Problem sheet 10

Due Monday, July 2nd at 17:30.

Question 1

Consider an integer $t \geq 3$.

(a) Prove that any orientation of K_t without oriented cycles induces a linear order of the vertices of K_t.

(b) Prove that for each $n < 2^{\frac{t-1}{2}}$ there is an orientation of the edges of K_n such that each set of t vertices contains an oriented cycle (of length at most t).

(c) Prove that $r_3(4, t) \geq 2^{\frac{t-1}{2}}$.

Question 2

For $t > k \geq 3$, $q \geq 2$ let $r_k(t; q)$ denote the *k-uniform q-color Ramsey number*, i.e., the smallest integer n such that any q-coloring of the edges of $K_n^{(k)}$ yields a monochromatic copy of $K_t^{(k)}$. Let $R = r_{k-1}(t-1; q)$ and prove that

$$r_k(t; q) \leq q^{\binom{R}{k-1}}.$$

(**Hint:** Follow the proof of Theorem 1 in (Ramsey) Lecture 4.)

Question 3

For $i \geq 1$, $x \in \mathbb{R}$, the tower-function $T_i(x)$ with base 2 is defined recursively by $T_1(x) = x$ and, for $i \geq 2$, $T_i(x) = 2^{T_{i-1}(x)}$. Prove that for all $t > k \geq 2$

$$2^{\frac{1}{k^k}t^{k-1}} < r_k(t) \leq T_k(4t + k).$$

(**Hint:** Use the statement from the previous problem.)