Problem sheet 11

Due Monday, July 9th at 17:30.

Question

Let $\rho \leq \frac{1}{16}$ and let H be a graph with t vertices, maximum degree ρt, and no isolated vertices. The goal of this exercise is to prove that for all $s \geq \rho t$ and $P = 12\log_2(2/\rho)\rho t$

$$r(K_s, H) \leq \left(\frac{2s}{\rho t}\right)^P.$$

1. Prove that for each bipartite graph of density $d \geq \frac{1}{2}$ with parts A and B there are at least $(1-d)|A|$ vertices in A of degree at least $(2d-1)|B|$.

2. Prove that each bipartite graph of density $d \geq \frac{3}{4}$ with parts A and B has a complete bipartite subgraph with parts $A' \subset A$ and $B' \subset B$ such that $|A'| \geq \frac{3}{4}|A|$, $|B'| \geq (4d-3)^{\frac{3}{4}|A|}|B|$.

3. Prove that $r(K_{\rho t}, H) \leq 2^P$. (This is the induction basis.)

4. Consider $s > \rho t$. You may divide the induction step into the following parts.

   (a) Use Lemma 2 from (Ramsey) Lecture 6 (with $\delta = \rho$) to find a blue copy of H or sets A and B, of size at least $\frac{1}{4}\rho^{\rho t}(\rho t)^{-2}N$ with the red graph between A and B of density at least $1-\rho$.

   (b) Apply part (1) for the red graph between A and B to find $A' \subseteq A$, $\rho|A|$, with all vertices of degree at least $(1-2\rho)|B|$. Use induction (or Thm. 2 from first Ramsey lecture) to find a red $K_{\frac{2}{3}s}$ within A'.

   (**Remark:** Use without proof: $\frac{1}{4}\rho^{\rho t-1}t^{-2}\left(\frac{2s}{\rho t}\right)^P \geq \left(\frac{2}{\rho}\right)^{2\rho t}\left(\frac{2(\frac{2}{3}s)}{\rho t}\right)^P$.)

   (c) Apply part (2) for the red graph between B and the red clique from part (3)b to find a large complete bipartite subgraph with parts $A'' \subseteq A'$, $B' \subseteq B$. Similarly to (b) find a red red $K_{\frac{1}{2}s}$ within B'.

   (**Remark:** Use without proof: $(1-8\rho)^{\frac{1}{2}s}\frac{1}{4}\rho^{\rho t}(\rho t)^{-2}\left(\frac{2s}{\rho t}\right)^P \geq \left(\frac{2}{\rho}\right)^{2\rho t}\left(\frac{2(\frac{1}{2}s)}{\rho t}\right)^P$.)

Solution

1. Assume not. Then there are more than $d|A|$ vertices in A of degree less than $(2d-1)|B|$. This yields a contradiction as the density of the graph is now less than

$$\frac{d|A|\,(2d-1)|B|+(1-d)|A||B|}{|A||B|}=\underbrace{d}_{\leq 1}\underbrace{(2d-1)}_{\geq 0}+1-d\leq(2d-1)+1-d=d.$$

2. Let $\ell=\frac{3}{4}|A|$ and let L denote the number of copies of $K_{1,\ell}$ that have one vertex in B and ℓ vertices in A. Recall that by standard arguments using Jensen's inequality we have

$$L\geq|B|\binom{d|A|}{\ell}.$$

Then

$$\frac{L}{\binom{|A|}{\ell}}\geq\frac{|B|\binom{d|A|}{\ell}}{\binom{|A|}{\ell}}=|B|\prod_{i=0}^{l-1}\frac{d|A|-i}{|A|-i}=|B|\prod_{i=0}^{l-1}\left(1-\frac{(1-d)|A|}{|A|-i}\right)$$
$$\geq|B|\left(1-\frac{(1-d)|A|}{|A|-\ell}\right)^{\ell}=|B|\,(4d-3)^{\frac{3}{4}|A|}.$$

Hence there is a set of ℓ vertices in A with at least $|B|\,(4d-3)^{\frac{3}{4}|A|}$ common neighbors. This proves (b).

3. If $s=\rho t$, then by Thm. 2 from the first Ramsey lecture

$$r(K_s,H)\leq r(K_s,K_t)\leq\binom{s+t}{s}=\binom{(1+\rho)t}{\rho t}\leq\left(\frac{e(1+\rho)}{\rho}\right)^{\rho t}\leq\left(\frac{4}{\rho^2}\right)^{\rho t}=\left(\frac{2}{\rho}\right)^{2\rho t}=2^{2\log(2/\rho)\rho t}.$$

This is the induction basis.

4. Consider $s>\rho t$, let $N=\left(\frac{2s}{\rho t}\right)^{P}$, and consider a coloring of K_N in red and blue with no blue copy of H. We shall prove that there is a red copy of K_s.

(a) Recall that H has maximum degree ρt. If the blue edges form a bi-$(\frac{1}{4}\rho^{\rho t}(\rho t)^{-2},\rho)$-dense graph, then by Lemma 2 from Ramsey Lecture 6, there is a blue copy of H. Hence the blue graph is not bi-$(\frac{1}{4}\rho^{\rho t}(\rho t)^{-2},\rho)$-dense and hence there are sets A, $B\subseteq V(K_N)$ with $|A|$, $|B|\geq\frac{1}{4}\rho^{\rho t}(\rho t)^{-2}N$ and the blue subgraph between A and B has density less than ρ. Let H_r be the red subgraph between A and B. Then H_r has density at least $1-\rho$.

(b) Note that $1-\rho>\frac{1}{2}$. By part (1) (with $d=1-\rho$) there is a set $A'\subseteq A$ of at least $\rho|A|$ vertices each of degree at least $(1-2\rho)|B|$ in H_r. Due to the hint

$$|A'|\geq\rho|A|\geq\frac{1}{4}\rho^{\rho t+1}(\rho t)^{-2}N=\frac{1}{4}\rho^{\rho t-1}t^{-2}\left(\frac{2s}{\rho t}\right)^{P}\geq\left(\frac{2}{\rho}\right)^{2\rho t}\underbrace{\left(\frac{2(\frac{2}{3}s)}{\rho t}\right)^{P}}_{\geq 1\text{ as }s\geq\rho t}\geq r(K_{\frac{2}{3}s},H).$$

The last inequality holds in case $\frac{2}{3}s \geq \rho t$ by induction on s and in case $\frac{2}{3}s < \rho t$ by Thm. 2 from the first Ramsey Lecture (like in the base case). Since there is no blue copy of H there is a red copy K of $K_{\frac{2}{3}s}$ within A'.

(c) Since $V(K) \subseteq A'$ and each vertex in A' has degree at least $(1-2\rho)|B|$ in H_r, the red subgraph between K and B has density at least $(1-2\rho)$. By part (b) (with $d = 1-2\rho$) there is a red complete bipartite subgraph between sets $A'' \subset V(K)$ and $B' \subseteq B$ with $|A''| \geq \frac{3}{4}|V(K)| = \frac{1}{2}s$ and

$$|B'| \geq (1-8\rho)^{\frac{1}{2}s}|B| \geq (1-8\rho)^{\frac{1}{2}s}\frac{1}{4}\rho^{\rho t}(\rho t)^{-2}\left(\frac{2s}{\rho t}\right)^{P} \overset{\text{Hint}}{\geq} \left(\frac{2}{\rho}\right)^{2\rho t} \underbrace{\left(\frac{2(\frac{1}{2}s)}{\rho t}\right)^{P}}_{\geq 1 \text{ as } s \geq \rho t} \geq r(K_{\frac{1}{2}s}, H).$$

The last inequality holds similarly to part (b). Since there is no blue copy of H there is a red copy K' of $K_{\frac{1}{2}s}$ within B'. Now $A'' \cup V(K')$ induces a red copy of K_s and the proof is complete.