Problem sheet 8
Due Monday, June 18th at 17:30.

Question 1

(a) Prove that if $r(m-1,n)$ and $r(m,n-1)$ are even, then

$$r(m,n) \leq r(m-1,n) + r(m,n-1) - 1.$$ 

(b) Calculate $r(n,2)$ for all $n \geq 2$.

(c) Calculate $r(4,3)$.

(d) Calculate $r(4,4)$.

(Remark: In part (c) and (d) a precise description of an appropriate coloring is sufficient without proof.)

Solution

(a) Let $r = r(m-1,n) + r(m,n-1) - 1$. For the sake of contradiction assume that $c$ is a coloring of $K_r$ in colors red and blue without red copy of $K_m$ and blue copy of $K_n$. Then each vertex is adjacent to at most $r(m-1,n) - 1$ red edges and at most $r(m-1,n) - 1$ blue edges. Since each vertex has degree $r - 1 = r(m-1,n) - 1 + r(m-1,n) - 1$, each vertex is adjacent to exactly $r(m-1,n) - 1$ red edges and exactly $r(m-1,n) - 1$ blue edges. Hence the graph on all red edges has all vertices of degree $r(m-1,n) - 1$ which is odd. This is a contradiction since this graph has $r$ vertices which is odd, too.

(b) We have $r(n,2) = n$. If in a 2-colouring of the edges of $K_n$ there is a blue edge then we are done. Otherwise all the edges are red and therefore we have a red $K_n$.

(c) We have $r(3,3) = 6$ and $r(4,2) = 4$. Hence by part (a)

$$r(4,3) \leq r(3,3) + r(4,2) - 1 = 9.$$
It remains to give a coloring of $K_8$ without red $K_4$ and blue $K_3$. Color all edges of a cycle $v_0, \ldots, v_7$ and all edges $v_i v_{i+4}$ (addition modulo 8) in blue. Color all the other edges in red. It is easy to see from the picture below that there is no red $K_4$ and no blue $K_3$.

(d) Due to Lemma 1 Lecture 1 and part (c) we have

$$r(4, 4) \leq 2r(4, 3) = 18.$$ 

We shall give a coloring of $K_{17}$ that shows that this bound is tight, i.e., $r(4, 4) = 18$. Color all edges of a cycle $v_0, \ldots, v_{16}$ blue and add additionally color all chords of length 2, 4, and 8 blue. Color all other edges, i.e., all chords of length 3, 5, 6, and 7 in red. See the following picture.

Choose four distinct vertices $v_a, v_b, v_c, v_d \in \{v_0, \ldots, v_{16}\}$ with $0 \leq a < b < c < d \leq 16$. A case distinction shows that these four vertices necessarily induce a chord with length in $\{1, 2, 4, 8\}$, i.e., a blue edge, as well as a chord with length in $\{3, 5, 6, 7\}$, i.e., a red edge. Thus there is no monochromatic $K_4$.

**Question 2**

Let $d \geq 1$. The $d$-cube $Q_d$ is the graph with vertex set $\{0, 1\}^d$, i.e., all binary vectors of length $d$, where two vertices are adjacent if and only if they differ in exactly one coordinate. Use the method of dependent random choice to prove that

$$r(Q_d) \leq 2^{3d} = |V(Q_d)|^3.$$


(Hint: Consider the majority color class in a coloring of $K_{2^d}$. Similar to Lemma 1 from Lecture 9, choose $\frac{3}{2}d$ random vertices and prove that their common neighborhood contains a subset of size $\frac{1}{2}|V(Q_d)|$ where each $d$-tuple has at least $|V(Q_d)|$ common neighbors.)

**Solution** Let $N = 2^{3d}$ and consider a 2-coloring of the edges of $K_N$. One of the color classes contains at least half of the edges of $K_N$, i.e., forms a graph $G$ on $n$ vertices with at least $\frac{1}{2}\binom{N}{2}$ edges. We shall prove that $G$ contains a monochromatic copy of $Q_d$ using the method of dependent random choice, similar as in Lemma 1 of Lecture 9. Choose $t = \frac{3}{2}d$ vertices $b_1, \ldots, b_t$ uniformly at random from $G$ and let $U$ denote the common neighborhood of these vertices in $G$.

The probability for a vertex $v \in V(G)$ to be in $U$ is the probability that the we chose vertices $b_1, \ldots, b_t$ from $N(v)$. The probability to choose a vertex $b$ from $N(v)$ is $\frac{d(v)}{N}$. Then (by Jensen’s inequality (I), $|E(G)| \geq \frac{N(N-1)}{4}$ (II), and $N = 2^{3d} \geq 8$ (III))

$$
\mathbb{E}(|U|) = \sum_{v \in V(G)} \left( \frac{d(v)}{N} \right) \geq N \left( \frac{2|E(G)|}{N^2} \right)^t \geq N \left( \frac{N-1}{2N} \right)^t \geq N \left( \frac{1}{2} \right)^t \left( \frac{7}{8} \right)^t = 2^{3d-4t} \tau^t.
$$

Call a set of $d$ vertices of $G$ **bad**, if it has less than $2^d$ common neighbors in $G$. Let $B$ denote the number of bad sets of size $d$ in $U$. For a fixed bad set $S$ the probability that $S$ is a subset of $U$ is at most the probability to choose $t$ vertices from the common neighborhood of $S$ and is less than $\left( \frac{2^d}{N} \right)^t = 2^{-2dt}$. Thus

$$
\mathbb{E}(B) < \left( \frac{N}{d} \right)^t 2^{-2dt} \leq 2^{3d^2-2dt} = 1.
$$

Altogether

$$
\mathbb{E}(|U| - B) > 2^{3d-4t} \tau^t - 1 = \left( \frac{2^3}{8} \right)^d - 1 \geq 2^d - 1 \geq 2^{d-1}.
$$

This shows that there is a choice of vertices $b_1, \ldots, b_t$ from $G$ such that we obtain a subset $U'$, with $|U'| > 2^{d-1}$, from the common neighborhood $U$ of these vertices, by deleting one vertex from each bad subset of $U$. This means that each subset of $U'$ of size $d$ has at least $2^d$ common neighbors in $G$.

The graph $Q_d$ is bipartite with bipartition $V(Q_d) = X \cup Y$, $|X| = 2^{d-1}$ (all vectors with an even number of 1’s), $|Y| = 2^{d-1}$ (all vectors with an odd number of 1’s). Moreover each vertex is of degree $d$. Embed the vertices from $X$ arbitrarily into $U'$. Then we can embed the vertices from $Y = \{y_1, \ldots, y_{2^{d-1}}\}$ greedily. Indeed, suppose that vertices $y_1, \ldots, y_i$, $i < 2^{d-1}$, are already embedded. The set $N(y_{i+1}) \subseteq U'$ corresponding to the neighbors of $y_{i+1}$ in $Q_d$ has size $d$ and thus the set $Y'$ of common neighbors of $N(y_{i+1})$ in $G$ has size at least $2^d = |V(Q_d)|$. Since less than $2^d$ vertices are embedded yet, we can pick a vertex from $Y'$ to embed $y_{i+1}$. 

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Question 3

For a graph $G$ let $\mathcal{R}(G)$ denote the set of all graphs $F$, such that there is a monochromatic copy of $G$ in any 2-coloring of the edges of $F$. A graph $F$ is called minimal Ramsey graph of $G$ if $F \in \mathcal{R}(G)$ but each proper subgraph of $F$ is not in $\mathcal{R}(G)$.

(a) Prove that each tree that is not a star has infinitely many minimal Ramsey graphs.  

(Hint: Use the existence of graphs of arbitrarily large girth and chromatic number.)

(b) Prove that for each graph $G$ of minimum degree $d$ each minimal Ramsey graph of $G$ has minimum degree at least $2d - 1$.

Solution

(a) Let $T$ be a tree on $t$ vertices that is not a star. For each $n \geq t$ let $G_n$ be a graph of chromatic number greater than $t^2$ and girth at least $n$. We claim that $G = G_n$ is a Ramsey graph of $T$. Consider an edge-coloring of $G$ in colors red and blue. If both color classes induce subgraphs of chromatic number at most $t$, then a product coloring of the vertices of $G$ yields $\chi(G) \leq t^2$, a contradiction. So assume that the red edges induce a subgraph of chromatic number greater than $t$. Then there is a red subgraph of minimum degree at least $t$ and hence a red copy of $T$ by Lemma 3, Lecture 2. This shows that $G_n \in \mathcal{R}(T)$. Iteratively remove edges from $G_n$ until we reach a subgraph $G'_n$ of $G_n$ that is a minimal Ramsey graph of $T$. Then $G'_n$ is not a tree, since otherwise we can 2-color its edges without creating a monochromatic copy of $T$, where we use that $T$ is not a star. So $G'_n$ contains a cycle and thus, since $G_n$ has girth at least $n$, $G'_n$ contains a cycle of length at least $n$. In particular $G'_n$ has at least $n$ vertices. Since infinitely many of the graphs $G'_n$ are distinct there are infinitely many minimal Ramsey graphs of $T$.

(b) Let $F$ be a minimal Ramsey graph of $G$. For the sake of contradiction assume that $v$ is a vertex in $F$ of degree at most $2d - 2$. Let $F'$ be the graph obtained by removing $v$ from $F$. Since $F$ is minimal Ramsey, $F' \notin \mathcal{R}(G)$. Thus there is a coloring of the edges of $F'$ in colors red and blue without monochromatic copy of $G$. Color up to $d - 1$ edges incident to $v$ in red and the remaining at most $d - 1$ edges in blue. This coloring of $F$ has no monochromatic copies of $G$, since $G$ has minimum degree $d$, a contradiction. Hence $F$ has minimum degree $2d - 1$. 