Problem sheet 4
Due Monday, May 14th at 17:30.

Question 1

A graph is triangle-formed if each edge is contained in exactly one triangle. Prove that for every \( \lambda > 0 \) there is \( n_0 \) such that each triangle-formed graph on \( n \) vertices, with \( n \geq n_0 \), has at most \( \lambda n^2 \) edges.

Solution  Let \( t \) denote the number of triangles in a graph \( G \). If \( G \) is triangles-formed, then \( |E(G)| = 3t \), since each edge is contained in exactly one triangle. Hence one needs to remove \( t = \frac{|E(G)|}{3} \) edges from \( G \) to make \( G \) triangle-free. Let \( \epsilon = \frac{\lambda}{3} \). Due to the triangle removal lemma, there is \( \delta > 0 \) such that every graph on \( n \) vertices with at most \( \delta n^3 \) triangles can be made triangle-free by removing at most \( \epsilon n^2 \) edges.

Let \( n_0 = \frac{1}{6\delta} \). Assume that \( G \) is triangle-formed with \( n \geq n_0 \) vertices and more than \( \lambda n^2 \) edges. Then more than \( \frac{\lambda n^2}{3} = \epsilon n^2 \) edges are needed to make \( G \) triangle-free, as argued above. Due to the triangle removal lemma, \( G \) has more than

\[
\delta n^3 \geq \delta n_0 n^2 = \frac{1}{3} \frac{1}{2} n^2 \geq \frac{1}{3} \left( \frac{n}{2} \right) \geq \frac{1}{3} |E(G)| = t
\]

triangles, a contradiction.

Question 2

(a) Prove that if \( G \) is a a graph on \( n \) vertices which consists of the union of \( n \) induced matchings. Then \( |E(G)| = o(n^2) \).

(Hint: Start the proof by removing edges as we did in the lecture for the triangle removal lemma and the alternative proof of Erdős-Stone-Simonovits. Moreover remove the following additional set of edges. For each of the given induced matchings \( M \), remove the edges which are incident to \( V_i \) and satisfying \( |V_i \cap V(M)| \leq \epsilon |V_i| \) for each \( i \).)
(b) Use part (a) to give an alternative proof of Roth’s theorem.

(c) Use part (a) to show that if $H$ is a 3-uniform hypergraph with no 6 vertices spanning at least 3 edges then $|E(H)| = o(n^2)$.

Solution  The result in the question was first proved by Ruzsa and Szemerédi.

(a) Let $\delta > 0$, we show that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, any graph on $n$ vertices as above has $|E(G)| \leq \delta n^2$. Assume to the contrary that $|E(G)| \geq \delta n$. Apply Szemerédi’s regularity lemma as in Problem sheet 3 with $0 < \epsilon < \frac{\delta}{8}$ and $k_0 > \frac{1}{\epsilon}$. Let $K$ be the resulting constant and let $V_0, V_1, ..., V_k$ be an $\epsilon$-regular partition such that $|V_0| \leq \epsilon n$ and $|V_1| = |V_2| = ... = |V_k|$. We remove from $G$ the following sets of edges,

1. Edges incident to vertices in $V_0$. There are at most $|V_0| \cdot n = \epsilon n^2$ such edges.
2. Edges contained in the sets $V_i$, $i \in [k]$. There are at most $k \cdot (\frac{n}{k})^2 \leq \epsilon n^2$ such edges.
3. Edges between the sets $V_i, V_j$ so $(V_i, V_j)$ is not an $\epsilon$-regular pair. There are at most $\epsilon n^2$ such edges.
4. Edges between the sets $V_i, V_j$ so $d(V_i, V_j) \leq 2\epsilon$. There are at most $2\epsilon (\frac{n}{k})^2 \cdot k^2 \leq 2\epsilon n^2$ such edges.
5. For each matching $M$, and $i \in [k]$, the edges which are incident to $V_i$ and satisfying $|V_i \cap M| \leq \epsilon |V_i|$. There are at most $n \cdot \epsilon n = \epsilon n^2$ such edges.

Let $G'$ be the resulting subgraph of $G$. Then $|E(G')| \geq (\delta - 6\epsilon)n^2$. By the pigeonhole principle there must be a matching $M$ so $|M \cap E(G')| \geq (\delta - 6\epsilon)n \geq 1$. Let $i \neq j \in [k]$ be such that there is an edge $e \in M$ such that $V_i \cap e \neq \emptyset$ and $V_j \cap e \neq \emptyset$. By (5) we know that $|V(M) \cap V_i| \geq \epsilon n$ and $|V(M) \cap V_j| \geq \epsilon n$. The pair $(V_i, V_j)$ is $\epsilon$-regular and $d(V_i, V_j) \geq 2\epsilon$, therefore $|E(V(M) \cap V_i, V(M) \cap V_j)| \geq \epsilon |V(M) \cap V_i| \cdot |V(M) \cap V_j| > \min\{|V(M) \cap V_i|, |V(M) \cap V_j|\}$ for $n$ large enough, in contradiction to the fact that $M$ is an induced matching.

(b) Let $A \subseteq [n]$ such that there is $\delta > 0$ such that $|A| \geq \delta n$. Let $G$ be a bipartite graph with a bipartition $(X, Y)$ so $X = [2n]$ and $Y = [3n]$. For each $x \in X$ define a matching $M_x = \{x + a, x + 2a\} | a \in A$ wherever the computation is modulo $3n$. Let $E(G) = \cup_{x \in X} M_x$. Then $|V(G)| = 5n$ and $|E(G)| \geq \frac{\delta}{5}|V(G)|$ edges. By part (a) this means that not all the matchings are induced. Let $x \in X$ so $M_x$ is not an induced matching. Let $a \neq b \in A$ so we have an edge $e$ between the edges $\{x + a, x + 2a\}, \{x + b, x + 2b\}$. Assume without loss of generality that $e = \{x + a, x + 2b\}$, then there are $x'$ and $c$ such that $x + a = x' + c$ and...
\[ x + 2b = x' + 2c. \] Hence \( c = 2b - a \) which can rewritten as \( c - b = b - a \). The sequence \( a, b, c \) is an arithmetic progression. Indeed, it can be rewritten as \( a, a + (b - a), b + (c - b) \)

and \( b + (c - b) = b + (b - a) = a + 2(b - a) \) as required.

(c) We can assume that every vertex is contained in at least 2 hyperedges. If there are vertices which are contained in at most 1 hyperedge then we remove them from the hypergraph. For each vertex \( v \in V(H) \), let \( M_v := \{ e \setminus \{ v \} \mid e \in E(H), e \cap \{ v \} \neq \emptyset \} \). If there is \( v \in V(H) \) such that \( M_v \) is not a matching then there are hyperedges \( e_1 = \{ v, x, y \} \) and \( e_2 = \{ v, x', y' \} \) in \( E(H) \), choosing \( e_1, e_2 \) together with some edge \( e_3 \) that contains \( y \) which is different from \( e_1 \) gives us the required set of vertices and hyperedges. A hyperedge \( e_3 \) exists due to our assumption that every vertex is contained in at least 2 hyperedges.

Let \( G \) be a graph with \( V(G) = V(H) \) and \( E(G) = \bigcup_{v \in V(H)} M_v \). By part (a) there must be a vertex \( v \in V(H) \) so \( M_v \) is not an induced matching. Let \( \{ x, y \}, \{ x', y' \} \in M_v \) and assume without loss of generality that \( \{ x', y' \} \in E(G) \). Then the hyperedges \( \{ v, x, y \}, \{ v, x', y' \}, \{ v', x, y \} \) gives us the required set of vertices and hyperedges.

**Question 3**

Suppose that \( \Delta \in \mathbb{N} \). Prove that there exists a constant \( c \) such that

\[
R(H) \leq c|V(H)|
\]

for every graph \( H \) with maximum degree \( \Delta(H) \leq \Delta \). We define \( R(H) \), the Ramsey number of \( H \), to be the smallest number \( n \) such that in any two colouring of \( K_n \) we get a monochromatic copy of \( H \).

(Hint: For a solution to this question you would need Szemerédi’s regularity lemma, Turán’s theorem, the fact that \( R(K_{\Delta+1}) \leq 4^\Delta \) and the counting lemma.)

**Solution** The result in the question was first proved by Chvátal, Rödl, Szemerédi and Trotter.

Let \( |V(H)| = h \), let \( 0 < \epsilon < \frac{1}{4} \), let \( k_0 = 4^\Delta \) and let \( 0 < \epsilon' < \min\{\frac{1}{k_0}, \frac{\epsilon'}{2\Delta} \} \). Let \( K = K(\epsilon') \) be the constant we get from Szemerédi’s regularity lemma. Let \( c \geq \frac{4K}{\epsilon} \).

Let \( G \) be a graph on \( n \geq ch \) vertices. We will show that either \( G \) or \( \overline{G} \) contains \( H \). By Szemerédi’s regularity lemma there is an \( \epsilon' \)-regular partition \( X_0 \cup X_1 \cup \ldots \cup X_k \) of \( V(G) \) so \( k_0 \leq k \leq K \), \( |X_0| \leq \epsilon'n \) and \( |X_1| = |X_2| = \ldots = |X_k| \).

Let \( R \) be the reduced graph where each vertex corresponds to a set \( X_i, i \in [k] \). There is an edge between the vertices which correspond to the sets \( X_i, X_j \) if and only if the pair \((X_i, X_j)\)
is \( \epsilon' \)-regular. Using the assumption on \( \epsilon' \), we have that,

\[
|E(R)| \geq \left( \frac{k}{2} \right) - \epsilon'k^2 = \left( 1 - \frac{1}{k} - 2\epsilon' \right) \frac{k^2}{2} = \left( 1 - \frac{1}{k_0(k_0 - 1)} \right) \frac{k^2}{2} = \left( 1 - \frac{1}{k_0 - 1} \right) \frac{k^2}{2}
\]

Hence by Turán’s theorem there are \( k_0 = 4^\Delta \) sets \( X_{i_1}, X_{i_2}, ..., X_{i_{4^\Delta}} \) sets such that every pair of such sets is \( \epsilon' \)-regular. For every pair of sets \( X_{ij}, X_{it}, j, t \in [4^\Delta] \), if the density \( d(X_{ij}, X_{it}) \geq \frac{1}{2} \), then we colour the edge between \( X_{ij}, X_{it} \) in \( R \) in red, and we colour it in blue otherwise. By the bound \( R(K_{\Delta+1}) \leq 4^\Delta \) there must be either red or blue clique of size \( \Delta + 1 \), let \( X_{j_1}, ..., X_{j_{\Delta+1}} \) be the corresponding sets. By part (a) in question 1 in problem sheet 3 we can assume without loss of generality that the clique is red.

For each \( t \in [\Delta + 1] \),

\[
|X_{jt}| \geq \frac{n - |X_0|}{k} \geq \frac{(1 - \epsilon')n}{k} \geq \frac{ch}{2k} \geq \frac{2h}{e^\Delta}.
\]

Each pair \( X_{j_i}, X_{j_i'} \) has density \( \frac{1}{2} \geq 2\epsilon \) and it is \( \epsilon' \)-regular and therefore \( \frac{\epsilon^2}{2^\Delta} \)-regular. Using that fact that \( \chi(H) \leq \Delta + 1 \) and applying the counting lemma, we get that \( G \) contains \( H \). This completes the proof.