Problem sheet 5

Due Monday, May 28th at 17:30.

Question 1

Let G be a bipartite graph with parts A and B and density d.

(a) Prove that there are $A' \subseteq A$, $B' \subseteq B$, with $|A'| \geq \lfloor d|A| \rfloor$, $|B'| \geq 2^{-|A|}|B|$ such that $A' \cup B'$ induces a complete bipartite graph in G.

(b) Prove that there is a constant c that depends on d only such that, if $|A| = |B| = n$, then there is a copy of $K_{a,b}$ in G with $a \geq c\log(n)$ and $b \geq \sqrt{n}$.

(**Hint:** Count pairs like in the proof of Theorem 1 in Lecture 8.)

Question 2

A 1-*subdivision* of a graph is obtained by subdividing each edge exactly once. Let $\epsilon \leq \frac{1}{2}$. Prove that each graph on $n \geq 16\epsilon^{-\frac{2}{3}}$ vertices and ϵn^2 edges contains a 1-subdivision of K_t with $t \geq \epsilon^{\frac{3}{2}} n^{\frac{1}{4}}$.

(**Hint:** Dependent Random Choice.)

Question 3

Prove that for each bipartite graph H with t vertices and $m \geq 2$ edges and for each $n \geq 2$

$$\mathrm{ex}(n, H) \geq \tfrac{1}{64}\, n^{2-\frac{t-2}{m-1}}.$$

Deduce Theorem 1 from Lecture 10.

(**Hint:** Consider a random graph.)

Question 4

Let $\text{ex}(n, C_{\geq k})$ denote the largest number of edges among all graphs on n vertices that do not contain any cycle of length at least k, $k \geq 3$.

(a) Prove that for each $k \geq 2$, each 2-connected graph with minimum degree k contains a cycle of length at least $2k$ or a Hamiltonian cycle.

(b) Prove that for all $k \geq 2$ and $n \geq 1$

$$\text{ex}(n, C_{\geq k+1}) \leq \frac{k}{2}(n-1).$$

(c) Prove that for each $k \geq 2$ there are infinitely many values of n with

$$\text{ex}(n, C_{\geq k+1}) = \frac{k}{2}(n-1).$$

Question 5

(a) Let G be a graph with n vertices such that $3 \leq d(v) \leq d$ for each $v \in V(G)$. Prove that the size of a set of vertices which intersect every cycle in G is at least $\frac{n+2}{d+1}$.

(b) Prove that if G has girth g and minimum degree at least 3 then the size of a set of vertices which intersect every cycle in G is at least $\frac{3}{8}2^{g/2}$.