Problem sheet 6

Due Monday, June 4th at 17:30.

Question 1

(a) Let H be a graph and let T be a breadth first search (BFS) tree in H starting from an arbitrary vertex $x \in V(H)$. Let V_i be the set of vertices at distance exactly i from x. Show that if either $H[V_i]$ or $H_{i+1} = H[V_i, V_{i+1}]$ contain a cycle of length k with a chord, then for some $m \leq i$, H contains cycles $C_{2m+1}, C_{2m+2}, ..., C_{2m+k-1}$ or cycles $C_{2m+2}, C_{2m+4}, ..., C_{2m+\ell}$ where ℓ is the largest even integer which is less than k.

(b) Let k be a positive integer. Prove using part (a) that if H is a graph with $|V(H)| = n$ and $|E(H)| \geq 8kn$, then for some integer r, H contains cycles $C_{2r}, C_{2r+2}, ..., C_{2r+2k-2}$.

(*Remark:* The bound in part (b) can be improved to $3kn$ using a theorem by Erdős and Gallai that says that for every $k \geq 3$, if G is a graph of average degree at least k then G contains a cycle of length at least $k+1$ with a chord. It is an open question to prove that this bound can be improved to $\frac{1}{2}(2k+1)(n-1)$. This bound would be tight.)

Solution

(a) Assume that $H[V_i]$ contains a cycle C of length k with a chord. Let y be the farthest vertex from x which still contains all of $V(C)$ as its descendants in T. Let a be one of the children of y. Let A be the (non-trivial) subset of vertices of C which are descendants of a and let $B = V(C) \setminus A$. If (A, B) is not a bipartition of $H[V(C)]$ then by Lemma 1 in Lecture 10, there are paths of every length $t < |V(C)| = k$ which begin in A and end in B. Let $\mathcal{P} = \{P_1, P_2, ..., P_{k-1}\}$ be a collection of those paths where P_j, $j \in [k-1]$ is a path of length j. We can extend each $P_j \in \mathcal{P}$, $j \in [k-1]$, to a cycle of length $j + 2(i - dist(x, y))$ by going through y. This way we obtain a set of cycles $C_{2m+1}, C_{2m+2}, ..., C_{2m+k-1}$ for $m = i - dist(x, y)$.

If (A, B) is a bipartition of $H[V(C)]$, then let z be a descendant of a so the descendants of z from $V(C)$ create a non-trivial subset A' of A and let $B' = V(C) \setminus A'$. By the choice of A',

(A', B') is not a bipartition of $V(C)$. Hence by Lemma 1 in Lecture 10, there are paths of every length $t < |V(C)| = k$ which begin in A' and end in B'. Let $\mathcal{P}' = \{P'_1, P'_2, ..., P'_{k-1}\}$ be the collection of those paths. Note that any even length path $\mathcal{P}'$ must start and end in A. We can extend each such even path P'_{2t} to a cycle of length $2k + 2(i - dist(x, a))$ by going through a. This way we obtain the cycles $C_{2m+2}, C_{2m+4}, ..., C_{2m+\ell}$ where ℓ is the largest even integer which is less than k and $m = i - dist(x, a)$.

The proof for H_{i+1} is similar.

(b) Let H be a graph with $|V(H)| = n$ and $|E(H)| \geq 8kn$. Note that by part (a) it is enough to show that H contains a cycle of length $(2k - 1)$ with a chord. Let T be a breadth first search (BFS) tree in H starting from an arbitrary vertex $x \in V(H)$. Let V_i be the set of vertices at distance exactly i from x. Then,

$$e(G) = \sum_{i=0}^{\infty} \left(e(V_i) + e(V_{i-1}, V_i)\right).$$

Assume to the contrary that H doesn't contain the required cycle. If for some i, $e(V_i) > 4k|V_i|$, then, as we saw in the proof of Claim 1 in lecture 11, $H[V_i]$ contains a cycle of length at least $(2k + 1)$ with a chord and so we are done by part (a). We are done in a similar way if there is an index i such that $e(V_{i-1}, V_i) \geq 2k(|V_{i-1}| + |V_i|)$. So,

$$e(G) = \sum_{i=0}^{\infty} \left(e(V_i) + e(V_{i-1}, V_i)\right) < 4kn + 4kn$$

which is a contradiction to our assumption.

Question 2

A graph G with $|V(G)| = n$ and $|E(G)| = m$ is *pancyclic* if it contains a cycle of every length $3 \leq \ell \leq n$. Prove that if G contains a Hamilton cycle and $m \geq \frac{n^2}{4}$ then G is pancyclic unless $G = K_{\frac{n}{2}, \frac{n}{2}}$.

(**Hint:** One way to prove the statement in the question is first to separate into two cases, where in the first case G contains C_{n-1} and in the other case G doesn't contain C_{n-1}. In the first case, G contains all the required cycles. In the second case, $G = K_{\frac{n}{2}, \frac{n}{2}}$.)

Solution Let G be a graph with $|V(G)| = n$, $|E(G)| = m$. Assume that G contains a Hamilton cycle and $m \geq \frac{n^2}{4}$. We prove that G is either pancyclic or $K_{\frac{n}{2}, \frac{n}{2}}$ by induction on n. It is not hard to check that the base case where $n = 3$ and $n = 4$ holds.

First assume that G contains C_{n-1}. Let $y_0, y_1, ..., y_{n-2}$ be the vertices of C_{n-1} and let $x \in V(G) \setminus V(C_{n-1})$.

If $d(x) \leq \frac{n}{2}$, then $e(G[C_{n-1}]) \geq \frac{n^2}{4} - \frac{n-1}{2} > \frac{(n-1)^2}{4}$, so by induction we get that all the cycles of the required lengths are in G.

If $d(x) > \frac{n}{2}$, then G contains also all cycles of length $3 \leq \ell \leq n-2$ (for $n \geq 5$, otherwise we are already done). Indeed, assume that there is no cycle of length ℓ in G. Then for each neighbour y_{i_j}, $j \in [d(x)]$, of x, then vertex $y_{i_{j+\ell}}$ is not a neighbour of x (the computations are modulo $n-1$), so $2d(x) \leq n-1$ which is a contradiction.

We assume now that G doesn't contain C_{n-1}. Let $x_0, x_1, ..., x_{n-1}$ be the vertices on the Hamilton cycle. All the following index computations are made modulo n. If there are indices $0 \leq i \neq j \leq n-1$ such that both $\{x_{i-1}, x_j\}$ and $\{x_i, x_{j+2}\}$ are edges in G then the cycle $x_{i-1}, x_j, x_{j-1}, ..., x_i, x_{j+2}, x_{j+3}, ..., x_{i-1}$ is a cycle of length $n-1$ in contradiction to our assumption. Therefore, for every $0 \leq i \leq n-1$, $d(x_{i-1}) + d(x_i) \leq n$ and $\sum_{i=0}^{n-1}(d(x_{i-1}) + d(x_i)) \leq n^2$. From this we can deduce that $4m \leq n^2$ and together with your assumption that $m \geq \frac{n^2}{4}$ we get that $m = \frac{n^2}{4}$ and n is even. Moreover for each $0 \leq i \leq n-1$ exactly one of the edges $\{x_{i-1}, x_j\}$ and $\{x_i, x_{j+2}\}$ is in G.

We define a partition $A \cap B$ of $V(G)$ where $A = \{x_{2k} | 0 \leq k \leq \frac{n}{2}\}$ and $B = \{x_{2k+1} | 0 \leq k \leq \frac{n}{2} - 1\}$. It is left to show that $G[A]$ and $G[B]$ are independent sets. Assume without loss of generality that $G[A]$ is not an independent set. Let $\{x_i, x_j\}$ be an edge in $G[A]$, then $i = j$ modulo 2. Let s be the smallest even positive integer such that for some k, $\{x_{k+1}, x_{k+1+s}\} \in E(G)$. Then $s \geq 4$ as otherwise $\{x_k, x_{k+1}\}$ is not an edge. Moreover $\{x_k, x_{k+1+s-2}\}$ is not an edge and $\{x_{k-1}, x_{(k-1)+s-2}\}$ is an edge. This is a contradiction to the choice of s.

Question 3

A graph H is edge-critical if $\chi(H \setminus e) < \chi(H)$ for any edge $e \in E(H)$. Let $T_r(n)$ be the complete r-partite graph on n vertices with parts of sizes as equal as possible. Prove that for n sufficiently large the extremal graph on n vertices which doesn't contain an edge-critical graph H is $T_{\chi(H)-1}(n)$.

For the solution of the above question you can use the following stability result:

Theorem. *Let $K_{r+1}(h)$ be the complete $(r+1)$-partite graph with parts of size h. Let $h, r \geq 2$ and let G be a $K_{r+1}(h)$-free graph with $|V(G)| = n$. If $|E(G)| = \left(1 - \frac{1}{r} + \mathrm{o}(1)\right)\binom{n}{2}$ then G contains an r-partite graph with minimum degree $\left(1 - \frac{1}{r} + \mathrm{o}(1)\right)n$.*

(**Hint:** Start from an H-free extremal graph G. Let G' be a subgraph of G that we get from the above theorem. Assign the $\mathrm{o}(n)$ vertices of $V(G) \setminus V(G')$ to the parts where they have the least number of neighbors. Show that actually each vertex has at most $\mathrm{o}(n)$ neighbors in its part and all but $o(n)$ vertices are its neighbors in each other part. From here reach a contradiction if G is not $T_{\chi(H)-1}(n)$.)

Solution Let G be an H-free graph on n vertices with maximal possible number of edges. Let $r = \chi(H) - 1$ and $|V(H)| = h$, then G is $K_{r+1}(h)$-free and with $|E(G)| \geq \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2}$. By the above Theorem G contains a subgraph G' with minimum degree $\left(1 - \frac{1}{r} + o(1)\right)n$. Let $V_1', V_2', ..., V_r'$ be the corresponding partition of G'. For each $i \in [r]$, $|V_i'| \geq \left(\frac{1}{r} + o(1)\right)n$. We assign the $o(n)$ vertices of $V(G) \setminus V(G')$ to the parts where they have the least number of neighbors. Let $V_1, V_2, ..., V_r$ be the resulting partition of $V(G)$.

Let $\epsilon > 0$ and assume that there is a vertex $x \in V(G)$ which is adjacent to at least ϵn vertices in its part, without loss of generality $x \in V_1$. Then x is also adjacent to at least ϵn vertices in each of the other parts V_i, $2 \leq i \leq r$. Let $U_i \subset N(x) \cap V_i$, $i \in [r]$, such that $|U_i| = \epsilon n$. Then $|E(\cup_{i=1}^{r} U_i)| \geq \left(1 - \frac{1}{r} + o(1)\right)\binom{r\epsilon n}{2}$ and therefore by Erdős-Stone-Simonovits G contains $K_r(h)$ and therefore $H \setminus x$, together with x we get a copy of H. This is a contradiction to our assumption. Hence x has at most $o(n)$ in its part.

Assume that there is $x \in V(G)$, without loss of generality $x \in V_1$, such that there is a part V_i, $2 \leq i \leq r$ where x has at most $o(n)$ neighbours. Then $d(v) \leq \left(1 - \frac{1}{r} - \epsilon\right)n$. Let $W \subset V_1'$ such that $|W| = h$. For each $2 \leq i \leq r$, the vertices in W have at least $\left(\frac{1}{r} + o(1)\right)n$ common neighbours in V_i'. Let G^* be a graph that we get by removing x and adding a new vertex v adjacent to all the vertices in W. So $|E(G^*)| > |E(G)|$ and therefore G^* contains H. The copy of H must contain v as G is H-free and not contain some vertex $v' \in V'$, but then we get that $G^* \cup \{v'\} \setminus \{v\}$ contains H as so does G, which is a contradiction. Hence for each $x \in V(G)$ in each of the parts, besides the part of x, all but $o(n)$ vertices are neighbours of x.

Assume that there is a part, without loss of generality V_1 which contains an edge $\{x, y\}$. Let $Z \subset V_1$ such that $\{x, y\} \subset Z$ and $|Z| = h$. For each $2 \leq i \leq r$, the vertices in Z have at least $\left(\frac{1}{r} + o(1)\right)n$ common neighbours in V_i. So we can find $K_r(h)$ in G such that one of the parts contains an edge. Due to the edge criticality of H, this implies that G contains H which is a contradiction to the choice of G.

We conclude that none of the parts can contain an edge and therefore G is $T_{\chi(H)-1}(n)$.