the constant we get from Semerédi's regularity lemma. Let $c \geq \frac{4K}{\epsilon^{\Delta}}$.

Let G be a graph on $n \geq ch$ vertices. We will show that either G or $\overline{G}$ contains H. By Semerédi's regularity lemma there is an ϵ'-regular partition $X_0 \cup X_1 \cup ... \cup X_k$ of $V(G)$ so $k_0 \leq k \leq K$, $|X_0| \leq \epsilon' n$ and $|X_1| = |X_2| = ... = |X_k|$.

Let R be the reduced graph where each vertex corresponds to a set X_i, $i \in [k]$. There is an edge between the vertices which correspond to the sets X_i, X_j if and only if the pair (X_i, X_j)

is ϵ'-regular. Using the assumption on ϵ', we have that,

$$|E(R)| \geq \binom{k}{2} - \epsilon' k^2 = \left(1 - \frac{1}{k} - 2\epsilon'\right)\frac{k^2}{2} \geq \left(1 - \frac{1}{k_0} - \frac{1}{k_0(k_0-1)}\right)\frac{k^2}{2} = \left(1 - \frac{1}{k_0 - 1}\right)\frac{k^2}{2}$$

Hence by Turán's theorem there are $k_0 = 4^\Delta$ sets $X_{i_1}, X_{i_2}, ..., X_{i_{4^\Delta}}$ sets such that every pair of such sets is ϵ'-regular. For every pair of sets X_{i_j}, X_{i_t}, $j, t \in [4^\Delta]$, if the density $d(X_{i_j}, X_{i_t}) \geq \frac{1}{2}$, then we colour the edge between X_{i_j}, X_{i_t} in R in red, and we colour it in blue otherwise. By the bound $R(K_{\Delta+1}) \leq 4^\Delta$ there must be either red or blue clique of size $\Delta+1$, let $X_{j_1}, ..., X_{j_{\Delta+1}}$ be the corresponding sets. By part (a) in question 1 in problem sheet 3 we can assume without loss of generality that the clique is red.

For each $t \in [\Delta + 1]$,

$$|X_{j_t}| \geq \frac{n - |X_0|}{k} \geq \frac{(1-\epsilon')n}{k} \geq \frac{ch}{2k} \geq \frac{2h}{\epsilon^\Delta}.$$

Each pair $X_{j_t}, X_{j'_t}$ has density $\frac{1}{2} \geq 2\epsilon$ and it is ϵ'-regular and therefore $\frac{\epsilon^\Delta}{2\Delta}$-regular. Using that fact that $\chi(H) \leq \Delta + 1$ and applying the counting lemma, we get that G contains H. This completes the proof.