Problem sheet 6

Due Monday, June 4th at 17:30.

Question 1

(a) Let H be a graph and let T be a breadth first search (BFS) tree in H starting from an arbitrary vertex $x \in V(H)$. Let V_i be the set of vertices at distance exactly i from x. Show that if either $H[V_i]$ or $H_{i+1} = H[V_i, V_{i+1}]$ contain a cycle of length k with a chord, then for some $m \leq i$, H contains cycles $C_{2m+1}, C_{2m+2}, \dots, C_{2m+k-1}$ or cycles $C_{2m+2}, C_{2m+4}, \dots, C_{2m+\ell}$ where ℓ is the largest even integer which is less than k.

(b) Let k be a positive integer. Prove using part (a) that if H is a graph with $|V(H)| = n$ and $|E(H)| \geq 8kn$, then for some integer r, H contains cycles $C_{2r}, C_{2r+2}, \dots, C_{2r+2k-2}$.

(*Remark:* The bound in part (b) can be improved to $3kn$ using a theorem by Erdős and Gallai that says that for every $k \geq 3$, if G is a graph of average degree at least k then G contains a cycle of length at least $k+1$ with a chord. It is an open question to prove that this bound can be improved to $\frac{1}{2}(2k+1)(n-1)$. This bound would be tight.)

Question 2

A graph G with $|V(G)| = n$ and $|E(G)| = m$ is *pancyclic* if it contains a cycle of every length $3 \leq \ell \leq n$. Prove that if G contains a Hamilton cycle and $m \geq \frac{n^2}{4}$ then G is pancyclic unless $G = K_{\frac{n}{2},\frac{n}{2}}$.

(**Hint:** One way to prove the statement in the question is first to separate into two cases, where in the first case G contains C_{n-1} and in the other case G doesn't contain C_{n-1}. In the first case, G contains all the required cycles. In the second case, $G = K_{\frac{n}{2},\frac{n}{2}}$.)

Question 3

A graph H is edge-critical if $\chi(H \setminus e) < \chi(H)$ for any edge $e \in E(H)$. Let $T_r(n)$ be the complete r-partite graph on n vertices with parts of sizes as equal as possible. Prove that for n

sufficiently large the extremal graph on n vertices which doesn't contain an edge-critical graph H is $T_{\chi(H)-1}(n)$.

For the solution of the above question you can use the following stability result:

Theorem. *Let $K_{r+1}(h)$ be the complete $(r+1)$-partite graph with parts of size h. Let $h, r \geq 2$ and let G be a $K_{r+1}(h)$-free graph with $|V(G)| = n$. If $|E(G)| = \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2}$ then G contains an r-partite graph with minimum degree $\left(1 - \frac{1}{r} + o(1)\right)n$.*

(**Hint:** Start from an H-free extremal graph G. Let G' be a subgraph of G that we get from the above theorem. Assign the $o(n)$ vertices of $V(G) \setminus V(G')$ to the parts where they have the least number of neighbors. Show that actually each vertex has at most $o(n)$ neighbors in its part and all but $o(n)$ vertices are its neighbors in each other part. From here reach a contradiction if G is not $T_{\chi(H)-1}(n)$.)