Problem sheet 7

Due Monday, June 11th at 17:30.

Question 1

A hypergraph is called r-*partite*, if there is a partition of the vertices into r parts such that each edge contains at most one vertex from each part. The *chromatic number* χ of a hypergraph is the smallest number of colors needed to color the vertices such that each edge contains at least two vertices of different colors.

(a) Prove that the Turán density is well-defined, i.e., prove that the following limit exists for each r-uniform hypergraph $\mathcal{H}$

$$\pi(\mathcal{H}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n,\mathcal{H})}{\binom{n}{r}}.$$

(b) Is there a uniform hypergraph $\mathcal{H}$ with $\pi(\mathcal{H}) = 1$?

(c) Is there an r-uniform hypergraph $\mathcal{H}$ with $\pi(\mathcal{H}) = 0$ that is not r-partite?

(d) Let $r \geq 3$. Is there an r-uniform hypergraph $\mathcal{H}$ with $\chi(\mathcal{H}) = 2$ and $\pi(\mathcal{H}) > 0$?

Solution

(a) The sequence $\frac{\mathrm{ex}(n,\mathcal{H})}{\binom{n}{r}}$ is clearly bounded from below by 0. We shall show that the sequence $\frac{\mathrm{ex}(n,\mathcal{H})}{\binom{n}{r}}$ is decreasing and thus eventually converges.

Consider an r-uniform hypergraph $\mathcal{F}$ with vertex set V, $|V| = n+1$, that has $\mathrm{ex}(n+1,\mathcal{H})$ edges and does not contain a copy of $\mathcal{H}$. Clearly there is no copy of $\mathcal{H}$ on each subset of n-vertices. For a set $U \subseteq V$ of n vertices let $e(U)$ denote the number of edges of $\mathcal{F}$ contained in U. Since each edge of $\mathcal{F}$ is contained in $\binom{n+1-r}{n-r} = n+1-r$ sets U of size n we have

$$\mathrm{ex}(n+1,\mathcal{H}) = |E(\mathcal{F})| = \frac{1}{n+1-r} \sum_{U\in\binom{V}{n}} e(U) \leq \frac{n+1}{n+1-r}\mathrm{ex}(n,\mathcal{H}).$$

Thus

$$\frac{\text{ex}(n+1,\mathcal{H})}{\binom{n+1}{r}} \le \binom{n+1}{r}^{-1} \frac{n+1}{n+1-r}\text{ex}(n,\mathcal{H}) = \frac{\text{ex}(n,\mathcal{H})}{\binom{n}{r}}.$$

This shows that the sequence decrease and hence is convergent as argued in the beginning.

(b) Consider an r-uniform hypergraph $\mathcal{H}$ on t vertices. A hypergraph on t vertices that does not contain a copy of $\mathcal{H}$ is not complete and thus has less than $\binom{t}{r}$ edges. Hence

$$\frac{\text{ex}(t,\mathcal{H})}{\binom{t}{r}} < 1.$$

As argued in part (a), the sequence $\frac{\text{ex}(n,\mathcal{H})}{\binom{n}{r}}$ does not increase and hence

$$\pi(\mathcal{H}) = \lim_{n\to\infty} \frac{\text{ex}(n,\mathcal{H})}{\binom{n}{r}} \le \frac{\text{ex}(t,\mathcal{H})}{\binom{t}{r}} < 1.$$

This shows that there is no hypergraph $\mathcal{H}$ with $\pi(\mathcal{H}) = 1$.

(c) Let $\mathcal{H}$ be a hypergraph that is not r-partite. The complete r-partite hypergraph $K^{(r)}_{t,\dots,t}$ with parts of size t does not contain $\mathcal{H}$. Since $K^{(r)}_{t,\dots,t}$ has $n = rt$ vertices and

$$t^r = \left(\frac{n}{r}\right)^r \ge \left(\frac{1}{e}\right)^r \binom{n}{r}$$

edges we have

$$\pi(\mathcal{H}) = \lim_{n\to\infty} \frac{\text{ex}(n,\mathcal{H})}{\binom{n}{r}} \ge \lim_{t\to\infty} \frac{|E(K^{(r)}_{t,\dots,t})|}{\binom{rt}{r}} \ge \left(\frac{1}{e}\right)^r > 0.$$

This shows that each r-uniform hypergraph $\mathcal{H}$ with $\pi(\mathcal{H}) = 0$ is r-partite.

(d) We will construct a hypergraph $\mathcal{H}$ with $\chi(\mathcal{H}) = 2$ and $\pi(\mathcal{H}) > 0$. Let A and B be disjoint sets with $|A \cup B| \ge r+1$ and let $\mathcal{H}$ denote an r-uniform hypergraph with vertex set $V = A \cup B$ that contains an edge $E \in \binom{V}{r}$ if and only if $A \cap E,\ B \cap E \ne \emptyset$. Clearly $\chi(\mathcal{H}) = 2$. We claim that $\mathcal{H}$ is not r-partite and hence $\pi(\mathcal{H}) > 0$ by part (c).

Consider an arbitrary partition of $A \cup B$ into r disjoint parts. Since $|A \cup B| \ge r+1$ there is a part P that contains at least two vertices u and v. Observe that each pair of vertices u, $v \in A \cup B$ is contained in an edge of $\mathcal{H}$. Thus the part P contains at least two vertices of some edge of $\mathcal{H}$. This shows that $\mathcal{H}$ is not r-partite. Hence $\pi(\mathcal{H}) > 0$ by part (c).

Question 2

Let V_1, V_2, V_3 be disjoint sets of size t and let $K^{(3)}_{t,t,t}$ be the hypergraph with vertex set $V = V_1 \cup V_2 \cup V_3$ that contains an edge $E \subseteq V$ if and only if $|E \cap V_i| = 1$ for all $i = 1, 2, 3$. Prove that there is a constant c (depending on t) such that for all n

$$\mathrm{ex}(n, K^{(3)}_{t,t,t}) \le c\, n^{3-\frac{1}{t^2}}.$$

(**Hint:** Consider some $\mathcal{H}$ without $K^{(3)}_{t,t,t}$ and count pairs (U, P), where U, $P \subseteq V(\mathcal{H})$ with $|U| = 2$, $|P| = t$, and $U \cup \{p\} \in E(\mathcal{H})$ for each $p \in P$. Apply Theorem 1 from Lecture 8.)

Solution Consider an arbitrary 3-uniform hypergraph $\mathcal{H}$ on n vertices that does not contain a copy of $K^{(3)}_{t,t,t}$. For a pair of vertices $U = \{u, v\}$ let $d(U)$ denote the number of edges in $\mathcal{H}$ that contain U. A *partner* of $\{u, v\}$ is a vertex w such that $\{u, v, w\}$ is an edge in $\mathcal{H}$. We shall count the number N of pairs (U, P) where U is a pair of distinct vertices and P is a set of t partners of U. We have by Jensen's inequality that

$$N = \sum_{U \in \binom{V(\mathcal{H})}{2}} \binom{d(U)}{t} \ge \binom{n}{2} \binom{\frac{1}{\binom{n}{2}} \sum_U d(U)}{t} = \binom{n}{2} \binom{\frac{1}{\binom{n}{2}} 3|E(\mathcal{H})|}{t} \ge \binom{n}{2} \left(\frac{3|E(\mathcal{H})|}{t\binom{n}{2}} \right)^t.$$

Remark: To ensure that these inequalities hold, we should assume that $\frac{1}{\binom{n}{2}} 3|E(\mathcal{H})| \ge t$. *But if this is not the case, then* $|E(\mathcal{H})| < tn^2$ *and we are done anyways.*

Recall that, by Theorem 1 from Lecture 8, there is a constant $\xi \ge 1$ such that any ordinary graph on n vertices and more than $\xi\, n^{2-\frac{1}{t}}$ edges contains a copy of $K_{t,t}$. For the sake of contradiction assume that there is a set E of more than $\xi\, n^{2-\frac{1}{t}}$ pairs of vertices of $\mathcal{H}$ with t common partners. I.e., there is a set P of t vertices of $\mathcal{H}$ such that each vertex in P is a partner of each pair in E. As argued above the ordinary graph formed by E contains a copy K of $K_{t,t}$. Thus the edgeset $\{\{u, v, w\} \mid uv \in E(K), w \in P\}$ forms a copy of $K^{(3)}_{t,t,t}$ in $\mathcal{H}$, a contradiction. This shows that each subset P of t vertices from $\mathcal{H}$ is contained in at most $\xi\, n^{2-\frac{1}{t}}$ of the considered pairs (U, P). Hence

$$N \le \binom{n}{t} \xi\, n^{2-\frac{1}{t}} \le \xi \left(\frac{e\, n}{t} \right)^t n^{2-\frac{1}{t}} = \xi \left(\frac{e}{t} \right)^t n^{2+t-\frac{1}{t}}.$$

Combining both inequalities yields

$$|E(\mathcal{H})|^t \le \xi \left(\frac{e}{t} \right)^t n^{2+t-\frac{1}{t}} \binom{n}{2}^{t-1} \left(\frac{t}{3} \right)^t \le \xi \left(\frac{e}{3} \right)^t n^{3t-\frac{1}{t}} \le \xi\, n^{3t-\frac{1}{t}}.$$

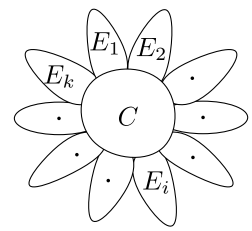
This shows that $|E(\mathcal{H})| \le c\, n^{3-\frac{1}{t^2}}$ for a constant $c = \xi^{\frac{1}{t}}$ and, since $\mathcal{H}$ was arbitrary,

$$\mathrm{ex}(n, K^{(3)}_{t,t,t}) \le c\, n^{3-\frac{1}{t^2}}.$$

Question 3

Let k, $r \geq 1$. An *r-uniform sunflower with k petals and core C* is a hypergraph that consists of k edges $E_1, \ldots, E_k$ of size r such that $E_i \cap E_j = C$, $1 \leq i < j \leq k$, and $|C| \leq r-1$. Let $\mathrm{ex}(n, S_k^{(r)})$ denote the largest number of edges in an r-uniform hypergraph that does not contain any sunflower with k petals. Prove that for each $n \geq r(k-1)$

$$(k-1)^r \leq \mathrm{ex}(n, S_k^{(r)}) \leq r!(k-1)^r.$$



(**Hint:** If there are only few disjoint edges, then many edges have a vertex in common.)

Solution First of all we shall prove the lower bound. Let $V_0, V_1, \ldots, V_r$ be disjoint sets of vertices with $|V_i| = k-1$, $1 \leq i \leq r$, and $|V_0| = n - r(k-1) \geq 0$. Consider the hypergraph $\mathcal{H}$ with vertex set $V = V_0 \cup V_1 \cup \cdots \cup V_r$ that contains an edge $E \subseteq V$ if and only if $E \cap V_0 = \emptyset$ and $|E \cap V_i| = 1$ for all $i = 1, \ldots, r$. Then $\mathcal{H}$ has n vertices and $(k-1)^r$ edges and is r-uniform. We claim that $\mathcal{H}$ contains no sunflower with k petals.

Consider a arbitrary set $S = \{E_1, \ldots, E_k\}$ of k distinct edges in $\mathcal{H}$. Without loss of generality assume that E_1 and E_2 differ in V_1, that is, $(E_1 \cap V_1) \cap (E_2 \cap V_1) = \emptyset$. Since $|V_1| = k-1$, there is a vertex v in V_1 that is contained in at least two distinct edges from S by pigeonhole principle. Therefore, if the edges in S form a sunflower, then v is contained in the core. Since v is contained in only one of E_1 or E_2, the edges $E_1, \ldots, E_k$ do not form a sunflower. Thus $\mathcal{H}$ does not contain any sunflower with k petals, since S was arbitrary.

Now we shall prove the upper bound for all positive integers n. Consider an arbitrary r-uniform hypergraph $\mathcal{H}$ on n vertices and more than $r!(k-1)^r$ distinct edges. We shall find a sunflower with k petals in $\mathcal{H}$ by induction on r. Note that k disjoint edges form a sunflower with k petals and empty core. If $r = 1$, then each edge in $\mathcal{H}$ consists of a single vertex. By assumption $\mathcal{H}$ contains at least k distinct such edges forming a desired sunflower. So consider $r \geq 2$. If $\mathcal{H}$ contains k disjoint edges, then these form a desired sunflower with k petals. So assume that $\mathcal{H}$ contains no k disjoint edges.

Let $S = \{E_1, \ldots, E_t\}$ be a largest set of disjoint edges from $\mathcal{H}$ and let $U = E_1 \cup \cdots \cup E_t$. Then $|U| \leq tr \leq (k-1)r$ and each edge from $\mathcal{H}$ intersects U (otherwise S is not maximal). By pigeonhole principle U contains a vertex v that is contained in at least

$$\frac{|E(\mathcal{H})|}{|U|} \geq \frac{r!(k-1)^r + 1}{(k-1)r} = (r-1)!(k-1)^{r-1} + \frac{1}{(k-1)r} > (r-1)!(k-1)^{r-1} \tag{1}$$

edges of $\mathcal{H}$. Consider the $(r-1)$-uniform hypergraph $\mathcal{H}'$ formed by the edgeset

$$E(\mathcal{H}') = \{E \setminus \{v\} \mid E \in E(\mathcal{H}), v \in E\}.$$

By (1), the hypergraph $\mathcal{H}'$ has more than $(r-1)!(k-1)^{r-1}$ edges and thus by induction contains k distinct edges $E'_1, \ldots, E'_k$ forming a sunflower with k petals and core C'. By construction $E'_i \cup \{v\}$ is an edge in $\mathcal{H}$ for each i, $1 \leq i \leq k$. Hence the edges $E'_1 \cup \{v\}, \ldots, E'_k \cup \{v\}$ form a sunflower with k petals and core $C = C' \cup \{v\}$ in $\mathcal{H}$.