Problem sheet 11

Due Monday, July 9th at 17:30.

Question

Let $\rho \leq \frac{1}{16}$ and let H be a graph with t vertices, maximum degree ρt, and no isolated vertices. The goal of this exercise is to prove that for all $s \geq \rho t$ and $P = 12 \log_2(2/\rho)\rho t$

$$r(K_s, H) \leq \left(\frac{2s}{\rho t}\right)^P.$$

1. Prove that for each bipartite graph of density $d \geq \frac{1}{2}$ with parts A and B there are at least $(1-d)|A|$ vertices in A of degree at least $(2d-1)|B|$.

2. Prove that each bipartite graph of density $d \geq \frac{3}{4}$ with parts A and B has a complete bipartite subgraph with parts $A' \subset A$ and $B' \subset B$ such that $|A'| \geq \frac{3}{4}|A|$, $|B'| \geq (4d-3)^{\frac{3}{4}|A|}|B|$.

3. Prove that $r(K_{\rho t}, H) \leq 2^P$. (This is the induction basis.)

4. Consider $s > \rho t$. You may divide the induction step into the following parts.

   (a) Use Lemma 2 from (Ramsey) Lecture 6 (with $\delta = \rho$) to find a blue copy of H or sets A and B, of size at least $\frac{1}{4}\rho^{\rho t}(\rho t)^{-2}N$ with the red graph between A and B of density at least $1-\rho$.

   (b) Apply part (a) for the red graph between A and B to find $A' \subseteq A$, $\rho|A|$, with all vertices of degree at least $(1-2\rho)|B|$. Use induction (or Thm. 2 from first Ramsey lecture) to find a red $K_{\frac{2}{3}s}$ within A'.

   (**Remark:** Use without proof: $\frac{1}{4}\rho^{\rho t-1}t^{-2}\left(\frac{2s}{\rho t}\right)^P \geq \left(\frac{2}{\rho}\right)^{2\rho t}\left(\frac{2(\frac{2}{3}s)}{\rho t}\right)^P$.)

   (c) Apply part (b) for the red graph between B and the red clique from part (c)ii to find a large complete bipartite subgraph with parts $A'' \subseteq A'$, $B' \subseteq B$. Similarly to (ii) find a red red $K_{\frac{1}{2}s}$ within B'.

   (**Remark:** Use without proof: $(1-8\rho)^{\frac{1}{2}s}\frac{1}{4}\rho^{\rho t}(\rho t)^{-2}\left(\frac{2s}{\rho t}\right)^P \geq \left(\frac{2}{\rho}\right)^{2\rho t}\left(\frac{2(\frac{1}{2}s)}{\rho t}\right)^P$.)