Problem sheet 8

Due Monday, June 18th at 17:30.

Question 1

(a) Prove that if $r(m-1,n)$ and $r(m,n-1)$ are even, then

$$r(m,n) \leq r(m-1,n) + r(m,n-1) - 1.$$

(b) Calculate $r(n,2)$ for all $n \geq 2$.

(c) Calculate $r(4,3)$.

(d) Calculate $r(4,4)$.

(*Remark:* In part (c) and (d) a precise description of an appropriate coloring is sufficient without proof.)

Question 2

Let $d \geq 1$. The *d-cube* Q_d is the graph with vertex set $\{0,1\}^d$, i.e., all binary vectors of length d, where two vertices are adjacent if and only if they differ in exactly one coordinate. Use the method of dependent random choice to prove that

$$r(Q_d) \leq 2^{3d} = |V(Q_d)|^3.$$

(**Hint:** Consider the majority color class in a coloring of $K_{2^{3d}}$. Similar to Lemma 1 from Lecture 9, choose $\frac{3}{2}d$ random vertices and prove that their common neighborhood contains a subset of size $\frac{1}{2}|V(Q_d)|$ where each d-tuple has at least $|V(Q_d)|$ common neighbors.)

Question 3

For a graph G let $\mathcal{R}(G)$ denote the set of all graphs F, such that there is a monochromatic copy of G in any 2-coloring of the edges of F. A graph F is called *minimal Ramsey graph* of G if $F \in \mathcal{R}(G)$ but each proper subgraph of F is not in $\mathcal{R}(G)$.

(a) Prove that each tree that is not a star has infinitely many minimal Ramsey graphs.

(**Hint:** Use the existence of graphs of arbitrarily large girth and chromatic number.)

(b) Prove that for each graph G of minimum degree d each minimal Ramsey graph of G has minimum degree at least $2d - 1$.