Question 1

For a positive integer $n$ let $nK_3$ denote the vertex disjoint union of $n$ copies of $K_3$.

(a) Prove that there is a unique coloring of $K_5$ without monochromatic $K_3$ (up to isomorphism and renaming of colors).

(b) Prove that $r(K_3, 2K_3) = 8$.

(c) Prove that $r(sK_3, tK_3) = 2s + 3t$ for all $t \geq s \geq 1, t \geq 2$.

(Hint: Use $r(2K_3, 2K_3) = 10$ without proof.)

Solution

(a) Consider a coloring of $K_5$ without monochromatic copies of $K_3$. It is easy to see that no vertex is incident to three red or three blue edges. Thus each color class forms a spanning 2-regular graph. The only such graph on 5 vertices is a cycle and hence the coloring is unique.

(b) To establish $r(K_3, 2K_3) \geq 8$ color a copy of $K_{2,5}$ in $K_7$ in red and the remaining edges blue. Then there is no red $K_3$ and no two vertex disjoint blue triangles.

Next we shall prove that any coloring of $K_8$ either contains a red triangle or two vertex disjoint blue triangles. Consider a coloring without red triangles. Since $r(K_3) = 6$ there is a blue triangle, say on vertices $u, v,$ and $w$. If the remaining five vertices do not contain another blue triangle, then by part (a) the blue and the red edges form a $C_5$ each. Say the blue edges form a cycle $C$ with vertices $x_1, \ldots, x_5$. It is easy to see that there are exactly two red and three blue edges between $\{x_1, \ldots, x_5\}$ and each vertex $y \in \{u, v, w\}$, and that the three blue neighbors of each $y \in \{u, v, w\}$ are consecutive on the blue cycle. An easy case distinction shows that, no matter how the blue neighborhoods are distributed in $C$, \
there are two vertex disjoint blue triangles, each with at most two vertices from \( \{u, v, w\} \) and at most two from \( C \).

(c) To establish \( r(sK_3, tK_3) \geq 2s + 3t \) we color the edges of \( K_{2s+3t-1} \) without red \( sK_3 \) and blue \( tK_3 \) as follows. Color a copy of \( K_{3t-1} \) and a vertex disjoint copy of \( K_{1,2s-1} \) blue and color the remaining edges red. Then all blue triangles are contained in the blue clique and hence there are at most \( t - 1 \) vertex disjoint blue triangles. The red subgraph forms a complete bipartite graph \( K_{3t-1,2s-1} \) where all edges are added in the part of size \( 2s - 1 \). Each red triangle contains at least two vertices from this part and hence there are at most \( s - 1 \) vertex disjoint red triangles.

Next we shall prove that for any \( t \geq s \geq 1, t \geq 2 \), any 2-coloring of \( K_{2s+3t} \) either contains \( t \) vertex disjoint red triangles or \( s \) vertex disjoint blue triangles by induction on \( t \). The hint yields the base cases \( s \in \{1, 2\} \) and \( t = 2 \) (and can be proven by case distinction).

So assume that \( t \geq 3 \) and consider a 2-coloring of \( K_{2s+3t} \). We treat the case \( s = 1 \) first. Since \( r(K_3) = 6 \) there is a monochromatic triangle \( K \). If \( K \) is red we are done. Otherwise consider the graph induced by vertices not in \( K \). It has \( 2 + 3t - 3 = 2 + 3(t - 1) \) vertices. Inductively there is either a red triangle or \( t - 1 \) vertex disjoint blue triangles and in both cases we are done.

So assume that \( s \geq 2 \). Iteratively removing vertices of a monochromatic \( K_3 \) as long as at least 6 vertices are left yields at least \( \left\lceil \frac{2s+3t-5}{3} \right\rceil = t + \left\lceil \frac{2s-5}{3} \right\rceil \geq t \) vertex disjoint monochromatic triangles. If these triangles are all of the same color we are done. So assume that there is a red triangle on vertices \( a, b, c \) and a vertex disjoint blue triangle on vertices \( u, v, w \). At least 5 out of the 9 edges between these triangles are of the same color, say blue. Then there is a blue triangle that shares exactly one vertex with the red triangle on \( \{a, b, c\} \) (they form a so-called bow-tie) like in the figure. Then the set \[ V = \{a, b, c\} \cup \{v, w, u\} \] of vertices not in these two triangles has size \( 2s + 3t - 5 = 2(s - 1) + 3(t - 1) \). Since \( t - 1 \geq s - 1 \geq 1 \) and \( t - 1 \geq 2 \), \( V \) induces either \( s - 1 \) vertex disjoint red triangles or \( t - 1 \) vertex disjoint blue triangles by induction. Together with a triangle from the bow-tie we have either \( s \) vertex disjoint red triangles or \( t \) vertex disjoint blue triangles. Hence \( r(sK_3, tK_3) \leq 2s + 3t \).
Question 2

(a) Calculate $r(C_3, C_4)$.

(b) Calculate $r(C_4, C_4)$.

Solution

(a) We claim that $r(C_3, C_4) = 7$.

First we show a coloring of $K_6$ without either red $C_3$ or blue $C_4$. Color the edges of $K_6$ such that the red edges form a copy of $K_{3,3}$. Then the blue edges form two vertex disjoint blue copies of $K_3 = C_3$ and there is no red $C_3$ and no blue $C_4$. This shows that 

$$r(C_3, C_4) \geq 7.$$ 

It remains to prove that every coloring of $K_7$ yields a red $C_3$ or a blue $C_4$. Consider a coloring of $K_7$ with no red copy of $C_3$. We shall prove that there is a blue copy of $C_4$. Since $r(3, 3) = 6$ there is a blue copy of $K_3 = C_3$, say with vertex set $U$. Let $V$ denote the remaining four vertices. If $V$ induces no red edge then $V$ induces a blue $C_4$. Otherwise let $xy$ be a red edge with $x, y \in V$. Then, for each $u \in U$ there is at least one blue edge between $\{x, y\}$ and $u$. By pigeonhole principle there are two blue edges between one of $x$ or $y$ and $U$, and hence a blue $C_4$.

(b) We claim that $r(C_4, C_4) = 6$.

Coloring the edges of $K_5$ with two edge disjoint copies of $C_5$ shows that 

$$r(C_4, C_4) \geq 6.$$ 

It remains to prove that every coloring of $K_6$ yields a monochromatic $C_4$. Since $r(3, 3) = 6$ there is a monochromatic copy of $K_3 = C_3$, say blue and with vertex set $U$. Let $V$ denote the remaining three vertices. If there are two blue edges between a vertex $v \in V$ and $U$, then there is a blue copy of $C_4$. Otherwise each vertex in $V$ sends at least two red edges to $U$. No two vertices from $V$ send two red edges to the same two vertices from $U$ otherwise we get a red $C_4$. This shows that the blue edges between $U$ and $V$ form a matching with three edges and the red edges form a copy of $C_6$. If there are at least two red edges induced on $V$, then at least one of those edges creates $C_4$ with 3 of the edges of the red $C_6$. So $V$ induces a blue edge that is contained in a blue $C_4$ together with two matching edges and one edge induced by $U$. 


Question 3

Let $G$ be a graph on $n$ vertices and let $P_3$ be the path with 3 vertices.

(a) Calculate $r(P_3, G)$ if there is a perfect matching in the complement of $G$.

(b) Calculate $r(P_3, G)$ if there is no perfect matching in the complement of $G$.

Solution  
Let $m$ denote the size of a largest matching in the complement of $G$. We claim that

$$r(P_3, G) = \begin{cases} n, & \text{if } 2m = n, \\ 2n - 2m - 1, & \text{if } 2m < n. \end{cases}$$

(a) Consider the case $2m = n$ first, i.e., there is a perfect matching in the complement of $G$. Coloring all edges of $K_{n-1}$ blue shows that $r(P_3, G) > n - 1$. Let $u_1, \ldots, u_m, v_1, \ldots, v_m$ denote the vertices of $G$ such that $u_iv_i$ is a non-edge of $G$. Consider a coloring of $K_n$ with no red copy of $P_3$. Then the red edges form a matching $x_1y_1, \ldots, x_ty_t$. Mapping $u_i$ to $x_i$ and $v_i$ to $y_i$ for $1 \leq i \leq t$ and mapping the remaining vertices from $G$ arbitrarily to the remaining vertices of $K_n$ yields a blue copy of $G$. Thus $r(P_3, G) \leq n$.

(b) Now assume that $2m < n$ and let $N = 2n - 2m - 2$. Color a perfect matching (i.e., with $n - m - 1$ edges) in $K_N$ red and all other edges blue. Consider a set $S$ of $n$ vertices in $K_N$. Then $S$ induces at least $m + 1$ red matching edges and hence not a blue copy of $G$. Hence there is no blue copy of $G$ and $r(P_3, G) > N$.

Finally consider a coloring of $K_{N+1}$ with no red copy of $P_3$. Then the red edges form a matching $x_1y_1, \ldots, x_ty_t$. Let $z_1, \ldots, z_{N+1-2t}$ denote the remaining vertices of $K_{N+1}$. Further let $u_1, \ldots, u_m, v_1, \ldots, v_m, w_1, \ldots, w_{n-2m}$ denote the vertices of $G$ such that $u_iv_i$ is a non-edge of $G$ and $w_1, \ldots, w_{n-2m}$ induce a complete graph in $G$. Observe that

$$|\{x_1, \ldots, x_t, z_1, \ldots, z_{N+1-2t}\}| \geq \left\lceil \frac{N+1}{2} \right\rceil = n - m = |\{u_1, \ldots, u_m, w_1, \ldots, w_{n-2m}\}|.$$

Mapping $u_i$ to $x_i$ and $v_i$ to $y_i$ for $1 \leq i \leq \min\{m, t\}$ and the remaining vertices from $G$ to (some of) the remaining vertices of $\{x_1, \ldots, x_t, z_1, \ldots, z_{N+1-2t}\}$ yields a blue copy of $G$. Hence $r(P_3, G) \leq N + 1$ and thus

$$r(P_3, G) = 2n - 2m - 1.$$