Problem sheet 8

Due Monday, June 18th at 17:30.

Question 1

(a) Prove that if $r(m-1,n)$ and $r(m,n-1)$ are even, then

$$r(m,n) \leq r(m-1,n) + r(m,n-1) - 1.$$

(b) Calculate $r(n,2)$ for all $n \geq 2$.

(c) Calculate $r(4,3)$.

(d) Calculate $r(4,4)$.

(*Remark:* In part (c) and (d) a precise description of an appropriate coloring is sufficient without proof.)

Solution

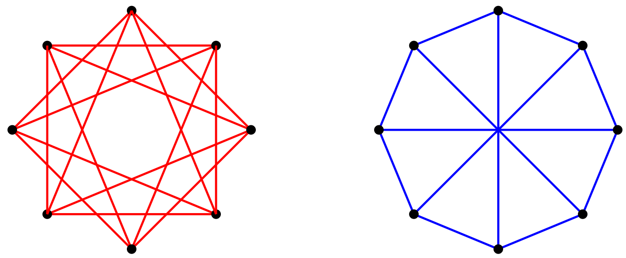
(a) Let $r = r(m-1,n) + r(m,n-1) - 1$. For the sake of contradiction assume that c is a coloring of K_r in colors red and blue without red copy of K_m and blue copy of K_n. Then each vertex is adjacent to at most $r(m-1,n) - 1$ red edges and at most $r(m-1,n) - 1$ blue edges. Since each vertex has degree $r - 1 = r(m-1,n) - 1 + r(m-1,n) - 1$, each vertex is adjacent to exactly $r(m-1,n) - 1$ red edges and exactly $r(m-1,n) - 1$ blue edges. Hence the graph on all red edges has all vertices of degree $r(m-1,n) - 1$ which is odd. This is a contradiction since this graph has r vertices which is odd, too.

(b) We have $r(n,2) = n$. If in a 2-colouring of the edges of K_n there is a blue edge then we are done. Otherwise all the edges are red and therefore we have a red K_n.

(c) We have $r(3,3) = 6$ and $r(4,2) = 4$. Hence by part (a)

$$r(4,3) \leq r(3,3) + r(4,2) - 1 = 9.$$

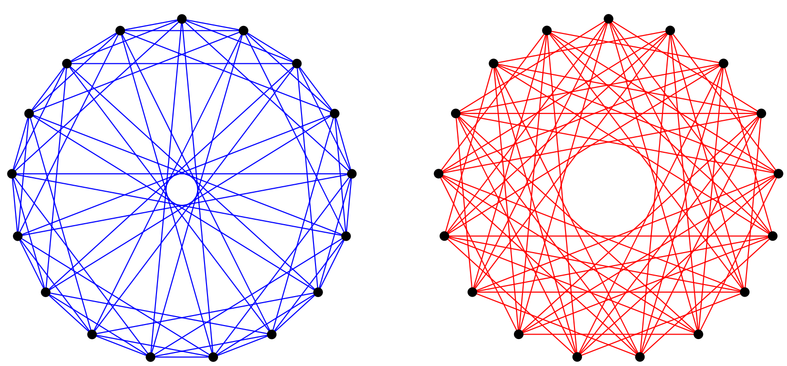
It remains to give a coloring of K_8 without red K_4 and blue K_3. Color all edges of a cycle $v_0, \ldots, v_7$ and all edges $v_i v_{i+4}$ (addition modulo 8) in blue. Color all the other edges in red. It is easy to see from the picture below that there is no red K_4 and no blue K_3.



(d) Due to Lemma 1 Lecture 1 and part (c) we have

$$r(4,4) \leq 2r(4,3) = 18.$$

We shall give a coloring of K_{17} that shows that this bound is tight, i.e., $r(4,4) = 18$. Color all edges of a cycle $v_0, \ldots, v_{16}$ blue and add additionally color all chords of length 2, 4, and 8 blue. Color all other edges, i.e., all chords of length 3, 5, 6, and 7 in red. See the following picture.



Choose four distinct vertices v_a, v_b, v_c, $v_d \in \{v_0, \ldots, v_{16}\}$ with $0 \leq a < b < c < d \leq 16$. A case distinction shows that these four vertices necessarily induce a chord with length in $\{1, 2, 4, 8\}$, i.e., a blue edge, as well as a chord with length in $\{3, 5, 6, 7\}$, i.e., a red edge. Thus there is no monochromatic K_4.

Question 2

Let $d \geq 1$. The *d-cube* Q_d is the graph with vertex set $\{0,1\}^d$, i.e., all binary vectors of length d, where two vertices are adjacent if and only if they differ in exactly one coordinate. Use the method of dependent random choice to prove that

$$r(Q_d) \leq 2^{3d} = |V(Q_d)|^3.$$

(**Hint:** Consider the majority color class in a coloring of $K_{2^{3d}}$. Similar to Lemma 1 from Lecture 9, choose $\frac{3}{2}d$ random vertices and prove that their common neighborhood contains a subset of size $\frac{1}{2}|V(Q_d)|$ where each d-tuple has at least $|V(Q_d)|$ common neighbors.)

Solution Let $N = 2^{3d}$ and consider a 2-coloring of the edges of K_N. One of the color classes contains at least half of the edges of K_N, i.e., forms a graph G on n vertices with at least $\frac{1}{2}\binom{N}{2}$ edges. We shall prove that G contains a monochromatic copy of Q_d using the method of dependent random choice, similar as in Lemma 1 of Lecture 9. Choose $t = \frac{3}{2}d$ vertices $b_1, \ldots, b_t$ uniformly at random from G and let U denote the common neighborhood of these vertices in G. The probability for a vertex $v \in V(G)$ to be in U is the probability that the we chose vertices $b_1, \ldots, b_t$ from $N(v)$. The probability to choose a vertex b from $N(v)$ is $\frac{d(v)}{N}$. Then (by Jensen's inequality (I), $|E(G)| \geq \frac{N(N-1)}{4}$ (II), and $N = 2^{3d} \geq 8$ (III))

$$\mathbb{E}(|U|) = \sum_{v \in V(G)} \left(\frac{d(v)}{N}\right)^t \overset{\text{(I)}}{\geq} N\left(\frac{2|E(G)|}{N^2}\right)^t \overset{\text{(II)}}{\geq} N\left(\tfrac{N-1}{2N}\right)^t \overset{\text{(III)}}{\geq} N\left(\tfrac{1}{2}\right)^t\left(\tfrac{7}{8}\right)^t = 2^{3d-4t}\,7^t.$$

Call a set of d vertices of G *bad*, if it has less than 2^d common neighbors in G. Let B denote the number of bad sets of size d in U. For a fixed bad set S the probability that S is a subset of U is at most the probability to choose t vertices from the common neighborhood of S and is less than $\left(\frac{2^d}{N}\right)^t = 2^{-2dt}$. Thus

$$\mathbb{E}(B) < \binom{N}{d} 2^{-2dt} \leq 2^{3d^2-2dt} = 1.$$

Altogether

$$\mathbb{E}(|U| - B) > 2^{3d-4t}\,7^t - 1 = \left(\frac{7^{\frac{3}{2}}}{8}\right)^d - 1 \geq 2^d - 1 \geq 2^{d-1}.$$

This shows that there is a choice of vertices $b_1, \ldots, b_t$ from G such that we obtain a subset U', with $|U'| > 2^{d-1}$, from the common neighborhood U of these vertices, by deleting one vertex from each bad subset of U. This means that each subset of U' of size d has at least 2^d common neighbors in G.

The graph Q_d is bipartite with bipartition $V(Q_d) = X \dot\cup Y$, $|X| = 2^{d-1}$ (all vectors with an even number of 1's), $|Y| = 2^{d-1}$ (all vectors with an odd number of 1's). Moreover each vertex is of degree d. Embed the vertices from X arbitrarily into U'. Then we can embed the vertices from $Y = \{y_1, \ldots, y_{2^{d-1}}\}$ greedily. Indeed, suppose that vertices $y_1, \ldots, y_i$, $i < 2^{d-1}$, are already embedded. The set $N(y_{i+1}) \subseteq U'$ corresponding to the neighbors of y_{i+1} in Q_d has size d and thus the set Y' of common neighbors of $N(y_{i+1})$ in G has size at least $2^d = |V(Q_d)|$. Since less than 2^d vertices are embedded yet, we can pick a vertex from Y' to embed y_{i+1}.

Question 3

For a graph G let $\mathcal{R}(G)$ denote the set of all graphs F, such that there is a monochromatic copy of G in any 2-coloring of the edges of F. A graph F is called *minimal Ramsey graph* of G if $F \in \mathcal{R}(G)$ but each proper subgraph of F is not in $\mathcal{R}(G)$.

(a) Prove that each tree that is not a star has infinitely many minimal Ramsey graphs.

(**Hint:** Use the existence of graphs of arbitrarily large girth and chromatic number.)

(b) Prove that for each graph G of minimum degree d each minimal Ramsey graph of G has minimum degree at least $2d - 1$.

Solution

(a) Let T be a tree on t vertices that is not a star. For each $n \geq t$ let G_n be a graph of chromatic number greater than t^2 and girth at least n. We claim that $G = G_n$ is a Ramsey graph of T. Consider an edge-coloring of G in colors red and blue. If both color classes induce subgraphs of chromatic number at most t, then a product coloring of the vertices of G yields $\chi(G) \leq t^2$, a contradiction. So assume that the red edges induce a subgraph of chromatic number greater than t. Then there is a red subgraph of minimum degree at least t and hence a red copy of T by Lemma 3, Lecture 2. This shows that $G_n \in \mathcal{R}(T)$.

Iteratively remove edges from G_n until we reach a subgraph G'_n of G_n that is a minimal Ramsey graph of T. Then G'_n is not a tree, since otherwise we can 2-color its edges without creating a monochromatic copy of T, where we use that T is not a star. So G'_n contains a cycle and thus, since G_n has girth at least n, G'_n contains a cycle of length at least n. In particular G'_n has at least n vertices. Since infinitely many of the graphs G'_n are distinct there are infinitely many minimal Ramsey graphs of T.

(b) Let F be a minimal Ramsey graph of G. For the sake of contradiction assume that v is a vertex in F of degree at most $2d - 2$. Let F' be the graph obtained by removing v from F. Since F is minimal Ramsey, $F' \notin \mathcal{R}(G)$. Thus there is a coloring of the edges of F' in colors red and blue without monochromatic copy of G. Color up to $d - 1$ edges incident to v in red and the remaining at most $d - 1$ edges in blue. This coloring of F has no monochromatic copies of G, since G has minimum degree d, a contradiction. Hence F has minimum degree $2d - 1$.