Problem sheet 9

Due Monday, June 25th at 17:30.

Question 1

For a positive integer n let nK_3 denote the vertex disjoint union of n copies of K_3.

(a) Prove that there is a unique coloring of K_5 without monochromatic K_3 (up to isomorphism and renaming of colors).

(b) Prove that $r(K_3, 2K_3) = 8$.

(c) Prove that $r(sK_3, tK_3) = 2s + 3t$ for all $t \geq s \geq 1$, $t \geq 2$.

(**Hint:** Use $r(2K_3, 2K_3) = 10$ without proof.)

Solution

(a) Consider a coloring of K_5 without monochromatic copies of K_3. It is easy to see that no vertex is incident to three red or three blue edges. Thus each color class forms a spanning 2-regular graph. The only such graph on 5 vertices is a cycle and hence the coloring is unique.

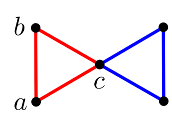
(b) To establish $r(K_3, 2K_3) \geq 8$ color a copy of $K_{2,5}$ in K_7 in red and the remaining edges blue. Then there is no red K_3 and no two vertex disjoint blue triangles.

Next we shall prove that any coloring of K_8 either contains a red triangle or two vertex disjoint blue triangles. Consider a coloring without red triangles. Since $r(K_3) = 6$ there is a blue triangle, say on vertices u, v, and w. If the remaining five vertices do not contain another blue triangle, then by part (a) the blue and the red edges form a C_5 each. Say the blue edges form a cycle C with vertices $x_1, \dots, x_5$. It is easy to see that there are exactly two red and three blue edges between $\{x_1, \dots, x_5\}$ and each vertex $y \in \{u, v, w\}$, and that the three blue neighbors of each $y \in \{u, v, w\}$ are consecutive on the blue cycle. An easy case distinction shows that, no matter how the blue neighborhoods are distributed in C, there are two vertex disjoint blue triangles, each with at most two vertices from $\{u, v, w\}$ and at most two from C.

(c) To establish $r(sK_3, tK_3) \geq 2s + 3t$ we color the edges of $K_{2s+3t-1}$ without red sK_3 and blue tK_3 as follows. Color a copy of K_{3t-1} and a vertex disjoint copy of $K_{1,2s-1}$ blue and color the remaining edges red. Then all blue triangles are contained in the blue clique and hence there are at most $t-1$ vertex disjoint blue triangles. The red subgraph forms a complete bipartite graph $K_{3t-1,2s-1}$ where all edges are added in the part of size $2s-1$. Each red triangle contains at least two vertices from this part and hence there are at most $s-1$ vertex disjoint red triangles.

Next we shall prove that for any $t \geq s \geq 1$, $t \geq 2$, any 2-coloring of K_{2s+3t} either contains t vertex disjoint red triangles or s vertex disjoint blue triangles by induction on t. The hint yields the base cases $s \in \{1, 2\}$ and $t = 2$ (and can be proven by case distinction). So assume that $t \geq 3$ and consider a 2-coloring of K_{2s+3t}. We treat the case $s = 1$ first. Since $r(K_3) = 6$ there is a monochromatic triangle K. If K is red we are done. Otherwise consider the graph induced by vertices not in K. It has $2 + 3t - 3 = 2 + 3(t-1)$ vertices. Inductively there is either a red triangle or $t-1$ vertex disjoint blue triangles and in both cases we are done.

So assume that $s \geq 2$. Iteratively removing vertices of a monochromatic K_3 as long as at least 6 vertices are left yields at least $\lceil \frac{2s+3t-5}{3} \rceil = t + \lceil \frac{2s-5}{3} \rceil \geq t$ vertex disjoint monochromatic triangles. If these triangles are all of the same color we are done. So assume that there is a red triangle on vertices a, b, c and a vertex disjoint blue triangle on vertices u, v, w. At least 5 out of the 9 edges between these triangles are of the same color, say blue. Then there is a blue triangle that shares exactly one vertex with the red triangle on $\{a, b, c\}$ (they form a so-called *bow-tie*) like in the figure. Then the set



V of vertices not in these two triangles has size $2s + 3t - 5 = 2(s-1) + 3(t-1)$. Since $t - 1 \geq s - 1 \geq 1$ and $t - 1 \geq 2$, V induces either $s-1$ vertex disjoint red triangles or $t-1$ vertex disjoint blue triangles by induction. Together with a triangle from the bow-tie we have either s vertex disjoint red triangles or t vertex disjoint blue triangles. Hence $r(sK_3, tK_3) \leq 2s + 3t$.

Question 2

(a) Calculate $r(C_3, C_4)$.

(b) Calculate $r(C_4, C_4)$.

Solution

(a) We claim that $r(C_3, C_4) = 7$.

First we show a coloring of K_6 without either red C_3 or blue C_4. Color the edges of K_6 such that the red edges form a copy of $K_{3,3}$. Then the blue edges form two vertex disjoint blue copies of $K_3 = C_3$ and there is no red C_3 and no blue C_4. This shows that

$$r(C_3, C_4) \geq 7.$$

It remains to prove that every coloring of K_7 yields a red C_3 or a blue C_4. Consider a coloring of K_7 with no red copy of C_3. We shall prove that there is a blue copy of C_4. Since $r(3, 3) = 6$ there is a blue copy of $K_3 = C_3$, say with vertex set U. Let V denote the remaining four vertices. If V induces no red edge then V induces a blue C_4. Otherwise let xy be a red edge with x, $y \in V$. Then, for each $u \in U$ there is at least one blue edge between $\{x, y\}$ and u. By pigeonhole principle there are two blue edges between one of x or y and U, and hence a blue C_4.

(b) We claim that $r(C_4, C_4) = 6$.

Coloring the edges of K_5 with two edge disjoint copies of C_5 shows that

$$r(C_4, C_4) \geq 6.$$

It remains to prove that every coloring of K_6 yields a monochromatic C_4. Since $r(3, 3) = 6$ there is a monochromatic copy of $K_3 = C_3$, say blue and with vertex set U. Let V denote the remaining three vertices. If there are two blue edges between a vertex $v \in V$ and U, then there is a blue copy of C_4. Otherwise each vertex in V sends at least two red edges to U. No two vertices from V send two red edges to the same two vertices from U otherwise we get a red C_4. This shows that the blue edges between U and V form a matching with three edges and the red edges form a copy of C_6. If there are at least two red edges induced on V, then at least one of those edges creates C_4 with 3 of the edges of the red C_6. So V induces a blue edge that is contained in a blue C_4 together with two matching edges and one edge induced by U.

Question 3

Let G be a graph on n vertices and let P_3 be the path with 3 vertices.

(a) Calculate $r(P_3, G)$ if there is a perfect matching in the complement of G.

(b) Calculate $r(P_3, G)$ if there is no perfect matching in the complement of G.

Solution Let m denote the size of a largest matching in the complement of G. We claim that

$$r(P_3, G) = \begin{cases} n, & \text{if } 2m = n, \\ 2n - 2m - 1, & \text{if } 2m < n. \end{cases}$$

(a) Consider the case $2m = n$ first, i.e., there is a perfect matching in the complement of G. Coloring all edges of K_{n-1} blue shows that $r(P_3, G) > n - 1$. Let $u_1, \ldots, u_m, v_1, \ldots, v_m$ denote the vertices of G such that $u_i v_i$ is a non-edge of G. Consider a coloring of K_n with no red copy of P_3. Then the red edges form a matching $x_1 y_1, \ldots, x_t y_t$. Mapping u_i to x_i and v_i to y_i for $1 \leq i \leq t$ and mapping the remaining vertices from G arbitrarily to the remaining vertices of K_n yields a blue copy of G. Thus $r(P_3, G) \leq n$.

(b) Now assume that $2m < n$ and let $N = 2n - 2m - 2$. Color a perfect matching (i.e., with $n - m - 1$ edges) in K_N red and all other edges blue. Consider a set S of n vertices in K_N. Then S induces at least $m + 1$ red matching edges and hence not a blue copy of G. Hence there is no blue copy of G and $r(P_3, G) > N$.

Finally consider a coloring of K_{N+1} with no red copy of P_3. Then the red edges form a matching $x_1 y_1, \ldots, x_t y_t$. Let $z_1, \ldots, z_{N+1-2t}$ denote the remaining vertices of K_{N+1}. Further let $u_1, \ldots, u_m, v_1, \ldots, v_m, w_1, \ldots, w_{n-2m}$ denote the vertices of G such that $u_i v_i$ is a non-edge of G and $w_1, \ldots, w_{n-2m}$ induce a complete graph in G. Observe that

$$|\{x_1, \ldots, x_t, z_1, \ldots, z_{N+1-2t}\}| \geq \left\lceil \tfrac{N+1}{2} \right\rceil = n - m = |\{u_1, \ldots, u_m, w_1, \ldots, w_{n-2m}\}|.$$

Mapping u_i to x_i and v_i to y_i for $1 \leq i \leq \min\{m, t\}$ and the remaining vertices from G to (some of) the remaining vertices of $\{x_1, \ldots, x_t, z_1, \ldots, z_{N+1-2t}\}$ yields a blue copy of G. Hence $r(P_3, G) \leq N + 1$ and thus

$$r(P_3, G) = 2n - 2m - 1.$$