Problem sheet 7

Due Monday, June 11th at 17:30.

Question 1

A hypergraph is called *r-partite*, if there is a partition of the vertices into r parts such that each edge contains at most one vertex from each part. The *chromatic number* χ of a hypergraph is the smallest number of colors needed to color the vertices such that each edge contains at least two vertices of different colors.

(a) Prove that the Turán density is well-defined, i.e., prove that the following limit exists for each r-uniform hypergraph $\mathcal{H}$

$$\pi(\mathcal{H}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, \mathcal{H})}{\binom{n}{r}}.$$

(b) Is there a uniform hypergraph $\mathcal{H}$ with $\pi(\mathcal{H}) = 1$?

(c) Is there an r-uniform hypergraph $\mathcal{H}$ with $\pi(\mathcal{H}) = 0$ that is not r-partite?

(d) Let $r \geq 3$. Is there an r-uniform hypergraph $\mathcal{H}$ with $\chi(\mathcal{H}) = 2$ and $\pi(\mathcal{H}) > 0$?

Question 2

Let V_1, V_2, V_3 be disjoint sets of size t and let $K^{(3)}_{t,t,t}$ be the hypergraph with vertex set $V = V_1 \cup V_2 \cup V_3$ that contains an edge $E \subseteq V$ if and only if $|E \cap V_i| = 1$ for all $i = 1, 2, 3$. Prove that there is a constant c (depending on t) such that for all n

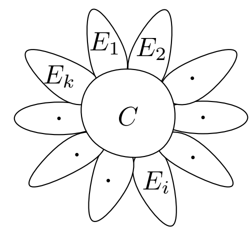
$$\mathrm{ex}(n, K^{(3)}_{t,t,t}) \leq c\, n^{3-\frac{1}{t^2}}.$$

(**Hint:** Consider some $\mathcal{H}$ without $K^{(3)}_{t,t,t}$ and count pairs (U, P), where U, $P \subseteq V(\mathcal{H})$ with $|U| = 2$, $|P| = t$, and $U \cup \{p\} \in E(\mathcal{H})$ for each $p \in P$. Apply Theorem 1 from Lecture 8.)

Question 3

Let k, $r \geq 1$. An *r-uniform sunflower with k petals and core C* is a hypergraph that consists of k edges $E_1, \ldots, E_k$ of size r such that $E_i \cap E_j = C$, $1 \leq i < j \leq k$, and $|C| \leq r-1$. Let $\mathrm{ex}(n, S_k^{(r)})$ denote the largest number of edges in an r-uniform hypergraph that does not contain any sunflower with k petals. Prove that for each $n \geq r(k-1)$

$$(k-1)^r \leq \mathrm{ex}(n, S_k^{(r)}) \leq r!(k-1)^r.$$



(**Hint:** If there are only few disjoint edges, then many edges have a vertex in common.)