Problem sheet 5

Due Monday, May 28th at 17:30.

Question 1

Let G be a bipartite graph with parts A and B and density d.

(a) Prove that there are $A' \subseteq A$, $B' \subseteq B$, with $|A'| \geq \lfloor d|A| \rfloor$, $|B'| \geq 2^{-|A|}|B|$ such that $A' \cup B'$ induces a complete bipartite graph in G.

(b) Prove that there is a constant c that depends on d only such that, if $|A| = |B| = n$, then there is a copy of $K_{a,b}$ in G with $a \geq c\log(n)$ and $b \geq \sqrt{n}$.

(**Hint:** Count pairs like in the proof of Theorem 1 in Lecture 8.)

Solution

(a) Let $d(v)$ denote the degree of a vertex v in G. We consider pairs (A', v) with $A' \subseteq A$, $|A'| = \lfloor d|A| \rfloor$, and $v \in B$, v adjacent to each vertex in A'. Let P be the number of such pairs. Then

$$\begin{aligned} P = \sum_{v \in B} \binom{d(v)}{\lfloor d|A| \rfloor} &\geq |B| \binom{\frac{1}{|B|}\sum_{v\in B} d(v)}{\lfloor d|A| \rfloor} \\ &= |B| \binom{\frac{1}{|B|}|E(G)|}{\lfloor d|A| \rfloor} = |B| \binom{d|A|}{\lfloor d|A| \rfloor} \geq |B| \\ &\geq 2^{-|A|}|B| \binom{|A|}{\lfloor d|A| \rfloor}. \end{aligned}$$

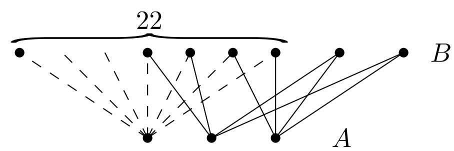
Here the second inequality is a variant of Jensen's inequality for the function $\binom{x}{d|A|}$. On the other hand,

$$P \leq \binom{|A|}{\lfloor d|A| \rfloor} \cdot M$$

where M is the maximum number of pairs (A', v) for some fixed A' and different $v \in B$ with $A' \subseteq N(v)$. Therefore, $M \geq 2^{-|A|}|B|$ and there is a set $A' \subseteq A$ of size $\lfloor d|A| \rfloor$ contained in at least $2^{-|A|}|B|$ such pairs. Hence there is a set B' of size at least $2^{-|A|}|B|$ such that each $v \in B$ is adjacent to each vertex in A'.

Remark: Here is an example of a graph that we can not ensure $|A'| \geq \lceil d|A| \rceil$.

Consider a bipartite graph G with parts A, $|A| = 3$, and B, $|B| = 24$, that is constructed from a vertex disjoint union of a star $K_{1,22}$ and a $K_{2,2}$ by adding four edges like in the figure.



This graph has density $\frac{30}{3 \cdot 24} = \frac{5}{12}$. Moreover $d|A| = \frac{5}{4}$, $\lceil \frac{5}{4} \rceil = 2$, and $2^{-|A|}|B| = 3$. But G contains no copy of $K_{2,3}$.

(b) Let $c = \frac{d}{2} \frac{1}{1-\log(d)}$ and $s = \lceil c \log(n) \rceil$. Then

$$\lceil c \log(n) \rceil \leq \tfrac{1}{2} dn \tag{1}$$

$$n \left(\tfrac{d}{2}\right)^{c \log(n)} = n^{1 + c \log(d/2)} = n^{1 - c(1 - \log(d))} = n^{1 - \frac{d}{2}} \geq n^{\frac{1}{2}}. \tag{2}$$

Similarly as in part (a) consider pairs (A', v) with $A' \subseteq A$, $|A'| = s$, and $v \in B$, v adjacent to each vertex in A'. The number of such pairs is given by

$$\sum_{v \in B} \binom{d(v)}{s} \geq n \binom{dn}{s} \overset{(1)}{\geq} n \frac{(\frac{1}{2}dn)^{c \log(n)}}{\lceil c \log(n) \rceil !} \geq n \left(\tfrac{d}{2}\right)^{c \log(n)} \binom{n}{s} \overset{(2)}{\geq} \sqrt{n} \binom{n}{s}.$$

Therefore, by pigeonhole principle, there is a set $A' \subseteq A$ of size s contained in at least $\sqrt{n}$ such pairs. Hence there is a set $B' \subseteq B$ of size at least $\sqrt{n}$ such that each $v \in B$ is adjacent to each vertex in A'.

Question 2

A 1-*subdivision* of a graph is obtained by subdividing each edge exactly once. Let $\epsilon \leq \frac{1}{2}$. Prove that each graph on $n \geq 16\epsilon^{-\frac{2}{3}}$ vertices and ϵn^2 edges contains a 1-subdivision of K_t with $t \geq \epsilon^{\frac{3}{2}} n^{\frac{1}{4}}$.

(**Hint:** Dependent Random Choice.)

Solution

Like in the proof of Theorem 1 in Lecture 9 there is a bipartite subgraph G' of G with parts A and B that contains at least $\frac{1}{2}\epsilon\, n^2$ edges. We apply the dependent random choice lemma with $r = 2$ and density $\alpha = \frac{1}{2}\epsilon$. This yields a subset $A' \subseteq A$ with

$$|A'| \geq \tfrac{1}{2}\alpha^2 n = \tfrac{1}{8}\epsilon^2 n = \tfrac{1}{8}\epsilon^2\, n^{\frac{3}{4}}\, n^{\frac{1}{4}} \geq \tfrac{1}{8}\epsilon^2\, 8\epsilon^{-\frac{1}{2}}\, n^{\frac{1}{4}} = \epsilon^{\frac{3}{2}} n^{\frac{1}{4}}$$

such that each pair of vertices in A' has at least

$$\tfrac{1}{4}\alpha n^{\frac{1}{2}} = \tfrac{1}{8}\epsilon n^{\frac{1}{2}} \geq \tfrac{1}{2}\epsilon^3 n^{\frac{1}{2}} > \binom{\epsilon^{\frac{3}{2}} n^{\frac{1}{4}}}{2}$$

common neighbors in B. Therefore we can embed a copy of a 1-subdivision of K_t with $t = \lceil \epsilon^{\frac{3}{2}} n^{\frac{1}{4}} \rceil$ in G' by choosing t arbitrary vertices from A' and the subdividing vertices from B.

Question 3

Prove that for each bipartite graph H with t vertices and $m \geq 2$ edges and for each $n \geq 2$

$$\mathrm{ex}(n, H) \geq \tfrac{1}{64}\, n^{2-\frac{t-2}{m-1}}.$$

Deduce Theorem 1 from Lecture 10.

(**Hint:** Consider a random graph.)

Solution

We will construct an H-free graph on n vertices with $cn^{2-\frac{t-2}{m-1}}$ edges randomly. Let $p = \frac{1}{8}n^{-\frac{t-2}{m-1}}$. Consider a graph $G = G_{n,p}$ with vertex set $[n]$ where each edge is contained in G with probability p. Let X be the random variable that is given by the number of copies of H in G. By linearity of expectation

$$\mathbb{E}(|E(G)|) = \sum_{\{u,v\}\in\binom{[n]}{2}} \mathbb{P}(uv \in E(G)) = \tbinom{n}{2}\, p \geq \tfrac{1}{4}\, p\, n^2,$$

$$\mathbb{E}(X) = \sum_{S\in\binom{[n]}{t}} \mathbb{P}(H \subseteq G[S]) \leq \binom{n}{t} t! p^m \leq n^t p^m = pn^t p^{m-1} \leq \tfrac{1}{8}\, p\, n^2.$$

Let G' be a graph obtained from G by removing one edge from each copy of H in G. Then G' is H-free and

$$\mathbb{E}(|E(G')|) \geq \mathbb{E}(|E(G)| - X) = \mathbb{E}(|E(G)|) - \mathbb{E}(X) \geq \tfrac{1}{8}\, p\, n^2 = \tfrac{1}{64}\, n^{2-\frac{t-2}{m-1}}.$$

Hence there is a graph G' on n vertices that is H-free and contains at least $\frac{1}{64}\, n^{2-\frac{t-2}{m-1}}$ edges.

For a cycle C_{2k} this yields Theorem 1 from Lecture 10 as

$$\mathrm{ex}(n, C_{2k}) \geq \tfrac{1}{64}\, n^{2-\frac{2k-2}{2k-1}} = \tfrac{1}{64}\, n^{1+\frac{1}{2k-1}}.$$

Question 4

Let $\mathrm{ex}(n, C_{\geq k})$ denote the largest number of edges among all graphs on n vertices that do not contain any cycle of length at least k, $k \geq 3$.

(a) Prove that for each $k \geq 2$, each 2-connected graph with minimum degree k contains a cycle of length at least $2k$ or a Hamiltonian cycle.

(b) Prove that for all $k \geq 2$ and $n \geq 1$

$$\mathrm{ex}(n, C_{\geq k+1}) \leq \frac{k}{2}\,(n-1).$$

(c) Prove that for each $k \geq 2$ there are infinitely many values of n with

$$\mathrm{ex}(n, C_{\geq k+1}) = \frac{k}{2}\,(n-1).$$

Solution

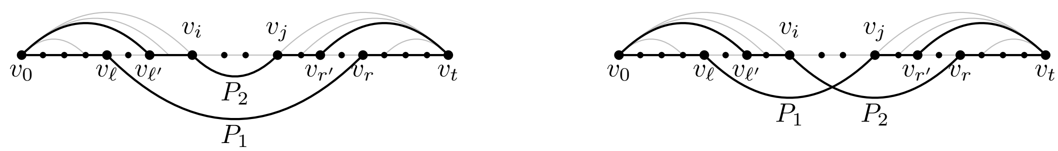
(a) Let G be a 2-connected graph with minimum degree at least k. Consider a longest path $P = v_0, \dots, v_t$ in G.

Case 1: If v_0 and v_t are adjacent then $v_0, \dots, v_t, v_0$ is a Hamiltonian cycle, since any vertex not in C that is adjacent to some vertex in C yields a longer path than P in G.

Case 2: There are indices i, j, with $0 < i < j < t$, such that v_0 is adjacent to v_j and v_t is adjacent to v_i. Let i, j be such indices with smallest difference $j - i$. Consider the cycle $C = v_0, \dots, v_i, v_t, \dots, v_j, v_0$. If $j = i + 1$, then C is Hamiltonian, since any vertex not in C that is adjacent to some vertex in C yields a longer path than P in G. If $j \geq i + 2$, then for each v_p, $2 \leq p \leq t$, adjacent to v_0, the vertex v_{p-1} is not adjacent to v_t. Moreover each vertex v_p with $i < p < j$ is neither adjacent to v_0 nor to v_t. Therefore C contains v_t, all neighbors of v_0, and for each neighbor v_p, $p \neq j$, the vertex v_{p-1}. Since v_t is not a neighbor of v_1, C has at least $2k$ vertices.

Case 3: We have $i \leq j$ whenever v_i adjacent to v_0 and v_j is adjacent to v_t. Let i be the largest integer with v_i adjacent to v_0, j the smallest integer with v_j adjacent to v_t.

Consider the cycles $C_1 = v_0, \ldots, v_i, v_0$ and $C_2 = v_j, \ldots, v_t, v_j$. Since G is 2-connected there are two vertex disjoint paths P_1 and P_2 connecting C_1 and C_2, each using only one vertex from C_1 and one from C_2. By walking along the subpath $v_i, \ldots, v_j$ of P, starting at v_i, until we meet either P_ν, $\nu = 1, 2$, or v_j for the first time, we can replace some part of P_ν such that P_ν starts at v_i and P_1 and P_2 are still disjoint paths connecting C_1 and C_2. Similarly we can replace some part of P_1 or P_2 such that one of the paths ends at v_j. Let $\{v_\ell, v_i\} = V(C_1) \cap (V(P_1) \cup V(P_2))$ and $\{v_j, v_r\} = V(C_2) \cap (V(P_1) \cup V(P_2))$. Further let ℓ' be the smallest index such that $\ell < \ell'$ and v_0 is adjacent to $v_{\ell'}$ and r' be the largest index such that $r' < r$ and v_t is adjacent to $v_{r'}$. Then the subpaths $v_i P v_{\ell'} v_0 P v_l$ and $v_j P v_{r'} v_t P v_r$ form a cycle with P_1 and P_2 that contains all neighbors of x_0 and all neighbors of x_t and hence has length larger than $2k$, see the picture below.



(b) Consider an arbitrary graph G on n vertices with more than $\frac{k}{2}(n-1)$ edges. Since $\frac{k}{2}(n-1) < |E(G)| \leq \binom{n}{2}$ we have $k < n$. We shall show that G contains a cycle of length at least $k+1$ by induction on $n = |V(G)|$. If $n = k+1$, then G has more than $\frac{k}{2}(n-1) = \frac{(n-1)^2}{2} = \binom{n}{2} - \frac{n-1}{2}$ edges. So there are at most $\lceil \frac{n-1}{2} \rceil - 1$ non-edges. Therefore G has minimum degree at least $(n-1) - \lceil \frac{n-1}{2} \rceil + 1 = \lfloor \frac{n-1}{2} \rfloor + 1 \geq \frac{n}{2}$. Thus G is hamiltonian by Dirac's theorem, that is, G contains a cycle of length $n = k+1$.

Consider $n > k+1$. If G is not 2-connected, let G_1 and G_2 denote subgraphs of G such that $|V(G_1) \cap V(G_2)| \leq 1$ and $E(G) = E(G_1) \cup E(G_2)$. Then

$$|E(G_1)| + |E(G_2)| = |E(G)| > \frac{k}{2}(n-1) \geq \frac{k}{2}(|V(G_1)| + |V(G_2)| - 2).$$

Therefore $|E(G_i)| > \frac{k}{2}(|V(G_i)| - 1)$ for some $i = 1, 2$, and hence G_i contains a cycle of length at least $k+1$ by induction. Clearly this cycle is contained in G.

If G contains a vertex v of degree at most $\frac{k}{2}$ in G, then

$$|E(G - v)| > \frac{k}{2}(n-1) - \frac{k}{2} = \frac{k}{2}(n-2).$$

Therefore $G - v$ contains a cycle of length at least $k+1$ by induction. Clearly this cycle is contained in G.

Hence G is 2-connected and each vertex has degree greater than $\frac{k}{2}$, then G contains a cycle of length more than $2\frac{k}{2} = k$ by the part (a).

(c) For each $k \geq 2$ and infinitely many values of n we construct graph that does not contains a cycle of length at least $k+1$. Suppose that $k-1$ divides $n-1$. Then construct a graph from $\frac{n-1}{k-1}$ vertex disjoint copies of K_{k-1} by adding a new vertex v that is connected to all other vertices. So this graph consists of $\frac{n-1}{k-1}$ edge disjoint copies of K_k all sharing exactly one vertex. This graph has n vertices, $\frac{n-1}{k-1}\binom{k}{2} = \frac{k}{2}(n-1)$ edges, and contains no cycle of length at least $k+1$.

Question 5

(a) Let G be a graph with n vertices such that $3 \leq d(v) \leq d$ for each $v \in V(G)$. Then the set of representatives of all the cycles in G is of size at least $\frac{n+2}{d+1}$. The set of representatives of all the cycles in G is a set of vertices which intersect every cycle in G.

(b) If G has girth g and minimum degree at least 3 then the set of representatives of all the cycles in G is of size at least $\frac{3}{8}2^{g/2}$.

Solution

(a) Let R be a set of representatives of all the cycles in G, then $G' = G[V(G) \setminus R]$ is a forest and $|E(G')| \leq n - |R| - 1$. The number of edges between R and $V(G) \setminus R$ is at least $3(n - |R|) - 2(n - |R| - 1) = n - |R| + 2$. On the other hand, the number of those edges is at most $d \cdot |R|$. Combining those two bounds we get the required lower bound on $|R|$.

(b) Let R be a set of representatives of all the cycles in G. We can assume that $g \geq 3$. Let $x \in V(G)$ and let $d = d(x)$. For each $i \in \{0, 1, 2, ..., \lfloor \frac{g-1}{2} \rfloor\}$, let V_i be all the points in G at distance i from x. For each $v \in V_i$ there is exactly one edge to V_{i-1}, as otherwise we get a cycle of length strictly smaller that g. Thus $|V_{i+1}| = (d-1) \cdot |V_i|$, $i \in \{1, 2, ..., \lfloor \frac{g-1}{2} \rfloor\}$. Hence,

$$|V(G)| = n \geq |V_0| + |V_1| + |V_3| + ... + |V_{\lfloor \frac{g-1}{2} \rfloor}| = 1 + d + d \cdot (d-1) + ... + d \cdot (d-1)^{\lfloor \frac{g-3}{2} \rfloor}$$

using $d \geq 3$ we get,

$$n \geq 1 + 3(2^{\lfloor \frac{g-3}{2} \rfloor} - 1).$$

By part (a),

$$|R| \geq \frac{n+2}{d+1} \geq \frac{3 + 3(2^{\lfloor \frac{g-3}{2} \rfloor} - 1)}{4} \geq \frac{3}{8} \cdot 2^{\frac{g}{2}}.$$