Graph Theory

Judith Stumpp

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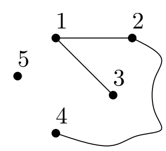
Bei dem folgenden Skript handelt es sich um einen Mitschrieb der Vorlesung Graph Theory vom Wintersemester 2011/2012. Sie wurde gehalten von Prof. Maria Axenovich Ph.D. . Der Mitschrieb erhebt weder Anspruch auf Vollständigkeit, noch auf Richtigkeit!

Kapitel 1

Definitions

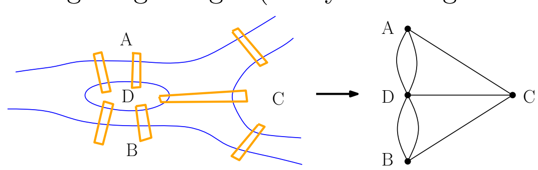
The *graph* is a pair V, E. V is a finite set and $E \subseteq \binom{V}{2}$ a pair of elements in V. V is called the set of vertices and E the set of edges.

Visualize: $G = (V, E),\ V = \{1, 2, 3, 4, 5\},\ E = \{\{1, 2\}, \{1, 3\}, \{2, 4\}\}$



History: word: Sylvester (1814-1897) and Cayley (1821-1895)
Euler - developed graph theory

Königsberg bridges (today Kaliningrad in Russia):

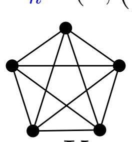


Problem: Travel through each bridge once, come back to the original point.

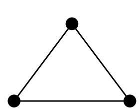
Impossible!

Notations:

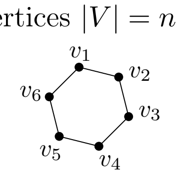
- $K_n = (V, \binom{V}{2})$ - complete graph on n vertices $|V| = n$



K_5



K_3

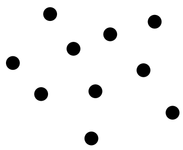


$C_6 = v_1v_2v_3v_4v_5v_6$

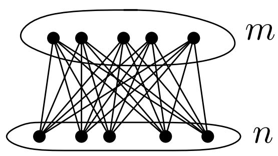
- C_n - cycle on n vertices
  $V = \{v_1, v_2, \ldots, v_n\},\ E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$

- P_n - path on n vertices (Note: P^n . path on n edges (Diestel))
  $V = \{v_1, v_2, \ldots, v_n\},\ E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}\}$

- Let P be a path from v_1 to v_n. The subpath of P from v_i to v_j is v_iPv_j and the subpath from v_{i+1} to v_j is $\mathring{v}_iPv_j$.
- E_n= $(V, \emptyset)$, $|V| = n$ isolated vertices

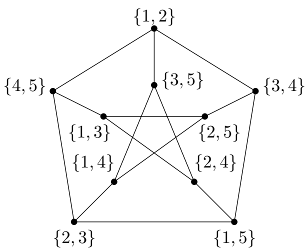


E_{10}

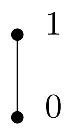
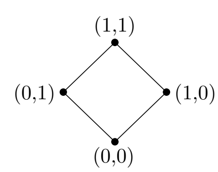
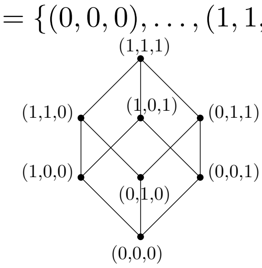


$K_{m,n}$

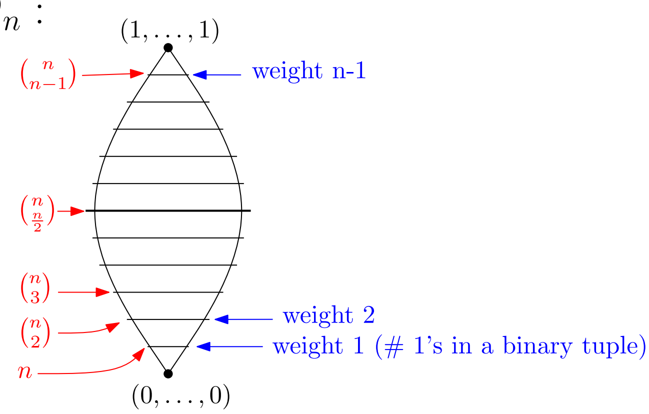
- $K_{n,m}$= $(A \cup B, A \times B)$, $A \cap B = \emptyset$ complete bipartite graph
- *Peterson graph*: $V = \binom{\{1,2,3,4,5\}}{2}$, $E = \{\{\{i,j\},\{k,l\} : \{i,j\} \cap \{k,l\} = \emptyset\}$

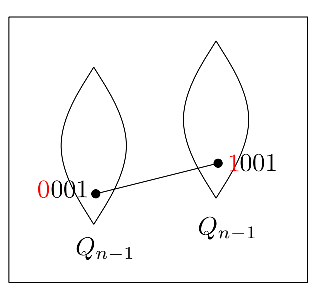


- *Kneser Graph $K(n,k)$*= $(\binom{V}{k}, E)$
  $|V| = n$, $E = \{\{A, B\} : A, B \in \binom{V}{k}$ and $A \cap B = \emptyset\}$.
  $\binom{V}{k}$ is the set of k-element subsets of V, $|\binom{V}{k}| = \binom{|V|}{k}$
- Q_n - hypercube of dimension n.
  $Q_n = \{2^{\{1,2,\dots,n\}}, E\}$, $E = \{\{A, B\} : |A \triangle B| = 1\}$ $(A \triangle B := (A \cup B) - (A \cap B))$
  V - set of binary n-tuples E - pairs of binary tuples different in 1 position

Q_1	Q_2	Q_3
$V = \{\emptyset, \{1\}\} = \{(0), (1)\}$	$V = \{(0,0), (0,1), (1,0), (1,1)\}$	$V = \{(0,0,0), \dots, (1,1,1)\}$
		

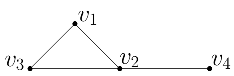
Q_n :





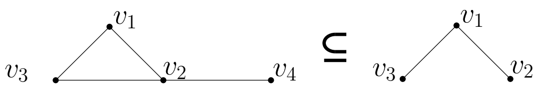
Parameter: Let $G = (V, E)$ be a graph. The *order* of G ist the number of vertices ($|V|$) and the *size* of G is the number of edges ($|E|$).

If the order of G is n, then $0 \le \text{size}(G) \le \binom{n}{2}$.
If $e = \{x, y\} \in E$, x is *adjacent* to y and x is *incident* to e.
There is a $n \times n$ matrix A of $G = (\{v_1, \dots, v_n\}, E)$ which is called the *adjacent matrix*.

For 

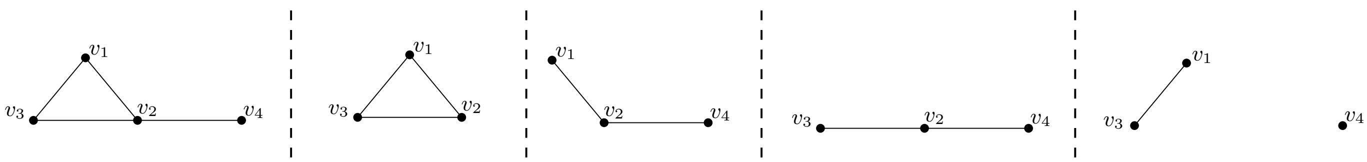
$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

Subgraph: $H \subseteq G,\ H = (V', E'),\ G = (V, E),\ V' \subseteq V,\ E' \subseteq E$



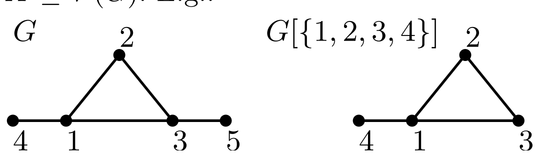
$H \underset{\text{ind}}{\subseteq} H$ is an *induced subgraph* of G if $H \subseteq G$ and for $v_1, v_2 \in V(H)$: $\{v_1, v_2\} \in E(H) \Leftrightarrow \{v_1, v_2\} \in E(G)$.
In the upper example it is no induced subgraph.

An induced subgraph is obtained from G by deleting vertices. E.g.:

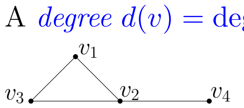


Let $G = (V, E)$ and $G' = (V', E')$ be graphs. Then we define $G \cup G' := (V \cup C', E \cup E')$ and $G \cap G' := (V \cap C', E \cap E')$.

$G[X] := (X, \{\{x, y\} : x, y \in X,\ \{x, y\} \in E(G)\})$ is called the subgraph of G *induced* by a vertex set $X \subseteq V(G)$. E.g.:



A *degree* $d(v) = \deg v$ of a vertex is the number of edges incident to that vertex.



$\deg v_1 = 2,\ \deg v_2 = 3,\ \deg v_3 = 2,\ \deg v_4 = 1$

In this example the *degree sequence* is $(2, 3, 2, 1)$, the *minimum degree* $\delta(G)$ is 1 and the *maximum degree* $\Delta(G)$ is 3.

Apparently $|E(G)| = \frac{1}{2} \sum_{i=1}^{n} \deg v_i$ is true.

Thus $\sum_{i=1}^{n} \deg v_i$ is even and therefore the number of vertices with odd degree is even.

$d(G) := \frac{1}{n} \sum_{i=1}^{n} \deg v_i$ is called the *average degree* of G.

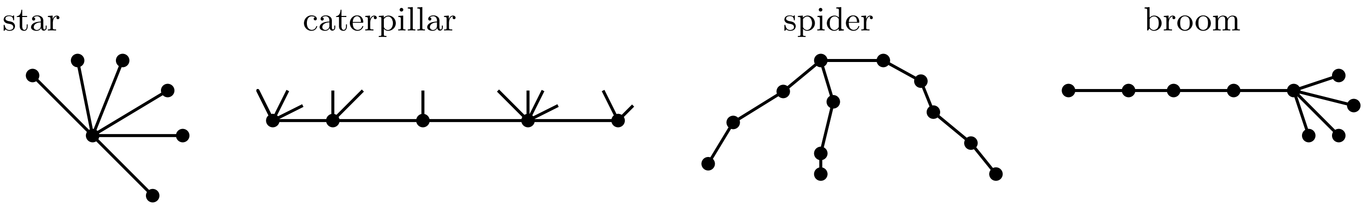
> **Extremal graph theorem:** *We'll prove that if G has n vertices and $> \left\lceil \frac{n^2}{4} \right\rceil$ edges $\Rightarrow$ G has a triangle.*

Let $A, B \subseteq V,\ A \cap B = \emptyset$. P is an *A-B-path* if $P = v_1 \dots v_k,\ V(P) \cap A = \{v_1\}$ and $V(P) \cap B = \{v_k\}$.

A graph is *connected* if any two vertices are linked by a path. A maximal connected subgraph of a graph is a *connected component*.

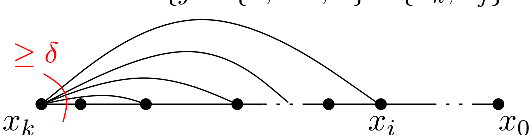
A connected graph without cycles is called a *tree*. A graph without cycles (*acyclic* graph) is called a *forest*.

Other „special named“ graphs:



> **Proposition:** *If a graph G has a minimum degree $\delta(G) \geq 2$ then G has a path of length $\delta(G)$ and a cycle with at least $\delta(G) + 1$ vertices.*

proof: Let $P = (x_0, \dots, x_k)$ be a longest path in G. Then all neighbors of x_k are in $V(P)$ (y is a neighbor of x if $\{x, y\} \in E$). In particular $k \geq \delta(G)$.
Let $i = \min\{j \in \{0, \dots, k\} : \{x_k, x_j\} \in E\}$. Then $x_i x_k x_{k-1} \dots x_i$ is a cycle of length at least $\delta + 1$.



■

The *girth* of a graph G is the length of a smallest cycle in G.

The *distance $d_G(v, w)$* of $v, w \in G$ is the length of the smallest path between them ($\min \emptyset = \infty$).
The *diameter* of G is $\max\{d_G(v, w) : v, w \in G\}$.

> **Proposition:** *Every nontrivial tree T has a leaf.*

proof: Assume T has no leaves. T has no isolated vertices $\Rightarrow \delta(T) \geq 2 \Rightarrow C_n \subseteq T$ ↯ ■

> **-** *A tree T of order $n \geq 1$ has $n - 1$ edges.*

proof: $T = K_1$ ✓

Assume it holds for all trees of order $< n$.

Let v be a leaf of T, $T' := T - v$.
$\Rightarrow |T'| = n - 1 < n$.
T' is acyclic.
Let $v', w \in T'$. $\exists P\ v' = v_0, v_1, \dots, v_n = w \subseteq T$.

To show: $v_i \neq v$ for all $i = 0, \dots, n$

$v_0, v_n \neq v$ because $v_0, v_n \in T',\ v \notin T'$
$v_i \neq v$ $(i = 1, \dots, n-1)$ because $d_T(v_i) \geq 2$, v_i is not a leaf.

$\Rightarrow P \subseteq T'$ connecting v_0 and $w \ \Rightarrow \ T'$ connected $\Rightarrow \ T$ is a tree.

With induction hypothesis T' has $(n-1)-1$ edges. Thus T has $(n-1)-1+2 = n-1$ edges. ■

A *walk* is an alternating sequence $v_0e_0v_1e_1\dots v_n$ of vertices and edges so that $e_i = v_iv_{i+1}$ for all $n = 0,\dots,n-1$. Compared to a path it is allowed to pass edges and vertices more than once. If $v_0 = v_n$, then the walk is a *closed walk*.

- *If G has a u-v-walk (between vertices u, v) $\Rightarrow G$ has a u-v-path.*

proof: Consider the shortest walk between u and v is W. Then W is a path. If not, W has a repeated vertex $W = \underbrace{ue_0v_1e_1\dots}_{=:W_1}\underbrace{v_i\dots}_{=:\tilde{W}}\underbrace{v_i\dots v}_{=:W_2}$, then $W' = W_1W_2$ is a shorter u-v-walk. ↯ ■

- *If G has an odd closed walk (i.e. odd # edges) then G has an odd cycle.*

proof: If there are no repeated vertices (except for first and last) $\Rightarrow$ we have an odd cycle.

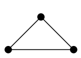
If there is a repeated vertex v_i, $W = \underbrace{v_0e_0v_1\dots}_{\text{1'st part of } W_2}\underbrace{v_i\dots v_i}_{W_1}\underbrace{\dots v_n = v_0}_{\text{2'nd part of } W_2}$.

W is a union of two closed walks W_1 and W_2. Either W_1 or W_2 is an odd closed walk

$\Rightarrow$ by induction it contains an odd cycle. ■

- *If G has a closed walk with a non-repeated edge $W = v_0e_0v_1\dots e_i\dots$ e_i is unique, then G contains a cycle.*

proof: Induction on # vertices.

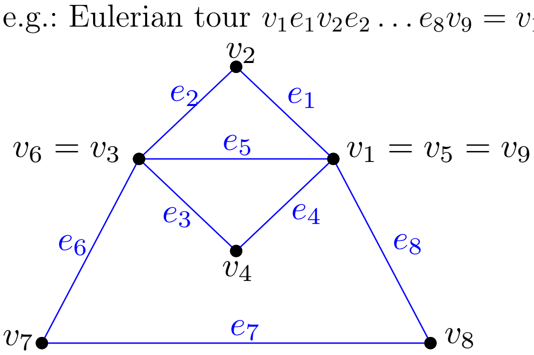
Basis: 

Step: $W = \underbrace{v_0e_0v_1\dots}_{\text{1'st part of } W_2}\underbrace{v_i\dots v_i}_{W_1}\underbrace{\dots v_n = v_0}_{\text{2'nd part of } W_2}$

(note, there is a repeated vertex v_j, otherwise W is a cycle)
So, W is a union of two closed walks W_1 and W_2 and either W_1 or W_2 has a non-repeated edge.
By induction, that walk contains a cycle. ■

Definition: An *Eulerian tour* is a closed walk containing all edges of a graph and repeating no edge.

e.g.: Eulerian tour $v_1e_1v_2e_2\dots e_8v_9 = v_1$ in



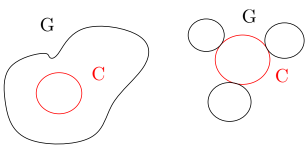
Theorem: *A connected graph G has an Eulerian tour iff (i.e. if and only if) each degree of vertex in G is even.*

proof:

„⇒“ : If there is an Eulerian tour then clearly the number of edges entering the vertex is the number of edges leaving the vertex.

„⇐“ : Assume that each degree is even.
Consider a walk with longest number of edges and no repeated edge, $W = v_0 \dots v_k$. Thus, there is no edge incident to v_0 that is not in W. Since $\deg v_0$ is even, v_0 must be v_n, i.e. W is a closed walk.
If all edges are in W, done. Otherwise, there is an edge e, not in W. Since G is connected, there is such e incident to a vertex in W. Say $e = v_i u$. Then $W' = uev_iWv_i$ is a longer walk with no repeated edges. $\lightning$

Other idea: all edges in G are even, $\delta(G) \geq 2 \ \Rightarrow G$ has a cycle C. Delete C from G (problem: $G - C$ maybe isn't connected).



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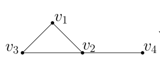
Connectivity:

We say that a Graph G is vertex *k-connected* if $|V(G)| > k$ and deleting any $(k-1)$ vertices does not disconnect the graph.

Any connected graph is 1-connected. If a graph is 2-connected then there exists no *cut-vertex* which is a vertex whose deletion disconnects a graph. Trees are not 2-connected.

If G is connected, $X \subseteq V,\ G - X$ disconnected $\Rightarrow X$ is called a *cut-set*.

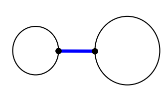
$$\kappa(G) = \max\{k : \ G \text{ is } k\text{-connected}\}$$

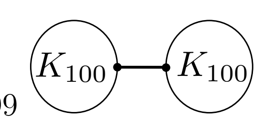
e.g.: $\kappa($  $) = 1,\ \kappa(C_n) = 2,\ \kappa(K_{n,m}) = \min\{m, n\}$.

G is called *l-edge connected* if $G \neq E_n$ and G does not become disconnected after deleting any $(l-1)$ edges.

$$\lambda(G)\,(= \kappa'(G)) = \max\{l : \ G \text{ is } l\text{-edge-connected}\}$$

e.g.: $\lambda(\text{tree}) = 1,\ \lambda(C_n) = 2$.

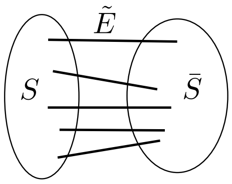
If $\lambda(G) = 1$ there exists a so called *bridge* (cut edge) .

Clearly $\lambda(G) \leq \delta(G)$. But it could be that $1 = \lambda(G) << \delta(G) = 99$  .

Lemma: *For any connected G: $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

proof: Idea: want to find the set of at most $\lambda := \lambda(G)$ vertices that disconnects the graph.

Let $\tilde{E}$ be a set of λ edges disconnecting G. Then $\tilde{E}$ is a cut, i.e. $\exists S \subseteq V:\ \forall e \in \tilde{E}$, one endpoint of e is in S, another is in $\bar{S} := V - S$.



If in G there are all edges between S and $\bar{S}$. $\lambda = |\tilde{E}| = |S| \cdot |\bar{S}| \geq |V(G)| - 1 \geq \kappa(G)$.

Otherwise $\exists x \in S, y \in \bar{S},\ x \not\sim y$ (i.e. $xy \notin E(G)$).

$$T := (N(x) \cap \bar{S}) \cup (\{z \in S:\ z \sim \bar{S}\} - \{x\})$$

T is a vertex cut, in particular after deleting T, x and y are in different connected components. We have $|T| \leq |\tilde{E}| = \lambda$ because

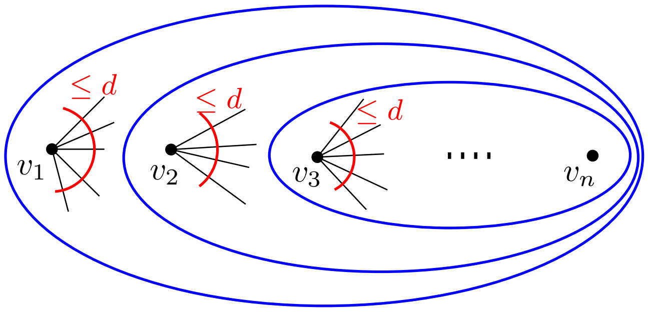
$$|N(x)| \leq \#(\text{edges incident to } x) \text{ and } |\{z \in S:\ z \sim \bar{S}\} - \{x\}| \leq \#(\text{edges incident to this set}).$$

■

Definition: A graph G is *d-degenerate* if there is a vertex order $v_1, v_2, \ldots, v_n$:

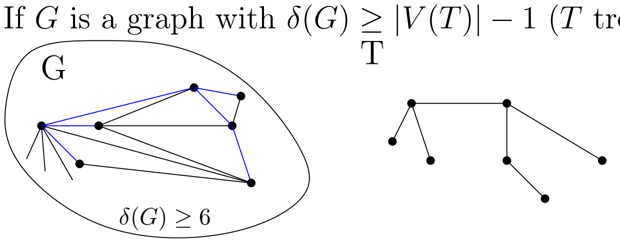
$$|N(v_i) \cap \{v_{i+1}, \ldots, v_n\}| \leq d.$$

I.e. we eliminate the graph by deleting a vertices sequence, s.t. at most d edges are gone at a time.



Let T be a graph. T is a *tree* if it is connected and acyclic.

- T is a tree iff T is connected and has $|V(T)| - 1$ edges.
- T is 1-degenerate.
- A *leaf* in a nontrivial tree is a vertex of degree 1.
- If G is a graph with $\delta(G) \geq |V(T)| - 1$ (T tree) then G contains T as a subgraph.



Lemma: *A graph is bipartite if and only if it has no odd cycles.*

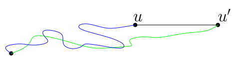
proof:

„⇒“ : Let G be a bipartite graph, then any cycle has a form $u_1v_1u_2v_2\dots u_kv_ku_1$, where $u_i \in U,\ v_i \in V,\ 1 \le i \le k,\ U, V$ are partite sets of G.

„⇐“ : Assume that G is connected and has no odd cycles. We shall prove that G is bipartite with partite sets U, V defined as follows.
Fix $x \in V(G)$.
Let $U = \{u : \text{dist}(x, u) \text{ is even}\},\ V\{v : \text{dist}(x, v) \text{ is odd}\}$
We need to verify that $G[U], G[V]$ are empty graphs.

Assume that $u, u' \in U$ and $\{u, u'\} \in E(G)$.

Consider a walk formed by shortest x-u-path, shortest x-u'-path and u, u'.



This is an odd closed walk that contains an odd cycle, a contradiction.

Thus $G[U]$ is an empty graph.
Similarly $G[V]$ is an empty graph.

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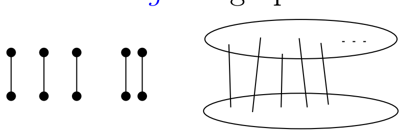
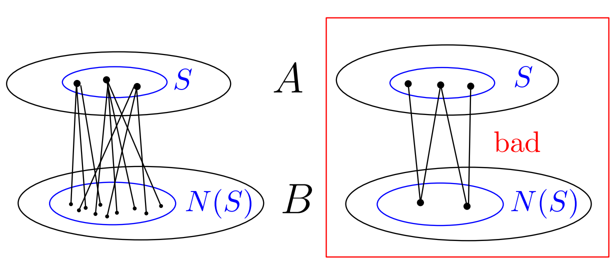
Matchings:
A *matching* is a graph that is a disjoint (vertex) union of edges.



Philip Hall (Apr. 1904 - Dec. 1982) Cambridge, UK

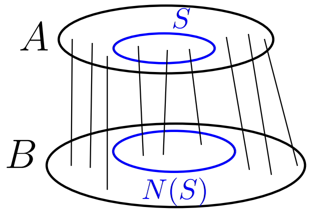
Recall that $N(S)$ for a set S of vertices is a set of neighbors of vertices in S.

Hall's matching theorem 1935: *Let G be a bipartite graph with partite sets A, B. Then G has a matching containing all vertices of A if and only if $|N(S)| \ge |S|$ for any $S \subseteq A$.*



proof:

„⇒“: obvious

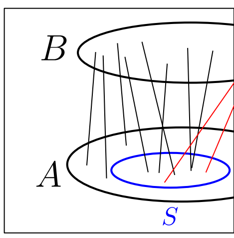


„⇐“: Assume that $|N(S)| \geq |S|$ for any $S \subseteq A$.
We shall proof that there is a matching containing all elements of A by induction on $|A|$.

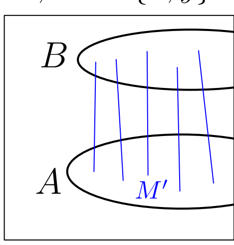
If $|A| = 1$, clear.

Assume that $|A| > 1$

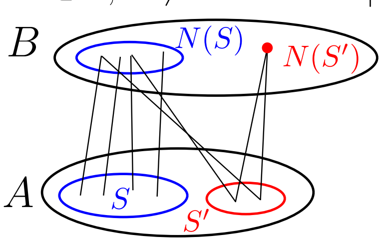
Case 1: $|N(S)| \geq |S| + 1$, for any $S \subset A,\ S \neq A$.
Let $\{x, y\} =: e \in E(G)$. Consider $G' = G - \{x, y\}$.
$|N_{G'}(S)| \geq |N_G(S)| - 1 \geq |S| + 1 - 1 = |S|$, for any $S \subseteq A - \{y\}$.



Thus, Hall's condition is true for G', and there is a matching M', containing all elements of $A - \{y\}$, by induction.
So, $M' \cup \{x, y\}$ is a matching saturating A in G.



Case 2: $\exists S \subset A,\ S \neq A$ such that $|N(S)| = |S|$.



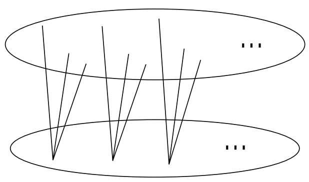
By induction, there is a matching containing all vertices of S. Let apply induction to $G[A - S, B - N(S)]$.
Assume that there is $S' \subseteq A - S$ such that $|N(S') \cap (B - N(S))| < |S'|$.
Then $|N(S' \cup S)| = |N(S) \cup (N(S') \cap (B - N(S)))| <\neq |S| + |S'|$.
A contradiction to Hall's condition applied to $S \cup S'$.
Thus for any $S' \subseteq A - S$, $|N(S') \cap (B - N(S))| \geq |S'|$, and there is a matching saturating $A - S$ in $G[A - S, B - N(S)]$. Together with a matching between S and $N(S)$, it gives a matching saturating A.

■

Corollaries of Hall's theorem:

1) *Let G be bipartite with partite sets A, B, such that $|N(S)| \geq |S| - d$ for all $S \subseteq A$, and some fixed positive integer d.*
Then G contains a matching of size at least $|A| - d$.

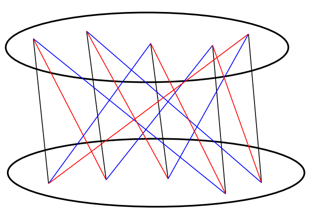
2) *A k-regular bipartite graph has a perfect matching, i.e. matching containing all vertices of a graph. Here k-regular is a graph with all degrees equal to k.*



G has partite sets A, B :

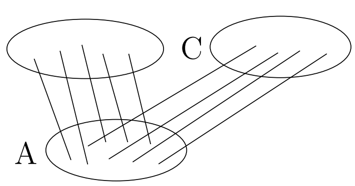
$$\begin{aligned} |E(G)| &= \#\textit{edges incident to } A = |A| \cdot k \\ &= \#\textit{edges incident to } B = |B| \cdot k \\ \Rightarrow |A| &= |B| \end{aligned}$$

3) *A k regular bipartite graph has a proper k-edge coloring.*



proof:

1) Construct G'.

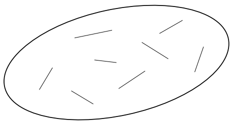


$|C| = d$, add all edges between A and C.
In G' $|N_{G'}(S)| \geq |N_G(S)| + d \geq |S| - d + d = |S|$.
By Hall's theorem, there is a matching in G' saturating A, with at most d edges not in G.

2) Let's verify Hall's condition.
Is it true that $|N(G)| \geq |S|$ for any $S \subseteq A$?
$\#$edges from S to B is $|S| \cdot k \ = \#$edges between S and $N(S) = q$
$\#$edges from $N(S)$ to A is $|N(S)| \cdot k \ \geq \#$edges between S and $N(S) = q$.
$|N(S)| \cdot \cancel{k} \geq q = |S| \cdot \cancel{k} \ \Rightarrow |N(S)| \geq |S|$.

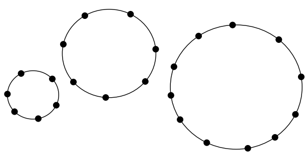
■

Non-bipartite graphs:



A *k-factor* in a graph is a *spanning* (containing each vertex) subgraph in which each vertex has degree k.

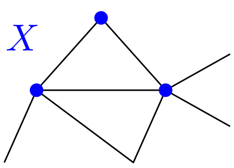
perfect matching = 1-factor



2-factor

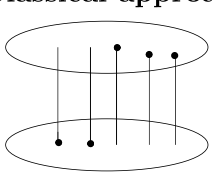
Denes König (Sep. 1884 - Oct. 1944)
Gyula König (Dec. 1849 - Apr. 1913)

Let $\nu(G)$ be the size of largest matching in G and $\tau(G)$ be the size of smallest *vertex cover*, i.e. set of vertices such that each edge is incident to some of this vertices, i.e. a set X of vertices such that $G - X$ is an empty graph.



König's theorem '31: *If G is a bipartite graph, then $\nu(G) = \tau(G)$.*

Classical approach: Given a maximal matching M and want to find a vertex cover of size $|M|$



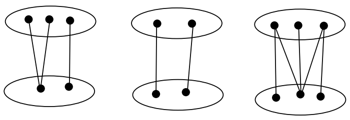
alternating path: starts with an unmatched vertex of M (alternating one point in A and one in B). Take the longest alternating path.

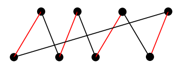
vertex cover: for any element of $\{a, b\} \in E(M)$, $a \in A, b \in B$ pick b if there is an alternating path ending in b, otherwise pick a.

proof: (by Romeo Rizzi '2000)

We want to prove that $\tau(G) \leq \nu(G)$ ($\tau(G) \geq \nu(G)$ trivial).

Assume that G is the smallest counterexample (#edges, #vertices).
Observe that G is connected, not a path, not a cycle, i.e. $\exists v : \deg(v) \geq 3$

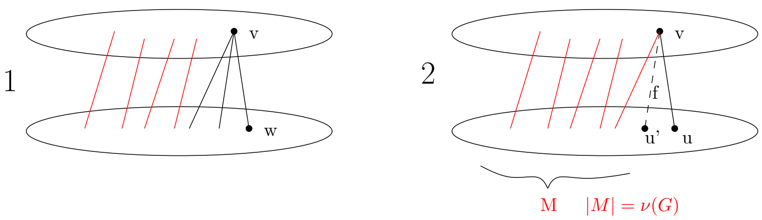




Let $v:\deg v \geq 3.$ $u \in N(v)$,

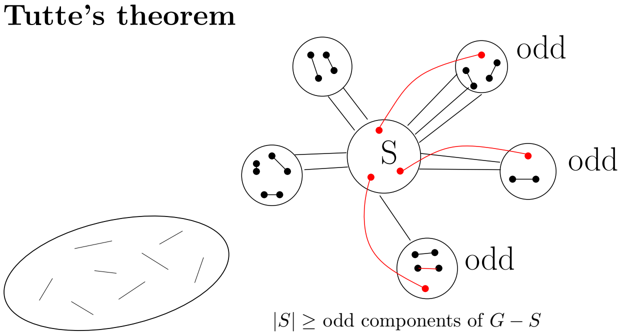
Case 1: $\nu(G\backslash u) < \nu(G)$:
Take a vertex cover X by König's theorem of $G-u$ of size $\leq \nu(G)-1$. Then $X \cup \{u\}$ is the vertex cover of G of size $\leq \nu(G)$.

Case 2: $\nu(G\backslash u) = \nu(G)$:
Then, in G there is a maximal matching, M, not containing u. There is $u' \in N(v)-\{u\}$, such that $f := \{v, u'\} \notin E(M)$.
Let W' be a cover of $G-f$ of size $\nu(G-f)=\nu(G)$. Then W' does not contain u (W' contains vertices of M only and $u \notin V(M)$). Thus W' contains v. So, W' covers f too. Thus W' covers G.



■

Tutte's theorem



For a subset S of vertices of G, let $q(S)$=#odd components of $G-S$.

> **Theorem:** *(Bill Tutte May 1917- May 2002)*
> *A graph G has a perfect matching (1-factor) if and only if $\forall S \subseteq V(G)$ $q(S) \leq |S|$.*

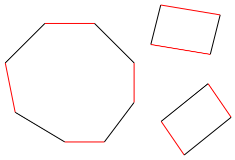
proof:

„⇒“ : trivial.

„⇐“ : Consider G, such that $\forall S \subseteq V(G)$, $q(S) \leq |S|$, and assume that G has no 1-factor. Add edge one-by-one, so the resulting graph G' is no 1-factor.

We shall show that in G' is a „bad“ set S, $q(S) > |S|$.
We shall show that S is also a bad set in G.

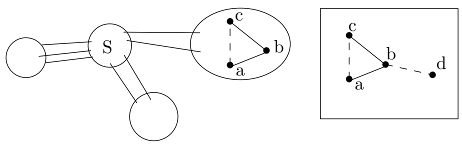
Observation: *If M_1, M_2 are perfect matchings in G, $M_1 \triangle M_2 = (M_1 \cup M_2) - (M_1 \cap M_2)$ are only cycles.*



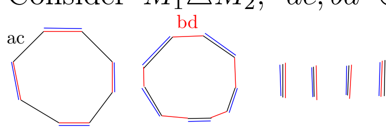
Let S be a set of vertices of degree $|V(G)|-1$. We shall show that S is bad in G'.

Claim: All components of $G'-S$ are complete.

Assume not, i.e. there is a non-complete component in $G'-S$.

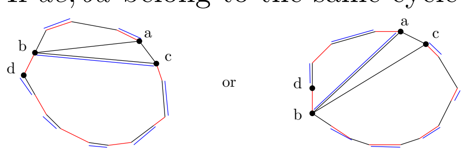


Then there is an induced path a,b,c in this component. Since $b \notin S$, $\deg b < |V(G')|-1$, there is $d \notin \{a,b,c\}$, such that $b \not\sim d$.
By maximality of G', there is a perfect matching M, in $G' \cup \{\{a,c\}\}$, there is a perfect matching M_2 in $G' \cup \{\{b,d\}\}$. Note $ac \in E(M_1), bd \in E(M_2)$. We shall create a perfect matching of G'.
Consider $M_1 \triangle M_2$, $ac, bd \in E(M_1 \triangle M_2)$. If ac, bd belong to different cycles of $M_1 \triangle M_2$:

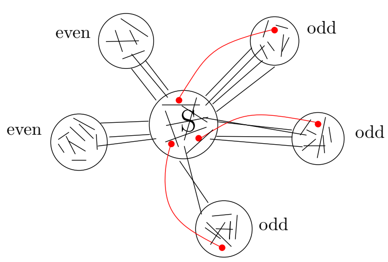


Take the edges of M_2 in a component containing ac, take edges of M_1 in a component with bd, otherwise take edges of M_1.

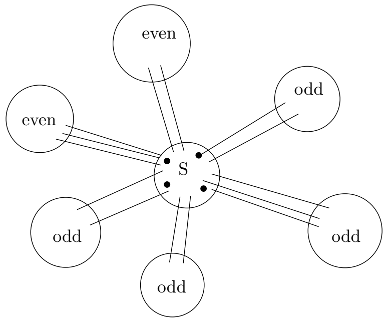
If ac, bd belong to the same cycle of $M_1 \triangle M_2$, then



A contradiction, since G' has no 1-factor, so all components of $G'-S$ are complete.□ *Claim*

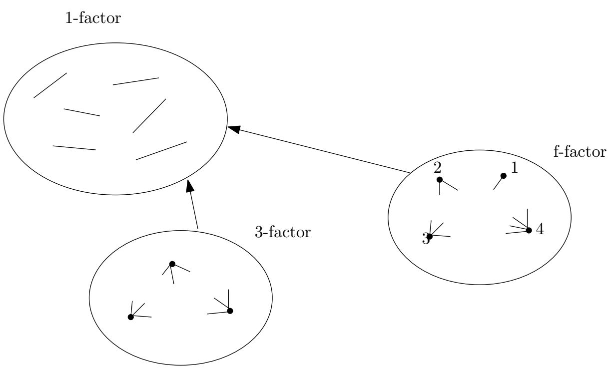


If S is not bad, i.e. $|q(S)| \leq |S|$, we can construct a perfect matching, a contradiction to the fact that G' has no perfect matching. Thus S is bad in G'.

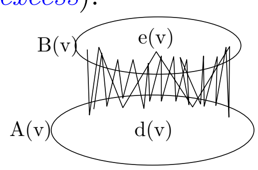


G is obtained from G' by deleting edges, so $q_G(S) \geq q_{G'}(S) > |S|$.

■



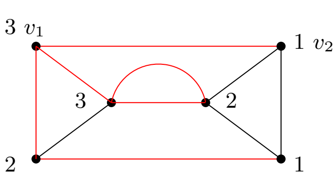
k-factor - spanning subgraph,
all degrees $= k$
f-factor: If $f : V \to \mathbb{N}$, an f-factor is a spanning subgraph H of G such that $\deg_H(v) = f(v)$.
Let *e(v)*$= \deg(v) - f(v) \geq 0$ (*excess*).

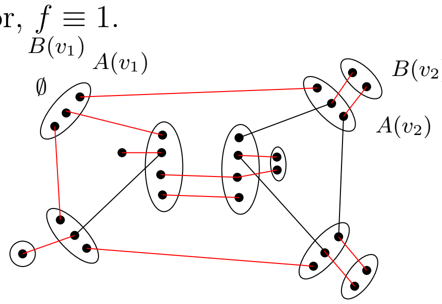
Replace each vertex of G with 

$K_{e(v),\, d(v)}$
For adjacent u and v, put an edge between $A(u)$ and $A(v)$, such that these edges form a matching.

An *f-factor*, in a graph G, for $f : V(G) \to \mathbb{N} \cup \{0\}$, such that $\forall v \in V \;\; f(v) \leq \deg(v)$, <u>is</u> a spanning subgraph H of G such that $\deg_H(v) = f(v)$.

1-factor or matching $\approx$ f-factor, $f \equiv 1$.





$f(v_1) = 3,\;\; f(v_2) = 1.$

For a graph G and a function $f : V(G) \to \mathbb{N} \cup \{0\}$, construct an auxiliary graph $T(G, f)$ by replacing each vertex v with vertex sets $A(v) \cup B(v)$, $|A(v)| = \deg(v)$, $|B(v)| = \deg(v) - f(v)$, and for adjacent vertices u, v placing an edge between $A(u)$ and $A(v)$, so that these edges are disjoint, and placing a complete bipartite graph between $A(u) \triangle B(u)$ for each vertex u.

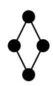
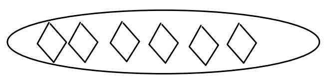
Claim: *G has an f-factor if and only if $T(G, f)$ has 1-factor.*

proof:

- Assume that M is an f-factor of G, to create a 1-factor in T, take the edges corresponding to M, and take missing edges between $A(u)$ and $B(u)$ $\forall u \in V$.
- Assume that M is a 1-factor in T, create an f-factor in G by deleting $B(u)$, $u \in V(G)$, contracting $A(u)$ into a single vertex, $u \in V(G)$.

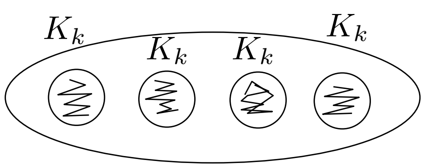
■

H-factor: Given a graph G, and a graph H, such that $|V(G)| \vdots |V(H)|$ ($\vdots$ = divisible). An H-factor of G is a spanning subgraph of G that is a vertex-disjoint union of copies of H.

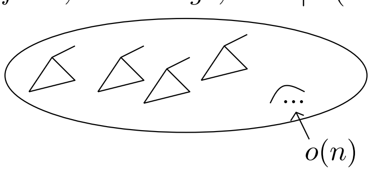
$H =$  $G =$ 

$H = K_2$ H-factor $\approx$ perfect matching.

Hajnal & Szemerédi '70: *If G satisfies $\delta(G) \geq \frac{k-1}{k} n$, $n \vdots k$, then G has a K_k-factor.*

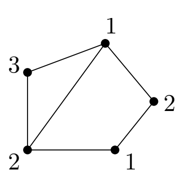


Alon-Yuster '95: *If G satisfies $\delta(G) \geq \frac{\chi(H)-1}{\chi(H)} n$. Then G contains at least $(1 - o(1)) \frac{n}{|V(H)|}$ (H is fixed, G is large, $n = |V(G)|$) copies of H vertex-disjoint.*



$\chi(H)$-chromatic number of a graph H := min #parts into which vertex sets can be partitioned, so that no two adjacent vertices are in same part.

$\chi(G)$:= min # colors assigned to $V(G)$ such that no two adjacent vertices get the same color.

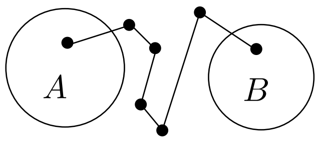
G 

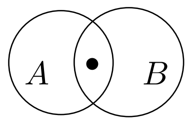
$\chi(G) = 3$

$\chi(K_k) = k$, $\chi(C_3) = 3$, $\chi(C_4) = 2$, $\chi(K_{m,n}) = 2$, $\chi(C_{2k+1}) = 3$

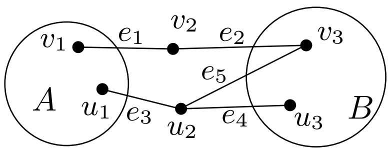
There are graphs with large $|V(G)|$ and small $\chi(G)$.

Connectivity: $A, B \subseteq V(G)$, A-B-path P is a path $v_0, v_1, \dots, v_k$ such that $V(P) \cap A = \{v_0\}$, $V(P) \cap B = \{v_k\}$.
$C \subseteq V \cup E$, we say that X separates A and B if each A-B-path contains an element of X.



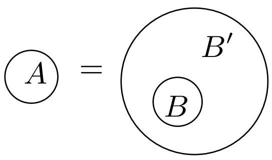


$v \in A \cap B \Rightarrow v$ is an $A - B$ path



$A - B$ sep. set : $\{v_2, u_2\}$
$\{e_1, e_5, e_4\}$
$\{e_1, u_1\}$

Note that a separating set must contain $A \cap B$.
Note $B' \supseteq B$ and X separates A and $B' \Rightarrow$ X separates A and B.

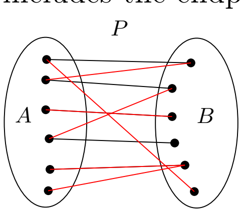


> **Menger's theorem (1927):** *(Karl Menger Jan. 1902 - Oct. 1985)*
> *Let G be a graph, $A, B \subseteq V(G)$. Min #vertices separating A and B = Max #vertex-disjoint A-B-paths.*

proof: Assume that $A \cap B = \emptyset$.
Let $k = k(G; A, B) = \min \#$vertices separating A and B, $k(G; A, B) \geq \max \#$ vertex-disjoint A-B-path (easy).

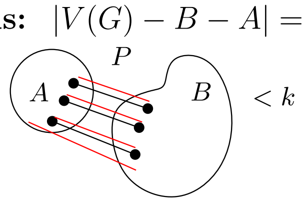
We shall prove that $\max \#$ vertex-disjoint A-B-path $\geq k(G; A, B) = k$ by stronger induction:

If P is any set of less than k disjoint A-B-paths then there is a set Q of disjoint A-B-paths that includes the endpoints of P and $|Q| = |P| + 1$.



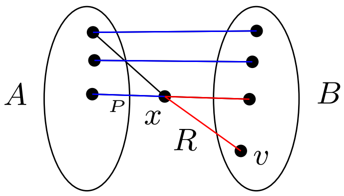
Lets prove this by induction on $|V(G) - B - A|$.

Basis: $|V(G) - B - A| = 0$.



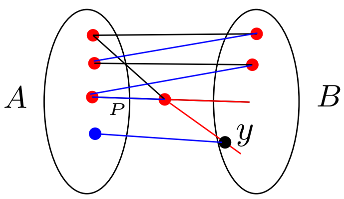
There is an edge between A and B, not adjacent to vertices of P, otherwise $|V(P) \cap A| < k$ is a vertex separating A and B.

Step: We have P, a set of less than k A-B-path, vertex disjoint.
There is an A-v-path for $v \in B \backslash (V(P))$, otherwise $V(P) \cap B$ is a set of less than k vertices separating A and B, call it R.

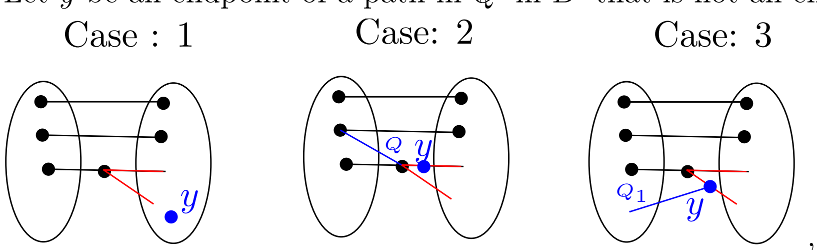


Let x be the last vertex of R that also belongs to a path in P call it P.

Let $B' = B \cup (V(xP) \cup V(xR))$.
$P' = P \backslash \{P\} \cup \{Px\}$.
note $k(G; A, B') \geq k(G; A, B)$.
By induction, there is a larger set of A-B'-paths, Q', $|Q'| \geq |P'| + 1$, Q' contains endpoints of P'.



Let y be an endpoint of a path in Q' in B' that is not an endpoint of P'.



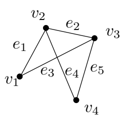
Case 1: $y \in B$:
Take $Q = Q' - \underbrace{\{Q\}}_{\text{path containing } x} \cup \{Q \cup xP\}$.

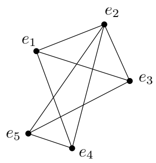
Case 2: $y \in xP$:
Take $Q = Q' - \{Q\} \cup \{Q \cup xR\} - \underbrace{\{Q_1\}}_{\text{path containing } y} \cup \{Q_1 \cup yP\}$.

Case 3: $y \in xR$:
Take $Q = Q' - \{Q\} \cup \{Q \cup xP\} - \{Q_1\} \cup \{Q_1 \cup yR\}$.

■

If $G = (V, E)$ a graph, then a *line graph $L(G)$* of G is a graph $L(G) = (E, E')$,
$E' = \{\{e, \tilde{e}\} : e, \tilde{e} \in E$ and $e, \tilde{e}$ are adjacent$\}$.

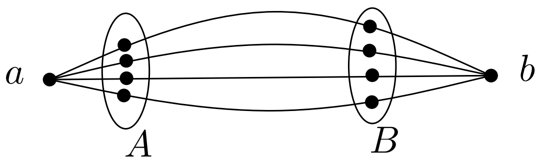




Corollary 1: *If $a, b \in V(G)$, $\{a, b\} \notin E(G)$.*

$$\min \#\textit{vertices separating } a \textit{ and } b = \max \#\textit{independent } a\text{-}b\textit{-paths}$$

(here independent means that they share only a and b)



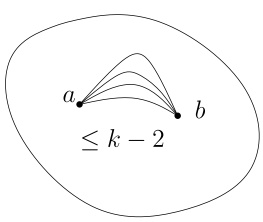
Apply Menger's theorem to $A = N(a)$ and $B = N(b)$.

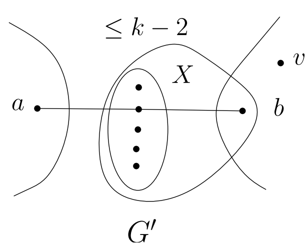
Corollary 2: *(Global version of Menger's theorem)*
Any graph G is k-connected if and only if for any two vertices a, b there are k independent paths between a and b.

outline of proof:

Suppose G contains k independent paths between any two vertices, thus we need $\geq k$ vertices to separate G. So $\kappa(G) \geq k$.

Let $\kappa(G) = k$, in particular $|(G)| > k$.
Assume that a and b are not connected by k independent paths. By corollary 1 a adjacent to b.
Let $G' = G - \{a, b\}$, then G' contains $\leq (k-2)$ independent a-b-paths.



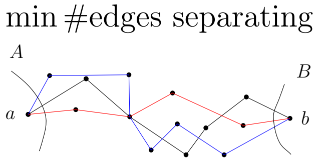


By corollary 1, we can separate a and b in G' by $\leq k-2$ vertices, X.
Since $|V(G)| > k$, there is $v \notin \{a, b\}$ and $v \notin$ component of a in $G' - X$.
Observe that v and a are separated by $X \cup \{b\}$ in G.

So, v and a are separated by $\leq k-1$ vertices, a contradiction to the fact that $\kappa(G) = k$. ■

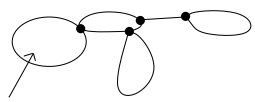
Edge-connectivity

1) $\min \#$edges separating a and b in G $=$ $\max \#$edge-disjoint a-b-paths.



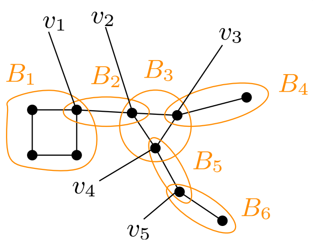
Apply Menger's theorem to $L(G)$ with $A = \{\text{edges incident to } a\}$, $B = \{\text{edges incident to } b\}$.

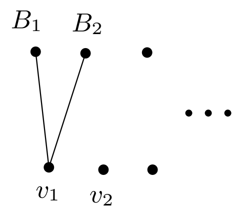
2) Global Menger's theorem (edge-connected)
A graph is k-edge-connected if and only if there are k edge-disjoint paths between any two vertices.



$\kappa(G) = 1$ blocks *block-cut-vertex tree*.

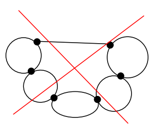
A block - either a bridge or maximal 2-connected subgraph.





$B_i \sim v_j$ if $v_j \in V(B_i)$.
Any two block intersect by at most 1 vertex.

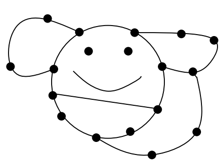
Block-cut-vertex graph is a tree.

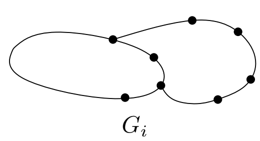


A block that is a leaf in a block-cut-vertex tree is a *block leaf*.

$\kappa(G) \geq 2 \;\Leftrightarrow\;$ G can be constructed using *ear-decomposition*

G is created using ear-decomposition if there is a sequence of graphs $G_0 \subseteq G_1 \subseteq \ldots G$, such that G_0 is a cycle, G_{i+1} is created from G_i by adding a G_i-path (*ear*) (i.e. a path with endpoints in G_i and no other vertices in G_i).

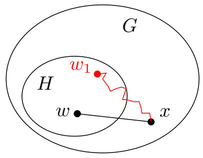




outline of proof:

„⇒“ : $\kappa(G) = 2$: We have that G has a cycle. Consider the largest subgraph H of G that is built as ear-decomposition.

Observe $H \underset{\text{ind}}{\subseteq} G$. If $u, v \in V(H)$, $v \not\sim_H u$, $v \sim_G u$, then add uv as a ear. If $H \neq G \Rightarrow \exists x \in V(G) - V(H)$, such that x is adjacent to a vertex $w \in V(H)$.
$G - w$ is connected, so in $G - w$ there is a path from x to H, call it P, call the first vertex of P in H, w_1.
So $wx \cup xPw$ is an H-ear. A contradiction to maximality of H, so $G = H$.

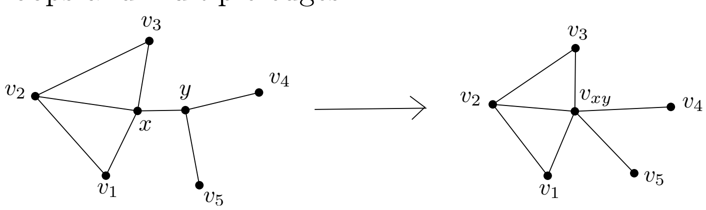


„⇐“ : Show that an ear-decomposition is 2-connected ...

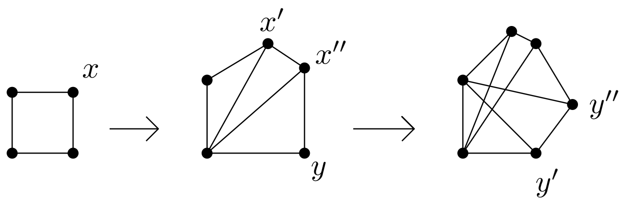
$\kappa(G) = 3: \ |V(G)| \geq 5.$

Observation: If $\kappa(G) = 3$ then there is an edge e of G such that $\kappa(G \circ e) \geq 3$.

Let $e = \{x, y\} \in E(G)$, $G \circ e$ is obtained from G by identifying x and y, removing (if necessary) loops and multiple edges.

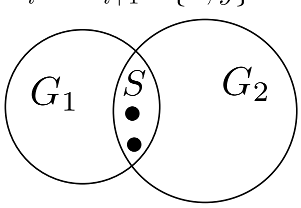


Tutte's theorem 1961: *A graph G is 3-connected if and only if it exists a sequence of graphs $G_0, G_1, \ldots, G_n$, such that $G_0 = K_4$, $G_n = G$, G_{i+1} is obtained from G_i: G_{i+1} has two vertices x, y of degree ≥ 3, $x \sim y$ and $G_i = G_{i+1} \circ \{x, y\}$.*

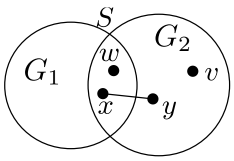


Lemma: *If G is 3-connected, then there exists an edge e such that $G \circ e$ is 3-connected. (without proof)*

proof: We want to prove hat if G_i is 3-connected, then G_{i+1} is also 3-connected. Assume not, i.e. $G_i = G_{i+1} \circ \{x, y\}$ and G_{i+1} is not 3-connected, i.e. there exists a cut-set S with $|S| \leq 2$.



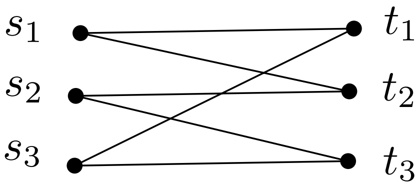
Let G_1 and G_2 be connected components of $G_{i+1} - S$. Observe, $\{x, y\} \neq S$, otherwise G_i is not 3-connected. But $\{x, y\} \cap S \neq \emptyset$, otherwise G_i is not 3-connected (disconnected by S). So, w.l.o.g. (without loss of generality) $x \in S$, $y \in V(G_2)$.

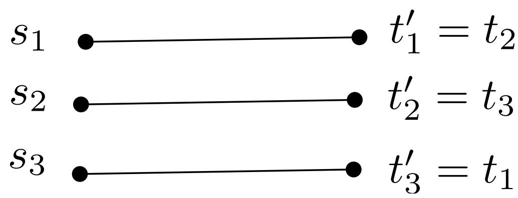


$|G_{i+1}| > |G_i|$

Assume that there exists a vertex $v \in V(G_2)\backslash\{y\}$, then in G_i $\{w, v_{xy}\}$ separates v from $V(G_1)$, a contradiction. So $V(G_2) = \{y\}$, so $\deg(y) \leq 2$, a contradiction. ■

A graph G is *k-linked*, if for any distinct vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$, there are vertex-disjoint s_i-t_i-paths, $i = 1, \ldots, k$.





G is k-linked $\Rightarrow$ G is k-connected (Menger's theorem)

G is $\underbrace{f(k)}_{22k}$-connected $\Rightarrow$ G is k-linked. (Bollobás-Thomason '96)