Problem sheet 10

Due Monday, July 2nd at 17:30.

Question 1

Consider an integer $t \geq 3$.

(a) Prove that any orientation of K_t without oriented cycles induces a linear order of the vertices of K_t.

(b) Prove that for each $n < 2^{\frac{t-1}{2}}$ there is an orientation of the edges of K_n such that each set of t vertices contains an oriented cycle (of length at most t).

(c) Prove that $r_3(4,t) \geq 2^{\frac{t-1}{2}}$.

Solution

(a) Let $u \prec v$ if the edge uv is oriented from u to v. Since the orientation has no cycles the ordering $\prec$ is a linear order of the vertices (one can iteratively find smallest elements).

(b) We shall prove that there is an orientation of the edges of K_n such that each set of t vertices contains an oriented cycle. To this end we shall calculate that the number of orientations without this property is less than the total number of orientations. By (a), we shall count the number of orientations of $E(K_n)$ that induce a transitive orientation on some t vertices.

For a fixed set S of t vertices, there are $t!$ possible linear orders of the vertices in S and thus $t!$ distinct transitive orientations of the edges induced by S. For a fixed orientation of the $\binom{t}{2}$ edges induced by S there are $2^{\binom{n}{2}-\binom{t}{2}}$ possible orientations of the remaining edges of K_n. This shows that there are at most

$$\binom{n}{t}\, t!\, 2^{\binom{n}{2}-\binom{t}{2}} \leq n^t 2^{\binom{n}{2}-\binom{t}{2}} < 2^{t\frac{t-1}{2}} 2^{\binom{n}{2}-\binom{t}{2}} = 2^{\binom{n}{2}}$$

orientations of the edges of K_n that induce a transitive orientation on some t vertices. The claim follows since there are $2^{\binom{n}{2}}$ distinct orientations of the edges of K_n in total.

(c) Let $n < 2^{\frac{t-1}{2}}$ and consider a set V of n vertices. We shall prove that there is a coloring of the triples from V in colors red and blue such that there is no red $K_4^{(3)}$ and no blue $K_t^{(3)}$.

Consider an orientation T of the edges of K_n (with vertex set V) such that each set of t vertices contains an oriented cycle (of length at most t). Color a triple red if it induces an oriented 3-cycle in T and blue otherwise.

First of all it is easy to see that among each set of 4 vertices there are 3 vertices that do not form an oriented 3-cycle. This shows that there is no red $K_4^{(3)}$.

Now consider a set U of t vertices. By assumption U induces an oriented cycle, so consider a shortest such cycle C. If C has more than 3 vertices, then each chord induces a shorter oriented cycle, a contradiction. Hence C is an oriented 3-cycle and thus the vertices from C form a red triple. This shows that there is no blue $K_t^{(3)}$.

Altogether, this coloring shows that $r_3(4,t) \geq 2^{\frac{t-1}{2}}$.

Question 2

For $t > k \geq 3$, $q \geq 2$ let $r_k(t;q)$ denote the *k-uniform q-color Ramsey number*, i.e., the smallest integer n such that any q-coloring of the edges of $K_n^{(k)}$ yields a monochromatic copy of $K_t^{(k)}$. Let $R = r_{k-1}(t-1;q)$ and prove that

$$r_k(t;q) \leq q^{\binom{R}{k-1}}.$$

(**Hint:** Follow the proof of Theorem 1 in (Ramsey) Lecture 4.)

Solution Let $R = r_{k-1}(t-1;q)$ and $N = q^{\binom{R}{k-1}}$ and consider a q-coloring c of the edges of $K_N^{(k)}$ with vertex set V.

Claim: For each i, $k-1 \leq i \leq R-1$, there is a vertex set $V_i = \{v_1, \ldots, v_i\}$ and a set $S_i \subseteq V \setminus V_i$ such that

- for all $1 \leq j_1 < \ldots < j_{k-1} \leq i$ all the edges in $K_N^{(k)}$ of the form $\{v_{j_1}, \ldots, v_{j_{k-1}}, x\}$, $x \in S_i \cup \{v_{j_{k-1}+1}, \ldots, v_i\}$, are of the same color,
- $|S_i| \geq N\, q^{-\binom{i}{k-1}} - k + 1$.

We prove the claim by induction on i. For $i = k-1$ choose an arbitrary vertex set $V_{k-1} = \{v_1, \ldots, v_{k-1}\}$ of size $k-1$. Consider the edges $\{v_1, \ldots, v_{k-1}, x\}$, $x \notin V_{k-1}$, and let S_{k-1} consist of those vertices x forming such edges of the majority color class. Then

$$|S_{k-1}| \geq \tfrac{1}{q}(N - (k-1)) \geq q^{-1}N - k + 1.$$

This proves the claim for $i = k-1$.

Now consider $i \geq k$ and assume that the claim holds for $i-1$. That is, there is a vertex set $V_{i-1} = \{v_1, \ldots, v_{i-1}\}$ and a set $S_{i-1} \subseteq V \setminus V_{i-1}$ satisfying the conditions of the claim. Choose an arbitrary vertex v_i from S_{i-1} and let $V_i = \{v_1, \ldots, v_i\}$. Let $p = \binom{i-1}{k-2}$ and let $X_1, \ldots, X_p$ denote the subsets of V_{i-1} of size $k-2$ in some order. Let $S_i^0 = S_{i-1}$ and for $j = 1, \ldots, p$ define sets S_i^j recursively as follows. Consider the edges $X_j \cup \{v_i, x\}$, $x \in S_i^{j-1}$, and let S_i^j consist of those vertices $x \in S_i^{j-1}$ forming such edges of the majority color class. Then

$$|S_i^j| \geq q^{-1}|S_i^{j-1}| \geq q^{-j}|S_i^0| = q^{-j}|S_{i-1} \setminus \{v_i\}|.$$

Finally let $S_i = S_i^p$. Then V_i and S_i clearly satisfy the first item of the claim and

$$\begin{aligned} |S_i| &\geq q^{-p}(|S_{i-1}| - 1) \geq q^{-p}(q^{-\binom{i-1}{k-1}}N - k) \geq q^{-\binom{i-1}{k-1}-p}N - q^{-1}k \\ &\geq q^{-\binom{i-1}{k-1}-\binom{i-1}{k-2}}N - (k-1) = q^{-\binom{i}{k-1}}N - k + 1. \end{aligned}$$

This finishes the proof of the claim.

Now consider the sets $V_{R-1} = \{v_1, \ldots, v_{R-1}\}$ and S_{R-1} satisfying the claim for $i = R-1$. By assumption $t > k \geq 3$. Thus $R = r_{k-1}(t-1; q) \geq t-1 \geq k$ and

$$|S_{R-1}| \geq q^{-\binom{R-1}{k-1}}N - k + 1 = q^{\binom{R}{k-1}-\binom{R-1}{k-1}} - k + 1 = q^{\binom{R-1}{k-2}} - k + 1 \geq 2^{k-1} - (k-1) \geq 2.$$

Choose two further vertices v_R, v_{R+1} from S_{R-1}. Then for all $1 \leq j_1 < \ldots < j_{k-1} \leq R$ and any v_j, $j > j_{k-1}$, all the edges in $K_N^{(k)}$ of the form $\{v_{j_1}, \ldots, v_{j_{k-1}}, v_j\}$, are of the same color. Coloring each tuple $\{v_{j_1}, \ldots, v_{j_{k-1}}\}$, $1 \leq j_1 < \ldots < j_{k-1} \leq R$, with this color yields a coloring c' of $K_R^{(k-1)}$ in q colors. Since $R = r_{k-1}(t-1; q)$ there are $t-1$ vertices $u_1, \ldots, u_{t-1} \in \{v_1, \ldots, v_R\}$ such that all edges induced by these vertices in $K_R^{(k-1)}$ are of the same color under c'. Hence all edges induced by $\{u_1, \ldots, u_{t-1}, v_{R+1}\}$ in $K_N^{(k)}$ are of the same color under c, i.e., form a monochromatic copy of $K_t^{(k)}$. Hence

$$r_k(t; q) \leq N = q^{\binom{r_{k-1}(t-1;q)}{k-1}}.$$

Question 3

For $i \geq 1$, $x \in \mathbb{R}$, the tower-function $T_i(x)$ with base 2 is defined recursively by $T_1(x) = x$ and, for $i \geq 2$, $T_i(x) = 2^{T_{i-1}(x)}$. Prove that for all $t > k \geq 2$

$$2^{\frac{1}{k^k}t^{k-1}} < r_k(t) \leq T_k(4t + k).$$

(**Hint:** Use the statement from the previous problem.)

Solution First we prove the lower bound. Let $n = 2^{\frac{1}{k^k} t^{k-1}}$ and color the edges of $K_n^{(k)}$ randomly, where an edge is red with probability $\frac{1}{2}$ and blue otherwise. Then the expected number of monochromatic copies of $K_t^{(k)}$ is (as $t \geq 3 > e$)

$$\binom{n}{t} \frac{2}{2^{\binom{t}{k}}} \leq \left(\frac{ne}{t}\right)^t \frac{2}{2^{\binom{t}{k}}} < 2^{\left(\frac{t}{k}\right)^k + 1 - \binom{t}{k}} \leq 2.$$

Since the expected number of monochromatic copies of $K_t^{(k)}$ is less than 2 there is a coloring with at most one monochromatic copy of $K_t^{(k)}$. Removing one vertex from this copy shows that $r_k(t) \geq 2^{\frac{1}{k^k} t^{k-1}}$.

Next we shall prove the upper bound. Observe that for $x \geq 1$

$$T_4(x+1) = 2^{2^{2^{x+1}}} = 2^{\left(2^{2^x} \cdot 2^{2^x}\right)} \geq 2^{\left(2^{2^x} \cdot 4\right)} = T_4(x)^4$$

and hence the following fact holds for $x \geq 1$ by induction on $i \geq 4$

$$T_i(x+1) = 2^{T_{i-1}(x+1)} \geq 2^{T_{i-1}(x)^{i-1}} \geq 2^{T_{i-1}(1)\, T_{i-1}(x)} \geq 2^{i\, T_{i-1}(x)} = T_i(x)^i. \tag{1}$$

Now we prove the upper bound on $r_k(t)$ by induction on k. Theorem 1 from (Ramsey-) Lecture 1 states that

$$r_2(t) \leq 2^{2t-3} < 2^{4t+2} = T_2(4t+2).$$

Previous problem states for $R = r_{k-1}(t-1)$ (and $q = 2$ colors) that

$$r_k(t) \leq 2^{\binom{R}{k-1}} \leq 2^{R^{k-1}}.$$

For $k = 3$ this yields, using the upper bound $r_2(t-1) \leq 2^{2t-5}$:

$$r_3(t) \leq 2^{(2^{2t-5})^2} = 2^{2^{4t-10}} \leq 2^{2^{4t+3}} = T_3(4t+3).$$

For $k = 4$ we get using the upper bound $r_3(t-1) \leq 2^{2^{4t-14}}$

$$r_4(t) \leq 2^{2^{3 \cdot 2^{4t-14}}} \leq 2^{2^{2^{4t+4}}} = T_4(4t+4).$$

For $k \geq 5$ we get inductively

$$r_k(t) \leq 2^{T_{k-1}(4t+k-5)^{k-1}} \overset{(1)}{\leq} 2^{T_{k-1}(4t+k-4)} \leq T_k(4t+k).$$