How to write a Proof

Recall that for a graph $G = (V, E)$ and a vertex $v \in V$ the *degree of* v is defined to be the number of edges of G that contain v. We call G *k-regular*, for a natural number k, when all vertices of G have degree exactly k.

Claim. *Let k and n be natural numbers, $n \geq k + 1$, such that if k is odd then n is even. Then there exists a k-regular graph on n vertices.*

Proof. We prove the existence of a k-regular n-vertex graph, which we will denote by $G(n, k)$, for two natural numbers n and k satisfying the assumptions in the claim by double induction, first on k and then on n.

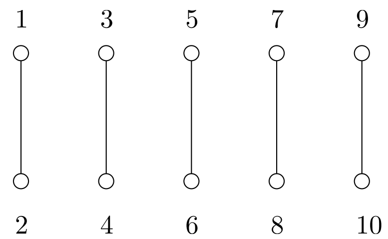
Outer Induction.

Induction base: $k = 0$.

Let $n \geq k + 1 = 1$. We define $G(n, k)$ to be the graph $([n], \emptyset)$, i.e., $G(n, k)$ has n vertices and no edges. Such a graph is often called an n-vertex *independent set*. By definition $G(n, k)$ is k-regular with $k = 0$ and consists of n vertices.

Induction base: $k = 1$.

Let $n \geq k + 1 = 2$, n even (since $k = 1$ is odd). We define the vertex set of $G(n, k)$ to be $[n]$ and the edge set to be $\{\{2i - 1, 2i\} \mid i = 1, \ldots, n/2\}$. That is, we group the n vertices into $n/2$ disjoint pairs (this is where we use the fact that n is even) and introduce an edge between any two vertices in the same pair as illustrated below. Such a graph is often called an n-vertex *matching*. By definition $G(n, k)$ is k-regular with $k = 1$ and consists of n vertices.



Induction step: $k \geq 2$.

Inner Induction.

Induction base: $n = k + 1$.

Note that if k is odd then $n = k+1$ is even. We define $G(n, k)$ to be the graph $([n], \binom{[n]}{2})$, i.e., $G(n, k)$ has n vertices and an edge between any two vertices. Such a graph is often called an n-vertex *clique*. Since every vertex $v \in [n]$ has an edge with every vertex in $[n] \setminus v$, we have that $G(n, k)$ is k-regular with $k = n - 1$. Moreover, $G(n, k)$ consists of n vertices.

Induction step: $n \geq k + 2$.

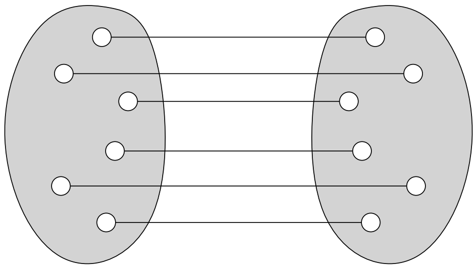
We distinguish two cases, namely $n \leq 2k$ and $n > 2k$. The idea is to use in the first case the graph $G(n, n - 1 - k)$, which exists by induction, and in the latter case to use disjoint copies of $G(\lfloor n/2 \rfloor, k)$ and $G(\lceil n/2 \rceil, k)$ when k is even and two disjoint copies of $G(n/2, k - 1)$ together with $n/2$ edges between corresponding vertices when k is odd.

Case 1. $n \leq 2k$.

Since $n \geq k + 1$ by assumption we have $k' = n - (k + 1) \geq 0$. Moreover with $k \geq 0$ we have $n \geq n - k = k' + 1$. And from $n \leq 2k$ (the assumption of Case 1) we conclude $k' = n - (k + 1) \leq 2k - (k + 1) = k - 1 < k$. Finally observe that in case n is odd we have that k is even by assumption of the claim, and that this implies that $k' = n - (k+1)$ is even. Thus the pair n and k' satisfies all requirements of the claim and additionally $k' < k$. Now by induction there exists a k'-regular n-vertex graph, which we denote by $G(n, k')$. We define $G(n, k)$ to be the complement of $G(n, k')$; the vertex set of both graph we denote by V. Clearly $G(n, k)$ has n vertices and every vertex $v \in V$ is adjacent to exactly those vertices in $G(n, k)$ to which v is non-adjacent in $G(n, k')$. Hence, because v has degree k' in $G(n, k')$, it has degree $n - 1 - k' = k$ in G. In other words $G(n, k)$ is a k-regular n-vertex graph as desired.

Case 2. $n > 2k$.
Let $n_1 = \lfloor n/2 \rfloor$ and $n_2 = \lceil n/2 \rceil$. Because $n > 2k$ (the assumption of Case 2) we have $n_2 \geq n_1 > k$. We further distinguish two subcases.

First, if k is odd, then we have that n is even by assumption of the claim, so $n_1 = n_2$. Let $k' = k - 1$, which is even since k is odd. By induction there exists an n_1-vertex k'-regular graph, which we denote by $G(n_1, k')$. We define $G(n, k)$ by taking two disjoint copies of $G(n_1, k')$ and adding an edge between each pair of corresponding vertices in the two copies as illustrated below. Because each vertex is contained in $k' = k - 1$ edges in its corresponding copy of $G(n_1, k')$ and in one additional edge with its corresponding vertex in the other copy, we see that $G(n, k)$ is $k' + 1 = k$-regular.



Now if k is even, then there exists by the inner induction an n_1-vertex k-regular graph $G(n_1, k)$ and an n_2-vertex k-regular graph $G(n_2, k)$. In this case we define $G(n, k)$ to be the disjoint union of $G(n_1, k)$ and $G(n_2, k)$. With $n_1 + n_2 = \lfloor n/2 \rfloor + \lceil n/2 \rceil = n$ it follows that $G(n, k)$ has n vertices and is k-regular, as desired.

This concludes the double induction and we have shown that whenever k and n are two natural numbers such that $n \geq k + 1$ and if k is odd then n is even, that there exists an n-vertex k-regular graph. □