Lecture Notes
Graph Theory

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Contents

Introduction

These brief notes include major definitions and theorems of the graph theory lecture held by Prof. Maria Axenovich at KIT in the winter term 2013/14. We neither prove nor motivate the results and definitions. You can look up the proofs of the theorems in the book "Graph Theory" by Reinhard Diestel [4]. A free version of the book is available at http://diestel-graph-theory.com.

Conventions:

- $G = (V, E)$ is an arbitrary (undirected, simple) graph
- $n := |V|$ is its number of vertices
- $m := |E|$ is its number of edges

Notations

notation	definition	meaning
$\binom{V}{k}$, V finite set, k integer	$\{S \subseteq V : \|S\| = k\}$	the set of all k-element subsets of V
$[n]$, n integer	$\{1, \ldots, n\}$	the set of the first n positive integers
2^S, S finite set	$\{T : T \subseteq S\}$	the power set of S, i.e. the set of all subsets of S
$S \triangle T$, S, T finite sets	$(S \cup T) \setminus (S \cap T)$	the symmetric difference of sets S and T, i.e. the set of elements that appear in exactly one of S or T
$A \sqcup B$	$A \cup B$	the union of the disjoint sets A and B

1 Preliminaries

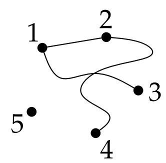
Definition. A *graph* G is an ordered pair (V, E), where V is a finite set and $E \subseteq \binom{V}{2}$ is a set of pairs of elements in V. graph, G

- The set V is called the set of *vertices* and E is called the set of *edges* of G. vertex, edge
- The edge $e = \{u, v\} \in \binom{V}{2}$ is also denoted by $e = uv$.
- If $e = uv \in E$ is an edge of G, then u is called *adjacent* to v and u is called *incident* to e. adjacent, incident

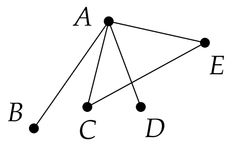
We can visualize graphs $G = (V, E)$ using pictures. For each vertex $v \in V$ we draw a point (or small disc) in the plane. And for each edge $uv \in E$ we draw a continuous curve starting and ending in the point/disc for u and v, respectively.

Several examples of graphs and their corresponding picture follow:

$V = [5]$, $E = \{12, 13, 24\}$



$V = \{A, B, C, D, E\}$,
$E = \{AB, AC, AD, AE, CE\}$



Definition (Graph variants)**.**

- A *directed graph* is $G = (V, A)$ where V is a finite set and $E \subseteq V^2$. The edges of a directed graph are also called *arcs*. directed graph, arc
- A *multigraph* is $G = (V, E)$ where V is a finite set and E is a multiset of elements from $\binom{V}{1} \cup \binom{V}{2}$, i.e. we also allow loops and multiedges. multigraph
- A *hypergraph* is $H = (X, E)$ where X is a finite set and $E \subseteq 2^X \setminus \{\emptyset\}$. hypergraph

Definition. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ we say that G_1 and G_2 are *isomorphic*, denoted by $G_1 \simeq G_2$, if there exists a bijection $\phi : V_1 \to V_2$ with $xy \in E_1$ if and only if $\phi(x)\phi(y) \in E_2$. isomorphic, $\simeq$

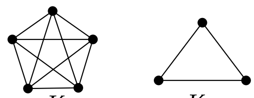
Loosely speaking, G_1 and G_2 are isomorphic if they are the same up to renaming of vertices. Hence, we may write $G_1 = G_2$ instead of $G_1 \simeq G_2$ whenever vertices are indistinguishable. $=$

Important graphs and graph classes

Definition. For all natural numbers n we define:

- the *complete graph* on n vertices as $K_n = \big([n], \binom{[n]}{2}\big)$. Complete graphs are also called *cliques*.

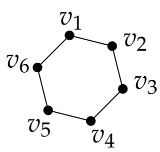
complete graph
clique



K_5 K_3

- for $n \geq 3$, the *cycle* on n vertices as $C_n = \big([n], \big\{\{i, i+1\} : i = 1, \ldots, n-1\big\} \cup \big\{n, 1\big\}\big)$. The *length of a cycle* is its number of edges.

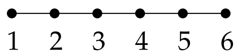
cycle



$C_6 = v_1v_2v_3v_4v_5v_6v_1$

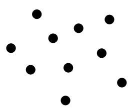
- the *path* on n vertices as $P_n = \big([n], \big\{\{i, i+1\} : i = 1, \ldots, n-1\big\}\big)$. The vertices 1 and n are called the *endpoints* or *ends* of the path. The *length of a path* is its number of edges.

path



- the *empty graph* on n vertices as $E_n = \big([n], \varnothing\big)$. Empty graphs are also called *independent sets*.

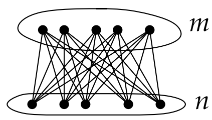
empty graph
independent set



E_{10}

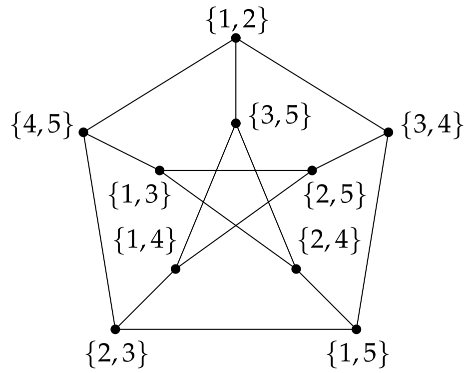
- for $m \geq 1$, the *complete bipartite graph* on n and m vertices as $K_{m,n} = (A \cup B, \{xy : x \in A, y \in B\})$, where $|A| = m$ and $|B| = n$, $A \cap B = \varnothing$.

complete bipartite graph



$K_{m,n}$

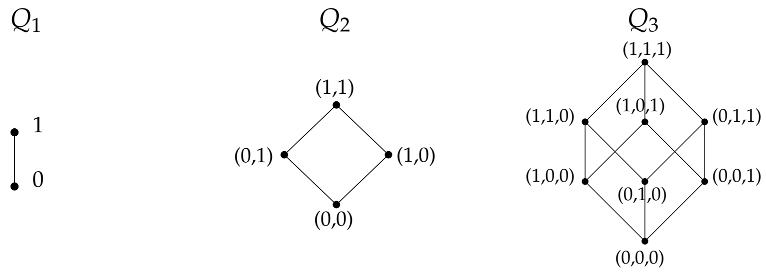
- the *Petersen graph* as $\left(\binom{[5]}{2}, \{\{S,T\} : S,T \in \binom{[5]}{2}, S \cap T = \emptyset\}\right)$. Petersen graph

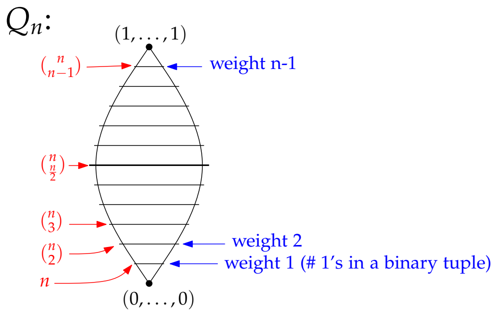


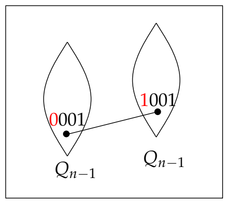
- for a natural number k, $k \leq n$, the *Kneser graph* as Kneser graph

$$K(n,k) = \left(\binom{[n]}{k}, \left\{\{S,T\} : S,T \in \binom{[n]}{k}, S \cap T = \emptyset\right\}\right).$$

  Note that $K(5,2)$ is the Petersen graph.
- the *n-dimensional hypercube* as $Q_n = \left(2^{[n]}, \{\{S,T\} : S,T \in 2^{[n]}, |S \triangle T| = 1\}\right)$. hypercube







Basic graph parameters and degrees

Definition. Let $G = (V, E)$ be a graph. We define the following parameters of G.

- The *order of* G, denoted by $|G|$, is the number of vertices of G, i.e. $|G| = |V|$.

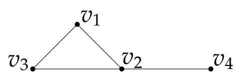
  order, $|G|$
- The *size of* G, denoted by $\|G\|$, is the number of edges of G, i.e. $\|G\| = |E|$. Note that if the order of G is n, then the size of G is between 0 and $\binom{n}{2}$.

  size, $\|G\|$
- Let $S \subseteq V$. The *neighbours of* S, denoted by $N(S)$, are the vertices in V that have an adjacent vertex in S. Instead of $N(\{v\})$ for $v \in V$ we usually write $N(v)$.

  neighbours, $N(v)$
- If the vertices of G are labeled $v_1, \ldots, v_n$, then there is an $n \times n$ matrix A with entries in $\{0, 1\}$, which is called the *adjacency matrix* and is defined as follows:

  adjacency matrix

$$v_i v_j \in E \qquad \Leftrightarrow \qquad A[i, j] = 1$$

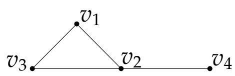


$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

A graph and its adjacency matrix.

- The *degree* of a vertex v of G, denoted by $d(v)$ or $\deg(v)$, is the number of edges incident to v.

  degree, $d(v)$



$$\deg(v_1) = 2, \deg(v_2) = 3, \deg(v_3) = 2, \deg(v_4) = 1$$

- A vertex of degree 1 in G is called a *leaf*, and a vertex of degree 0 in G is called an *isolated vertex*.

  leaf
  isolated vertex
- The *degree sequence* of G is the multiset of degrees of vertices of G, e.g. in the example above the degree sequence is $\{1, 2, 2, 3\}$.

  degree sequence
- The *minimum degree of* G, denoted by $\delta(G)$, is the smallest vertex degree in G (it is 1 in the example).

  minimum degree, $\delta(G)$
- The *maximum degree of* G, denoted by $\Delta(G)$, is the highest vertex degree in G (it is 3 in the example).

  maximum degree, $\Delta(G)$
- G is called *k-regular* for a natural number k if all vertices have degree k. Graphs that are 3-regular are also called *cubic*.

  regular
  cubic

- The *average degree of* G is defined as $d(G) = \left(\sum_{v \in V} \deg(v)\right)/|V|$. Clearly, we have $\delta(G) \leq d(G) \leq \Delta(G)$ with equality if and only if G is k-regular for some k.

average degree, $d(G)$

Lemma 1 (Handshake lemma, 1.2.1)**.** For every graph $G = (V, E)$ we have

$$2|E| = \sum_{v \in V} d(v).$$

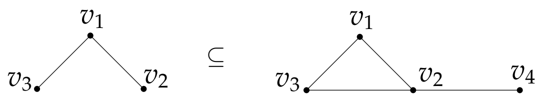
Corollary 2. In particular, the sum of all vertex degrees is even and therefore the number of vertices with odd degree is even.

Subgraphs

Definition.

- A graph $H = (V', E')$ is a *subgraph of* G, denoted by $H \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E$. In particular, $G_1 = G_2$ if and only if $G_1 \subseteq G_2$ and $G_2 \subseteq G_1$.

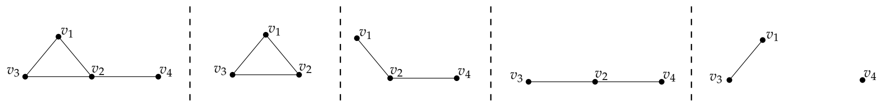
subgraph, $\subseteq$



- A subgraph H of G is called an *induced subgraph* of G if for every two vertices $u, v \in V(H)$ we have $uv \in E(H) \Leftrightarrow uv \in E(G)$. In the example above H is not an induced subgraph of G. Every induced subgraph of G can be obtained by deleting vertices (and all incident edges) from G.

  Examples:

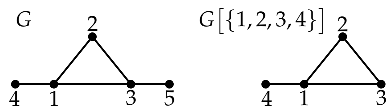
induced subgraph



- Every induced subgraph of G is uniquely defined by its vertex set. We write $G[X]$ for the induced subgraph of G on vertex set X, i.e. $G[X] = \left(X, \{xy : x, y \in X, \ xy \in E(G)\}\right)$. Then $G[X]$ is called the *subgraph of* G *induced by the vertex set* $X \subseteq V(G)$.

  Example:

$G[X]$



- A subgraph $H = (V', E')$ of $G = (V, E)$ is called a *spanning subgraph* of G if $V' = V$.

spanning subgraph

- G is called *bipartite* if there exists natural numbers m, n such that $G \subseteq K_{m,n}$.

bipartite

- A *cycle (path, clique, independent set) in* G is a subgraph H of G that is isomorphic to a cycle (path, clique, independent set).
- A *walk* (of length k) is a non-empty alternating sequence $v_0 e_0 v_1 e_1 \cdots e_{k-1} v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. If $v_0 = v_k$, the walk is *closed*. — walk; closed walk
- Let $A, B \subseteq V$, $A \cap B = \emptyset$. A path P in G is called an *A-B-path* if $P = v_1 \ldots v_k$, $V(P) \cap A = \{v_1\}$ and $V(P) \cap B = \{v_k\}$. When $A = \{a\}$ and $B = \{b\}$, we simply call P an a-b-path. If G contains an a-b-path we say that the vertices a and b are *linked by a path*. — A-B-path
- Two paths P, P' in G are called *independent* if every vertex in both P and P' is an endpoint of P and P'. — independent paths
- G is called *connected* if any two vertices are linked by a path. — connected
- A maximal connected subgraph of G is called a *connected component* of G. — component
- G is called *acyclic* if G does not have any cycle. Acyclic graphs are also called *forests*. — acyclic; forest
- G is called a *tree* if G is connected and acyclic. — tree

Proposition 3. If a graph G has minimum degree $\delta(G) \geq 2$, then G has a path of length $\delta(G)$ and a cycle with at least $\delta(G) + 1$ vertices.

Proposition 4. If a graph has an u-v-walk, then it has an u-v-path.

Proposition 5. If a graph has a closed walk of odd length, then it contains an odd cycle.

Proposition 6. If a graph has a closed walk with a non-repeated edge (at least one edge appears in the walk with multiplicity one), then the graph contains a cycle.

Proposition 7. A graph is bipartite if and only if it has no cycles of odd length.

Definition. An *Eulerian tour of* G is a closed walk containing all edges of G, each with multiplicity one. — Eulerian tour

Theorem 8 (Eulerian tour condition, 1.8.1)**.** A connected graph has an Eulerian tour if and only if every vertex has even degree.

Lemma 9. Every tree on at least two vertices has a leaf.

Lemma 10. A tree of order $n \geq 1$ has exactly $n - 1$ edges.

Lemma 11. Every connected graph contains a spanning tree.

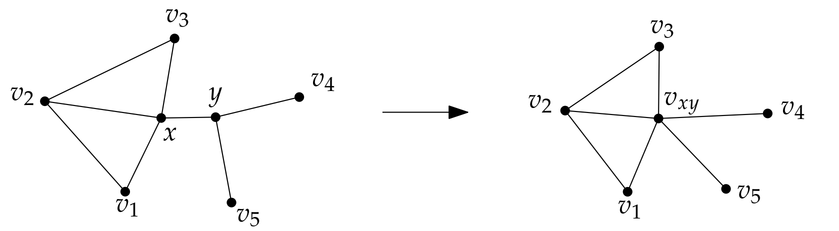
Lemma 12. A connected graph on $n \geq 1$ vertices and $n - 1$ edges is a tree.

Lemma 13. The vertices of every connected graph can be ordered $(v_1, \ldots, v_n)$ so that for every $i \in \{1, \ldots, n\}$ the graph $G\big[\{v_1, \ldots, v_i\}\big]$ is connected.

Operations on graphs

Definition. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs, $U \subseteq V$ be a subset of vertices of G and $F \subseteq \binom{V}{2}$ be a subset of pairs of vertices of G. Then we define

- $G \cup G' := (V \cup V', E \cup E')$ and $G \cap G' := (V \cap V', E \cap E')$. Note that $G, G' \subseteq G \cup G'$ and $G \cap G' \subseteq G, G'$. Sometimes, we also write $G + G'$ for $G \cup G'$. — $G \cup G', G \cap G'$
- $G - U := G[V \setminus U]$, $G - F := (V, E \setminus F)$ and $G + F := (V, E \cup F)$. If $U = \{u\}$ or $F = \{e\}$ then we simply write $G - u$, $G - e$ and $G + e$ for $G - U$, $G - F$ and $G + F$, respectively. — $G - U$, $G - F$, $G + F$
- For an edge $e = xy$ in G we define $G \circ e$ as the graph obtained from G by identifying x and y and removing (if necessary) loops and multiple edges. — $G \circ e$



- The *complement of* G, denoted by $\overline{G}$ or G^C, is defined as the graph $(V, \binom{V}{2} \setminus E)$. In particular, $G + \overline{G}$ is a complete graph, and $\overline{G} = (G + \overline{G}) - E$. — complement, $\overline{G}$

More graph parameters

Definition. Let $G = (V, E)$ be any graph.

- The *girth of* G, denoted by $g(G)$, is the length of a shortest cycle in G. If G is acyclic, its girth is said to be ∞. — girth, $g(G)$
- The *circumference* of G is the length of a longest cycle in G. If G is acyclic, its circumference is said to be 0. — circumference
- G is called *Hamiltonian* if G has a spanning cycle, i.e. there is a cycle in G that contains every vertex of G. In other words, G is Hamiltonian if and only if its circumference is $|V|$. — Hamiltonian
- G is called *traceable* if G has a spanning path, i.e. there is a path in G that contains every vertex of G. — traceable
- For two vertices u and v in G, the *distance between u and v*, denoted by $d(u, v)$, is the length of a shortest u-v-path in G. If no such path exists, $d(u, v)$ is said to be ∞. — distance, $d(u, v)$

- The *diameter of* G of G, denoted by $\mathrm{diam}(G)$, is the maximum distance among all pairs of vertices in G, i.e. diameter, $\mathrm{diam}(G)$

$$\mathrm{diam}(G) = \max_{u,v \in V} d(u,v).$$

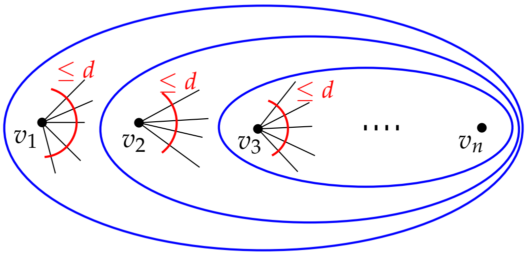
- The *radius of* G of G, denoted by $\mathrm{rad}(G)$, is defined as radius, $\mathrm{rad}(G)$

$$\mathrm{rad}(G) = \min_{u \in V} \max_{v \in V} d(u,v).$$

- If there is a vertex ordering $v_1, \dots, v_n$ of G for a $d \in \mathbb{N}$ such that

$$|N(v_i) \cap \{v_{i+1}, \dots, v_n\}| \leq d,$$

for all $i \in [n]$ then G is called *d-degenerate*. The minimum d for which G is d-degenerate is called the *degeneracy* of G. d-degenerate degeneracy



We remark that the 1-degenerate graphs are precisely the forests.

Proposition. For any graph $G = (V, E)$ each of the following is equivalent.

(i) G is a tree, that is, G is connected and acyclic.

(ii) G is minimally connected.

(iii) G is maximally acyclic.

(iv) G is connected and 1-degenerate.

(v) G is connected and $|E| = |V| - 1$.

(vi) G is acyclic and $|E| = |V| - 1$.

(vii) G is connected and every non-trivial subgraph of G has a vertex of degree at most 1.

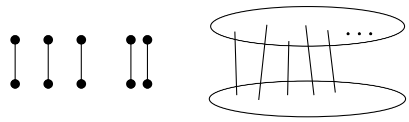
(viii) Any two vertices are joined by a unique path in G.

2 Matchings

Definition.

- A *matching* (*independent edge set*) is a vertex-disjoint union of edges.

  matching



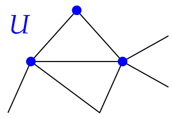
- A *matching in G* is a subgraph of G isomorphic to a matching. We denote the size of the largest matching in G by $\nu(G)$.

  $\nu(G)$

- A *vertex cover in G* is a set of vertices $U \subseteq V$ such that each edge in E is incident to at least one vertex in U. We denote the size of the smallest vertex cover in G by $\tau(G)$.

  vertex cover

  $\tau(G)$



- A *k-factor of G* is a k-regular spanning subgraph of G.

  k-factor

- A *1-factor of G* is also called a *perfect matching* since it is a matching of largest possible size in a graph of order $|V|$. Clearly, G can only contain a perfect matching if $|V|$ is even.

  perfect matching

- A *k-edge colouring* is an assignment $c' \colon E \to [k]$ of edges to colours in $[k]$ such that no two edges incident to the same vertex receive the same colour. The *chromatic index of G* is the minimal k such that G has a k-edge colouring. It is denoted by $\chi'(G)$.

  edge colouring

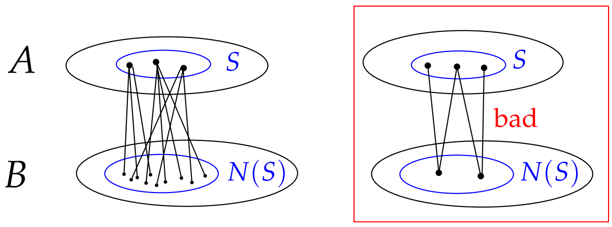
  chromatic index, $\chi'(G)$

- A *k-vertex colouring* is an assignment $c \colon V \to [k]$ of vertices to colours in $[k]$ such that no two adjacent vertices receive the same colour. The *chromatic number of G* is the minimal k such that G has a k-vertex colouring. It is denoted by $\chi(G)$.

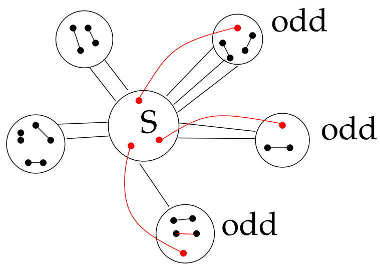
  vertex colouring

  chromatic number, $\chi(G)$

Theorem 14 (Hall's marriage theorem 1935, 2.1.2)**.** Let G be bipartite with partite sets A and B. Then G has a matching containing all vertices of A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.



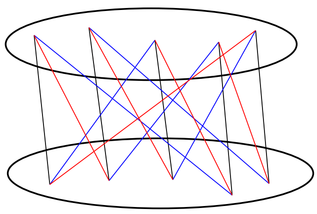
Theorem 15 (Tutte's theorem 1947, 2.2.1)**.** For $S \subseteq V$ define $q(S)$ to be the number of odd components of $G - S$. A graph G has a perfect matching if and only if $q(S) \leq |S|$ for all $S \subseteq V$.



$|S| \geq$ odd components of $G - S$

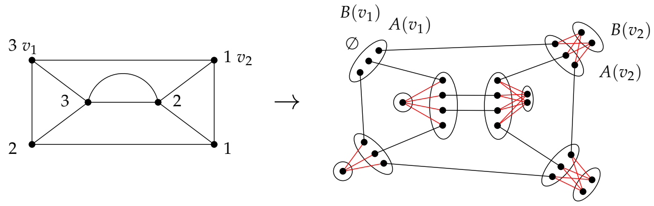
Corollary 16.

- Let G be bipartite with partite sets A and B such that $|N(S)| \geq |S| - d$ for all $S \subseteq A$, and a fixed positive integer d. Then G contains a matching of size at least $|A| - d$.
- A k-regular bipartite graph has a perfect matching.
- A k-regular bipartite graph has a proper k-edge coloring.

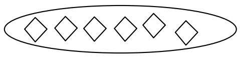


Definition.

- For all functions $f\colon V \to \mathbb{N}$ a *f-factor of G* is a spanning subgraph H of G such that $\deg_H(v) = f(v)$ for all $v \in V$. f-factor
- Let $f\colon V \to \mathbb{N}$ be a function with $f(v) \le \deg(v)$ for all $v \in V$. We can construct the auxiliary graph $T(G, f)$ by replacing each vertex v with vertex sets $A(v) \cup B(v)$ such that $|A(v)| = \deg(v)$ and $|B(v)| = \deg(v) - f(v)$. For adjacent vertices u and v we place an edge between $A(u)$ and $A(v)$ such that the edges between the A-sets are independent. We also insert a complete bipartite graph between $A(v)$ and $B(v)$ for each vertex v. $T(G, f)$



- Let H be a graph. A *H-factor of G* is a spanning subgraph of G that is a vertex-disjoint union of copies of H. H-factor

$H =$  $G =$ 

Lemma 17. Let $f\colon V \to \mathbb{N}$ be a function with $f(v) \le \deg(v)$ for all $v \in V$. G has a f-factor if and only if $T(G, f)$ has a 1-factor.

Theorem 18 (König's theorem 1931, 2.1.1)**.** Let G be bipartite. Then $\nu(G) = \tau(G)$, i.e. the size of a largest matching is the same as the size of a smallest vertex cover.

Theorem (Hajnal and Szemerédi 1970)**.** If G satisfies $\delta(G) \ge (1 - 1/k)n$, where k is a divisor of n, then G has a K_k-factor.

Theorem (Alon and Yuster 1995)**.** Let H be a graph. If G satisfies

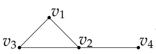
$$\delta(G) \ge \left(1 - \frac{1}{\chi(H)}\right) n,$$

then G contains at least $\big(1 - o(1)\big) \cdot n/|V(H)|$ vertex-disjoint copies of H.

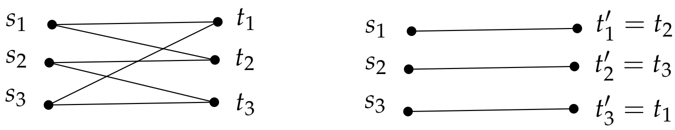
3 Connectivity

Definition.

- For a natural number $k \geq 1$, a graph G is called *k-connected* if $|V(G)| \geq k+1$ and for any $(k-1)$-set U of vertices in G the graph $G - U$ is connected. — k-connected
- The maximum k for which G is k-connected is called the *connectivity of* G, denoted by $\kappa(G)$. — connectivity, $\kappa(G)$

  $\kappa($  $) = 1,\ \kappa(C_n) = 2,\ \kappa(K_{n,m}) = \min\{m, n\}.$

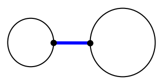
- For a natural number $k \geq 1$, a graph G is called *k-linked* if for any $2k$ distinct vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ there are vertex-disjoint s_i-t_i-paths, $i = 1, \ldots, k$. — k-linked



- If G is connected, but $G - X$ is disconnected for a subset X of vertices of G, then X is called a *cut set* of G. If a cut set consists of a single vertex v, then v is called a *cut vertex* of G. — cut set, cut vertex
- For a natural number $\ell \geq 1$, a graph G is called *ℓ-edge-connected* if $E(G) \neq \emptyset$ and for any $(\ell-1)$-set F of edges in G the graph $G - F$ is connected. — ℓ-edge-connected
- The *edge-connectivity of* G is the maximum ℓ such that G is ℓ-edge-connected. It is denoted by $\kappa'(G)$ or $\lambda(G)$. — edge-connectivity, $\kappa'(G)$

$$G \text{ non-trivial tree} \Rightarrow \lambda(G) = 1,\ G \text{ cycle} \Rightarrow \lambda(G) = 2.$$

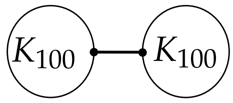
- If G is connected, but $G - e$ is disconnected for an edge e of G, then e is called a *cut edge or bridge* of G. — cut edge, bridge



Clearly, for every $k, \ell \geq 2$ if a graph is k-connected, k-linked or ℓ-edge-connected, then it is also $(k-1)$-connected, $(k-1)$-linked or $(\ell-1)$-edge-connected, respectively. Moreover, for a non-trivial graph is it equivalent to be 1-connected, 1-linked, 1-edge-connected, or connected.

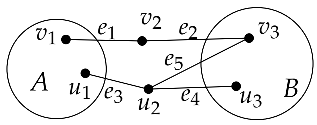
Lemma 19. For any connected, non-trivial graph G we have

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$



A graph G with $\kappa(G), \lambda(G) \ll \delta(G)$.

Definition. For a subset X of vertices and edges of G and two vertex sets A, B we say that X *separates A and B* if each A-B-path contains an element of X. separate



Some sets separating A and B: $\{e_1, e_4, e_5\}$, $\{e_1, u_2\}$, $\{u_1, u_3, v_3\}$

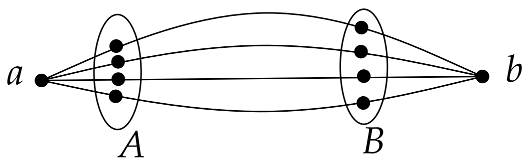
Note that a separating set of vertices must contain $A \cap B$.

Theorem 20 (Menger's theorem 1927, 3.3.1)**.** For any graph G and any vertex set $A, B \subseteq V(G)$ we have

$$\min \#\text{vertices separating } A \text{ and } B = \max \#\text{independent } A\text{-}B\text{-paths}.$$

Corollary 21. If a, b are vertices of G, $\{a, b\} \notin E(G)$, then

$$\min \#\text{vertices separating } a \text{ and } b \;=\; \max \#\text{independent } a\text{-}b\text{-paths}$$

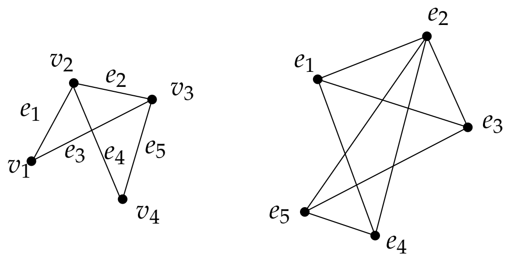


Theorem 22 (Global version of Menger's theorem, 3.3.6)**.** A graph G is k-connected if and only if for any two vertices a, b in G there exist k independent a-b-paths.

Note that Menger's Theorem implies that if G is k-linked, then G is k-connected. Moreover, Bollobás and Thomason proved in 1996 that if G is $22k$-connected, then G is k-linked.

Definition. The *line graph* $L(G)$ of G is the graph $L(G) = (E, E')$, where line graph $L(G)$

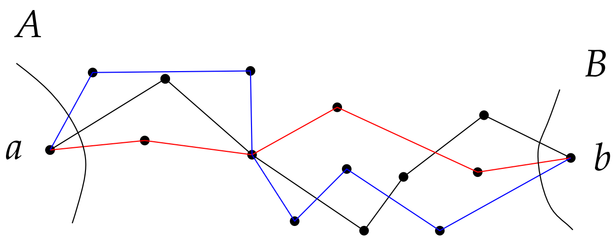
$$E' = \left\{ \{e_1, e_2\} \in \binom{E}{2} : e_1 \text{ incident to } e_2 \text{ in } G \right\}.$$



A graph and its line graph.

Corollary. If a, b are vertices of G, then

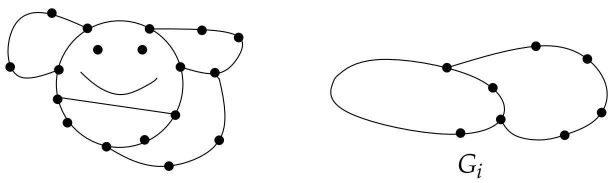
$$\min \#\text{edges separating } a \text{ and } b \;=\; \max \#\text{edge-disjoint } a\text{-}b\text{-paths}$$



Moreover, a graph is k-edge-connected if and only if there are k edge-disjoint paths between any two vertices.

Definition. An *ear-decomposition* of a graph G is a sequence $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_k$ of graphs, such that ear-decomposition

- G_1 is a cycle
- for each $i = 2, \ldots, k$ the graph G_i is given by $G_{i-1} + P_i$, where P_i, called an *ear*, is a G_i-path ear
- $G_k = G$

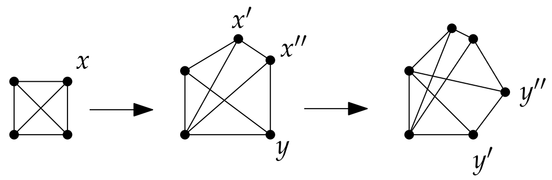


Theorem 23. A graph is 2-connected if and only if it has an ear-decomposition.

Lemma. If G is 3-connected, then there exists an edge e of G such that $G \circ e$ is also 3-connected.

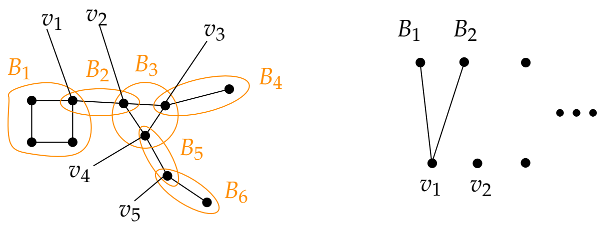
Theorem 24. A graph G is 3-connected if and only if there exists a sequence of graphs $G_0, G_1, \ldots, G_n$, such that

- $G_0 = K_4$
- for each $i = 2, \ldots, k$ the graph G_i has two adjacent vertices x', x'' of degree at least 3, so that $G_i = G_{i+1} \circ x'x''$
- $G_n = G$



Definition. The *block-cut-vertex graph or block graph* of G is a bipartite graph H whose partite sets are the *blocks* of the G, the bridges and maximal 2-connected subgraphs of G, and the cut vertices of G. There is an edge between a block and a cut vertex if and only if the block contains the cut vertex.

block-cut-vertex graph



The leaves of this graph are called *block leaves*.

block leaf

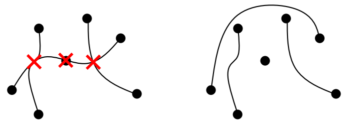
Theorem 25. The block-cut-vertex graph of a connected graph is a tree.

4 Planar graphs

This section deals with graph drawings. We restrict ourselves to graph drawings in the plane $\mathbb{R}^2$. It is also feasible to consider graph drawings in other topological spaces, such as the torus $\mathbb{R}^2/\mathbb{Z}^2$.

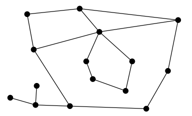
Definition.

- The *line segment* between $p \in \mathbb{R}^2$ and $q \in \mathbb{R}^2$ is the set $\{p + \lambda(q - p) : 0 \leq \lambda \leq 1\}$. line segment
- A *homeomorphism* is a continuous function that has a continuous inverse function. homeomorphism
- Two sets $A \in \mathbb{R}^2$ and $B \in \mathbb{R}^2$ are said to be *homeomorphic* if there is a homeomorphism $f\colon A \to B$. homeomorphic
- An *arc* in $\mathbb{R}^2$ is a homeomorphic image of a line segment. arc
- A set $A \subseteq \mathbb{R}^2$ is *path-connected* if there is a continuous path between any two points in A. path-connected
- Let $A \subseteq \mathbb{R}^2$ be a set of points. A *region of* A is a maximal path-connected subset R of A. Its boundary δR is also called its *frontier*. region frontier
- A *polygon* is a union of finitely many line segments that is homeomorphic to the circle $S^1 := \{x \in \mathbb{R}^2 : \|x\| = 1\}$ polygon
- A *plane graph* $G = V \cup E$ is a subset of $\mathbb{R}^2$ consisting of a set of vertices V and a set of edges E such that plane graph
  1. V is a finite set of points in $\mathbb{R}^2$,
  2. E is a finite set of arcs between vertices,
  3. different edges have different endpoints,
  4. the interior of an edge contains no vertex and no points of another edge.

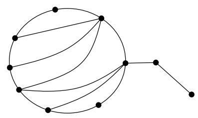
  

  The regions of $\mathbb{R}^2 \setminus G$, denoted by $F(G)$, are called the *faces of* G. Faces with three vertices are called *triangles*. If all of the faces in $F(G)$ are triangles, then G is called a *plane triangulation*. faces, $F(G)$ triangle triangulation
- A plane graph is *maximally plane* if one cannot add edges and still obtain a plane graph. maximally plane

- $G = (V, E)$ is *planar* if it has a plane embedding, i.e. if there is a plane graph $G' = V' \cup E'$ and a bijection $f\colon V \to V'$ such that $uv \in E$ if and only if G' has an edge between $f(u)$ and $f(v)$. planar graph



- $G = (V, E)$ is *outerplanar* if it has a plane embedding such that the boundary of the outer face contains all of the vertices V. outerplanar graph

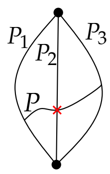


Theorem (Fáry's theorem)**.** Every planar graph has a plane embedding with straight line segments as edges.

Lemma (Jordan curve theorem)**.** Let $P \subseteq \mathbb{R}^2$ be a polygon. Then $\mathbb{R}^2 \setminus P$ has exactly two regions. One of the regions is unbounded, the other is bounded. Each of the two regions has P as frontier.

Lemma. Let P_1, P_2 and P_3 be internally disjoint arcs that have the same endpoints. Then

1. $\mathbb{R}^2 \setminus (P_1 \cup P_2 \cup P_3)$ has exactly three regions with boundaries $P_1 \cup P_2$, $P_1 \cup P_3$ and $P_2 \cup P_3$, respectively.
2. Let P be an arc from the interior of P_1 to the interior of P_3 whose interior lies in the region of $\mathbb{R}^2 \setminus (P_1 \cup P_3)$ containing the interior of P_2. Then P contains a points of P_2.



Lemma. Let G be a plane graph and e an edge of G. Then

- A frontier X of a face of G either contains e or is disjoint to the interior of e.
- If e is on a cycle in G, then e is on the frontier of exactly two faces.
- If e is on no cycle in G, then e is on the frontier of exactly one face.

Lemma 26. A plane graph is maximally plane if and only if each of its faces is a triangle.

Theorem 27 (Euler's formula, 4.2.9)**.** Let G be a connected plane graph with v vertices, e edges and f faces. Then

$$v - e + f = 2.$$

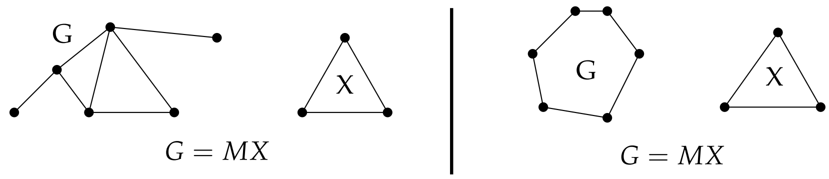
Corollary. Let $G = V \cup E$ be a plane graph. Then

- $|E| \leq 3|V| - 6$ with equality exactly if G is a plane triangulation.
- $|E| \leq 2|V| - 4$ if no face in $F(G)$ is a triangle.

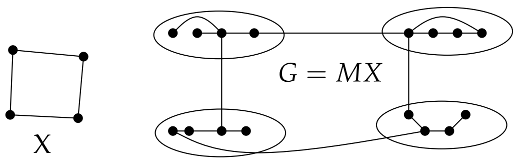
Lemma (Pick's formula)**.** Let P be a polygon with corners on the grid $\mathbb{Z}^2$, A its area, I the number of grid points inside of P and B be the number of grid points on the boundary of P. Then $A = I + B/2 - 1$.

Definition. Let G and X be two graphs.

- We say that X *is a minor of* G, denoted by $G = MX$, if X can be obtained from G by successive vertex deletions, edge deletions and edge contractions. minor, $G = MX$



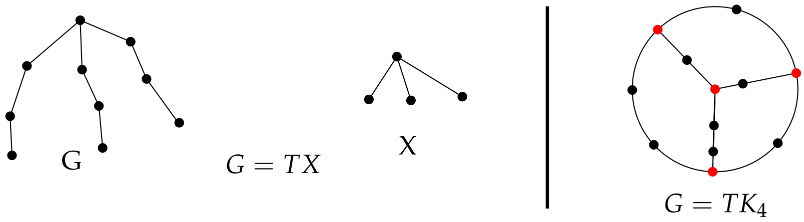
  An alternative characterisation is that the vertices V' of a subgraph of G can be partitioned into sets $V' = V_1 + \cdots + V_{|V(X)|}$ such that $G[V_i]$ is connected and $V_i \sim V_j$ if and only if the corresponding vertices $v_i, v_j \in V(X)$ are adjacent for each $i, j \in [|V(X)|]$.



- X is a *single-edge subdivision of* G if $V(X) = V(G) \cup \{v\}$ and $E(X) = E(G) - xy + xv + vy$ for $xy \in E(G)$ and $v \notin V(G)$. X is a *subdivision of* G if it can be obtained from G by a series of single-edge subdivisions. subdivision

- We say that X *is a topological minor of* G, denoted by $G = TX$, if a subgraph of G is a subdivision of X.

topological minor, $G = TX$



Theorem 28 (Kuratowski's theorem 1930, 4.4.6)**.** A graph is planar if and only if it does not have K_5 or $K_{3,3}$ as topological minors.

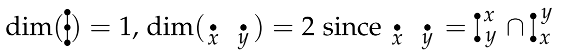
Definition.

- Let X be a set and $\leq \subseteq X^2$ a relation on X. Then $\leq$ is a *partial order* if it is reflexive, antisymmetric and transitive. A partial order is *total* if $x \leq y$ or $y \leq x$ for every $x, y \in X$.

partial order
total order

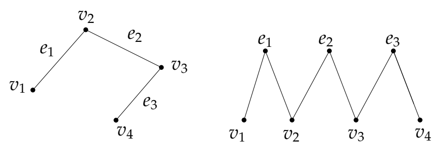
- Let $\leq$ be a partial order on a set X. The pair $(X, \leq)$ is called a *poset* (partially ordered set). If $\leq$ is clear from context, the set X itself is called a poset. The *poset dimension of* $(X, \leq)$ is the smallest number d such that there are total orders $R_1, \ldots, R_d$ on X with $\leq = R_1 \cap \cdots \cap R_d$.

poset
poset dimension, $\dim(X, \leq)$



- The *incidence poset* $(V \cup E, \leq)$ on a graph $G = (V, E)$ is given by $v \leq e$ if and only if e is incident to v for all $v \in V$ and $e \in E$.

incidence poset



Theorem (Schnyder)**.** Let G be a graph and P be its incidence poset. Then G is planar if and only if $\dim(P) \leq 3$.

Theorem 29 (5-coloring theorem, 5.1.2)**.** Every planar graph is 5-colorable.

The more well-known 4-coloring theorem is much harder to prove. Interestingly, it is one of the first theorems that has been proved using computer assistance. The computer-generated proof uses an enormous case distinction. Some mathematicians have philosophical problems with this approach since the resulting proof cannot be easily verified by humans. A shorter proof is still outstanding.

Theorem (4-coloring theorem)**.** Every planar graph is 4-colorable.

Definition.

- Let $L(v) \subseteq \mathbb{N}$ be a list of colors for each vertex $v \in V$. We say that G is *L-list-colorable* if there is coloring $c\colon V \to \mathbb{N}$ such that $c(v) \in L(v)$ for each $v \in V$ and adjacent vertices receive different colors.
- Let $k \in \mathbb{N}$. We say that G is *k-list-colorable or k-choosable* if G is L-list-colorable for each list L with $|L(v)| = k$ for all $v \in V$.
- The *choosability*, denoted by $\mathrm{ch}(G)$, is the smallest k such that G is k-choosable.
- The *edge choosability*, denoted by $\mathrm{ch}'(G)$, is defined analogously.

L-list-colorable

k-list-colorable

choosability, $\mathrm{ch}(G)$

edge choosability, $\mathrm{ch}'(G)$

Theorem 30 (Thomassen's theorem 1994, 5.4.2)**.** Every planar graph is 5-choosable.

5 Colorings

Lemma (Greedy estimate for the chromatic number)**.**
Let G be graph. Then $\chi(G) \leq \Delta(G) + 1$.

Theorem 31 (Brook's theorem 1924, 5.2.4)**.** Let G be a connected graph. Then $\chi(G) \leq \Delta(G)$ unless G is a complete graph or an odd cycle.

Definition.

- The *clique number $\omega(G)$ of G* is the largest order of an induced complete subgraph of G. — clique number, $\omega(G)$
- The *co-clique number $\alpha(G)$ of G* is the largest order of an induced empty subgraph of G. — co-clique number, $\alpha(G)$
- A graph G is called *perfect* if $\chi(H) = \omega(H)$ for each induced subgraph H of G. For example, bipartite graphs are perfect with $\chi = \omega = 2$. — perfect graph

Lemma (Small coloring results)**.**

- $\chi(G) \geq \max\{\omega(G), n/\alpha(G)\}$ since each color class is an empty induced subgraph and $\chi(K_k) = k$.
- $\|G\| \geq \binom{\chi(G)}{2} \Leftrightarrow \chi(G) \leq 1/2 + \sqrt{2\|G\| + 1/4}$ since there must be at least one edge between any two color classes.
- The chromatic number $\chi(G)$ of G is at most one more than the length of a longest directed path in any orientation of G.

Theorem (Perfect graph theorem 1972)**.** A graph G is perfect if and only if its complement $\overline{G}$ is perfect.

Theorem (Strong perfect graph theorem 2002)**.** A graph G is perfect if and only if it does not contain an odd cycle on at least 5 vertices (an *odd hole*) or the complement of an odd hole as an induced subgraph.

Definition. Let A be the adjacency matrix of a graph G.

- By the spectral theorem the symmetric matrix A has an orthonormal basis of eigenvectors and all of its eigenvalues are real.
- The *spectrum $\lambda(G)$ of G* is the multiset of eigenvalues of A. — spectrum, $\lambda(G)$
- The *spectral radius of* G is $\lambda_{\max}(G) := \max\{\lambda : \lambda \in \lambda(G)\}$. Analogously, $\lambda_{\min}(G) := \min\{\lambda : \lambda \in \lambda(G)\}$. — spectral radius, $\lambda_{\max}(G)$

Lemma (Small results about the eigenvalues of G)**.** Let A be the adjacency matrix of G and let H be an induced subgraph of G. Then

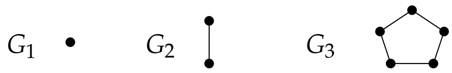
- $\lambda_{\min}(G) \leq \lambda_{\min}(H) \leq \lambda_{\max}(H) \leq \lambda_{\max}(G)$,

- $\delta(G) \leq 2\|G\|/n \leq \lambda_{\max}(G) \leq \Delta(G)$,
- $\operatorname{trace}(A) = 0$, $\operatorname{trace}(A^2) = 2\|G\|$, $\operatorname{trace}(A^3) = 6 \cdot \#$ triangles in G.

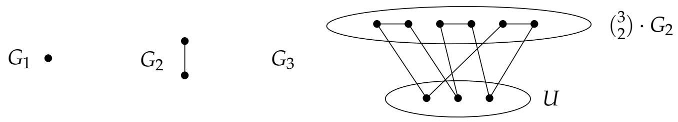
Theorem 32 (Spectral estimate for the chromatic number from folklore)**.**
Let G be a graph. Then $\chi(G) \leq \lambda_{\max}(G) + 1$.

Example (Mycielski's construction)**.**
We can construct a family $\big(G_k = (V_k, E_k)\big)_{k\in\mathbb{N}}$ of triangle-free graphs with $\chi(G_k) = k$ as follows:

- G_1 is the single-node graph, G_2 is the single-edge graph.
- $V_{k+1} := V_k \cup U \cup \{w\}$ where $V_k \cap \big(U \cup \{w\}\big) = \emptyset$, $V = \{v_1, \dots, v_n\}$ and $U = \{u_1, \dots, u_n\}$.
- $E_{k+1} := E_k \cup \{wu_i : i = 1, \dots, k\} \cup \bigcup_{i=1}^{n} \{u_i v : v \in N_{G_k}(v_i)\}$.



Example (Tutte's construction)**.** We can construct a family $(G_k)_{k\in\mathbb{N}}$ of triangle-free graphs with $\chi(G_k) = k$ as follows: G_1 is the single-node graph. To get from G_k to G_{k+1}, take an independent set U of size $k(|G_k| - 1) + 1$ and $\binom{|U|}{|G_k|}$ copies of G_k. For each subset of size $|G_k|$ in U then introduce a perfect matching to exactly one of the copies of G_k.



Theorem 33 (König's theorem 1916, 5.3.1)**.**
Let G be a bipartite graph. Then $\chi'(G) = \Delta(G)$.

Theorem 34 (Vizing's theorem 1964, 5.3.2)**.**
Let G be a graph. Then $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$.

Thus, the chromatic index of a graph can only take one of two possible values. Determining which of the two values occurs is NP-complete.

Lemma. We have $\operatorname{ch}(K_{n,n}) \geq c \cdot \log(n)$ for some constant $c > 0$. In particular,

$$\operatorname{ch}\left(K_{\binom{2k-1}{k},\binom{2k-1}{k}}\right) \geq c \cdot k.$$

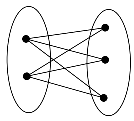
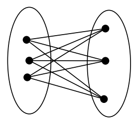
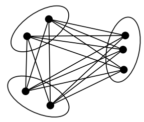
Theorem 35 (Galvin's theorem 1995, 5.4.4)**.**
Let G be a bipartite graph. Then $\operatorname{ch}'(G) = \chi'(G)$.

6 Extremal graph theory

In this section $c, c_1, c_2, \ldots$ always denote unspecified constants in $\mathbb{R}_{>0}$.

Definition.

- Let n be a positive integer and H a graph. By $\mathrm{ex}(n, H)$ we denote the maximum size of a graph of order n that does not contain H as a subgraph. $\mathrm{EX}(n, H)$ is the set of such graphs.
- Let n and r be integers with $1 \leq r \leq n$. The *Turàn graph* $T(n, r)$ is the complete r-partite graph of order n whose partite sets differ by at most 1 in size. It does not contain K_{r+1}. We denote $\|T(n, r)\|$ by $t(n, r)$.

$T(5,2)$  $T(6,2)$  $T(7,3)$ 

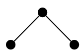
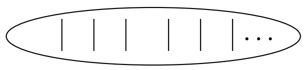
$\mathrm{ex}(n, H)$

$\mathrm{EX}(n, H)$

Turàn graph, $T(n, r)$ $t(n, r)$

Example.

- $\mathrm{ex}(n, K_2) = 0$, $\mathrm{EX}(n, K_2) = \{E_n\}$
- $\mathrm{ex}(n, P_3) = \lfloor n/2 \rfloor$, $\mathrm{EX}(n, P_3) = \{\lfloor n \rfloor \cdot K_2 + (n \bmod 2) \cdot E_1\}$

$H =$  $EX(n, H)$ 

Lemma (On Turàn graphs)**.**

- Among all r-partite graphs on n vertices the Turàn graph $T(n, r)$ has the largest number of edges.
- We have the recursion

$$t(n, r) = t(n - r, r) + (n - r)(r - 1) + \binom{r}{2}.$$

- A Turàn graph lacks a ratio of $1/r$ of the edges:

$$\lim_{n \to \infty} \frac{t(n, r)}{\binom{n}{2}} = \left(1 - \frac{1}{r}\right).$$

Theorem 36 (Turàn's theorem 1941, 7.1.1)**.** For all integers $r > 1$ and $n \geq 1$, every graph G with n vertices, $\mathrm{ex}(n, K_r)$ edges and $K_r \not\subseteq G$ is a $T_{r-1}(n)$.

Definition. Let $X, Y \subseteq V(G)$ be vertex sets and $\epsilon > 0$.

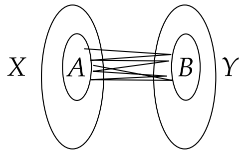
- We define $\|X, Y\|$ to be the number of edges between X and Y and *the density $d(X, Y)$ of (X, Y)* to be

density, $d(X, Y)$

$$d(X, Y) := \frac{\|X, Y\|}{|X||Y|}.$$

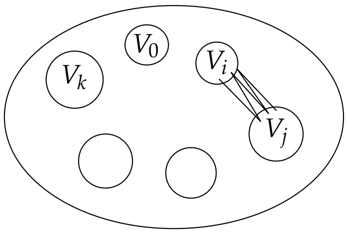
- For $\epsilon > 0$ the pair (X, Y) is an *ϵ-regular pair* if we have $|d(X, Y) - d(A, B)| \leq \epsilon$ for all $A \subseteq X$, $B \subseteq Y$ with $|A| \geq \epsilon|X|$ and $|B| \geq \epsilon|Y|$.

ϵ-regular pair



- An *ϵ-regular partition* of the graph $G = (V, E)$ is a partition of the vertex set $V = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_k$ with the following properties:
  1. $|V_0| \leq \epsilon|V|$
  2. $|V_1| = |V_2| = \cdots = |V_k|$
  3. All but at most ϵk^2 of the pairs (V_i, V_j) for $1 \leq i < j \leq k$ are ϵ-regular.

ϵ-regular partition



Theorem 37 (Erdős-Stone theorem 1946, 7.1.2)**.** For all integers $r > s \geq 1$ and every $\epsilon > 0$ there exists an integer n_0 such that every graph with $n \geq n_0$ vertices and at least

$$t_{r-1}(n) + \epsilon n^2$$

edges contains K_s^r as a subgraph.

Theorem (Chvátal-Szemerédi theorem 1981)**.** Chvátal and Szemerédi proved a more quantitative version of the Erdős-Stone theorem: For every $\epsilon > 0$ and every integer $r \geq 3$, every graph on n vertices and at least $\big(1 - 1/(r-1) + \epsilon\big)\binom{n}{2}$ edges contains K_r^t as a subgraph. Here t is given by

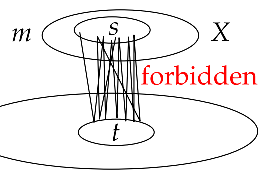
$$t = \frac{\log n}{500 \cdot \log(1/\epsilon)}.$$

Furthermore, there is a graph G on n vertices and $\big(1 - (1+\epsilon)/(r-1)\big)\binom{n}{2}$ edges that does not contain K_r^t for

$$t = \frac{5 \cdot \log n}{\log(1/\epsilon)},$$

i.e. the choice of t is asymptotically tight.

Definition. The *Zarankiewicz* function $z(m, n; s, t)$ denotes the maximum number of edges that a bipartite graph with parts of size m and n can have without containing $K_{s,t}$.



Zarankiewicz, $z(m, n; s, t)$

Theorem 38 (Kővári-Sós-Turán theorem 1954)**.**
We have the upper bound

$$z(m, n; s, t) \leq (s-1)^{1/t}(n - t + 1)m^{1-1/t} + (t-1)m$$

for the Zarankiewicz function. In particular,

$$z(n, n; t, t) \leq c_1 \cdot n \cdot n^{1-1/t} + c_2 \cdot n = \mathcal{O}(n^{2-1/t})$$

for $m = n$ and $t = s$.

Corollary.
For $t \geq s \geq 1$ we can bound the extremal number of $K_{t,s}$ using the Kővári–Sós–Turán theorem

$$\text{ex}(n, K_{t,s}) \leq \frac{1}{2} \cdot z(n, n; s, t) \leq cn^{2-1/s}.$$

For $t = s = 2$ this bound yields

$$\text{ex}(n, C_4) \leq \frac{n}{4}(1 + \sqrt{4n-3}).$$

This bound is actually tight, i.e. $\text{ex}(n, C_4) = 1/2 \cdot n^{3/2} \cdot (1 + o(1))$.

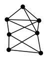
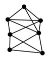
Lemma. $\mathrm{ex}(n, K_{r,r}) \geq cn^{2-2/(r+1)}$ for all $n, r \in \mathbb{N}$.

Theorem 39. For all $n \in \mathbb{N}$ we have $\mathrm{ex}(n, P_{k+1}) \leq \big(n \cdot (k-1)\big)/2$.

Theorem 40 (Szemerédi's regularity lemma 1970, 7.4.1)**.** For every $\epsilon > 0$ and every integer $m \geq 1$ there is an $M \in \mathbb{N}$ such that every graph of order at least m has an ϵ-regular partition $V_0 \sqcup \cdots \sqcup V_k$ with $m \leq k \leq M$.

Corollary 41. Erdős-Stone together with $\lim_{n\to\infty} t(n,r)/\binom{n}{2} = 1 - 1/r$ yields an asymptotic formula for the extremal number of every graph H on at least one edge:

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} = \frac{\chi(H)-2}{\chi(H)-1}$$

For example, $\mathrm{ex}(n,$  $) \simeq 2/3 \cdot \binom{n}{2}$ since $\chi($  $) = 4$.

Conjecture (Hadwiger conjecture)**.** Let r be a natural number and G be a graph. Then $\chi(G) \geq r$ implies $MK_r \subseteq G$.

For $r \in \{1,2,3,4\}$ this is easy to see. For $r \in \{5,6\}$ the conjecture has been proven using the 4-color-theorem. It is still open for $r \geq 7$.

Theorem 42. Every graph G of average degree at least cr^2 contains K_r as a topological minor.

Theorem. Let G be a graph of minimum degree $\delta(G) \geq d$ and girth $g(G) \geq 8k+3$ for $d, k \in \mathbb{N}$ and $d \leq 3$. Then G has a minor H of minimum degree $\delta(H) \geq d(d-1)^k$.

Theorem 43 (Thomassen's theorem 1983, 7.2.5)**.** For all $r \in \mathbb{N}$ there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every graph of minimum degree at least 3 and girth at least $f(r)$ has a K_r minor.

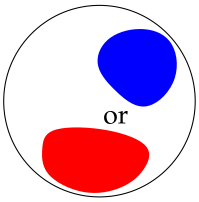
Theorem (Kühn-Osthus 2002)**.** Let $r \in \mathbb{N}$. Then there is a constant $g \in \mathbb{N}$ such that we have $TK_r \subseteq G$ for every graph G with $\delta(G) \geq r-1$ and $g(G) \geq g$.

7 Ramsey theory

In every 2-coloring in this section we use the colors red and blue.

Definition.

- In an edge-coloring of a graph, a set of edges is
  - *monochromatic* if all edges have the same color,
  - *rainbow* if no two edges have the same color,
  - *lexical* if two edges have the same color if and only if they have the same lower endpoint in some ordering of the vertices.
- Let k be a natural number. Then the *Ramsey number* $R(k) \in \mathbb{N} \cup \{\infty\}$ is the smallest n such that every 2-edge-coloring of K_n contains a monochromatic K_k.



Color $E(K_n)$ in 2 colors.

- Let k and l be natural numbers. Then the *asymmetric Ramsey number* $R(k,l)$ is the smallest $n \in \mathbb{N} \cup \{\infty\}$ such that every 2-edge-coloring of a K_n contains a red K_k or a blue K_l.
- Let G and H be graphs. Then the *graph Ramsey number* $R(G,H)$ is the smallest $n \in \mathbb{N} \cup \{\infty\}$ such that every 2-edge-coloring of K_n contains a red G or a blue H.
- Let $r, l_1, \ldots, l_k$ be natural numbers. Then the *hypergraph Ramsey number* $R_r(l_1, \ldots, l_k)$ is the smallest $n \in \mathbb{N} \cup \{\infty\}$ such that for every k-coloring of $\binom{[n]}{r}$ there is an $i \in \{1, \ldots, k\}$ and a $V \subseteq [n]$ with $|V| = l_i$ such that $\binom{V}{r}$ has color i.
- Let G and H be graphs. Then the *induced Ramsey number* $R_{ind}(G,H)$ is the smallest $n \in \mathbb{N} \cup \{\infty\}$ such that there is a graph F on n vertices every 2-coloring of which contains a red G or a blue H.
- For $n \in \mathbb{N}$ and a graph H, the *anti-Ramsey number* $AR(n,H)$ is the maximum number of colors that an edge-coloring of K_n can have without containing a rainbow copy of H.

monochromatic

rainbow

lexical

Ramsey, $R(k)$

asymmetric Ramsey, $R(k,l)$

graph Ramsey, $R(G,H)$

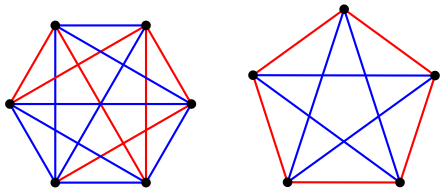
hypergraph Ramsey, $R_r(l_1, \ldots, l_k)$

induced Ramsey, $R_{\text{ind}}(G,H)$

anti-Ramsey, $AR(n,H)$

Lemma.

- $R(3) = 6$, i.e. every 2-edge-colored K_6 contains a monochromatic triangle and there is a 2-coloring of a K_5 without monochromatic triangles.



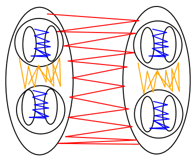
- Clearly, $R(2,k) = R(k,2) = k$.

Theorem 44 (Ramsey theorem 1930, 9.1.1)**.** For every $k \in \mathbb{N}$ we have $\sqrt{2}^k \leq R(k) \leq 4^k$. In particular, the Ramsey numbers, the asymmetric Ramsey numbers and the graph Ramsey numbers are finite.

Theorem 45. For every $k, l \in \mathbb{N}$ we have $R(k,l) \leq R(k-1,l) + R(k,l-1)$. This implies $R(k,l) \leq \binom{k+l-2}{k-1}$ by induction.

Lemma 46. For every $r, p, q \in \mathbb{N}$ we have $R_r(p,q) \leq R_{r-1}\big(R_r(p-1,q), R_r(p,q-1)\big) + 1$.

Lemma. We have $c_1 \cdot 2^k \leq R_2(\underbrace{3,\ldots,3}_{k}) \leq c_2 \cdot k!$ for some constants $c_1, c_2 > 0$.



Applications of Ramsey theory

Theorem (Erdős-Szekeres 1935)**.** Any sequence of $(r-1)(s-1)+1$ distinct real numbers contains an increasing subsequence of length r or a decreasing subsequence of length s.

Theorem 47 (Erdős-Szekeres 1935)**.** For every $m \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that every set of at least N points in $\mathbb{R}^2$ contains a convex m-gon.

Theorem (Schur 1916)**.** Let $c\colon \mathbb{N} \to [r]$ be a coloring of the natural numbers with $r \in \mathbb{N}$ colors. Then there are monochromatic $x, y, z \in \mathbb{N}$ with $x + y = z$.

Definition. Let $r \in \mathbb{N}$ and $A \in \mathbb{Z}^{n \times k}$.

- A is said to be *r-regular* if there is a monochromatic solution of $Ax = 0$ for every r-coloring $c\colon \mathbb{N} \to [r]$ of $\mathbb{N}$. r-regular matrix

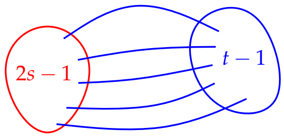
- A fulfils the *column condition* if there is a partition $C_1 \sqcup \cdots \sqcup C_l$ of the columns of A such that the following holds: Let $s_i := \sum_{c \in C_i} c$ for $i \in [l]$ be the sum of columns in C_i. Then we must have $s_1 = 0$ and every s_i for $i \in \{2, \dots, l\}$ is a rational linear combination of the columns in $C_1 \sqcup \ldots \sqcup C_{i-1}$.

  For example, $2x_1 + x_2 + x_3 - 4x_4$ fulfils the column condition since $2 + 1 + 1 - 4 = 0$.

column condition

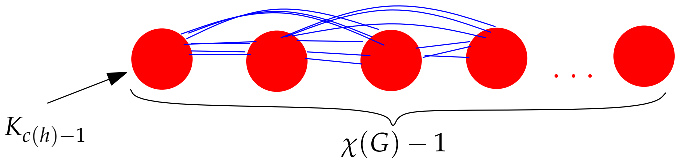
Theorem 48 (Rado 1933)**.** Let $A \in \mathbb{Z}^{n \times k}$. If A fulfils the column condition, then A is r-regular for every $r \in \mathbb{N}$.

Lemma 49. For every $s, t \in \mathbb{N}$ with $s \geq t \geq 1$ we have $R(sK_2, tK_2) = 2s + t - 1$.



Lemma 50. For every $s, t \in \mathbb{N}$ with $s \geq t \geq 1$ and $s \geq 2$ we have $R(sK_3, tK_3) = 3s + 2t$.

Theorem 51 (Chvátal, Harary 1972)**.** Let G and H be graphs. Then $R(G, H) \geq (\chi(G) - 1)(c(H) - 1) + 1$ where $c(H)$ is the cardinality of the largest component of H.



Theorem 52 (Induced Ramsey theorem)**.**
$R_{\text{ind}}(G, H)$ is finite for all graphs G and H.

Theorem 53 (Canonical Ramsey theorem, Erdős-Rado 1950)**.** For all $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that any edge coloring of K_n with arbitrarily many colors contains a K_k that is monochromatic, rainbow or lexical.

Theorem 54 (Chvátal-Rödl-Szemerédi-Trotter 1983)**.** For every positive integer Δ there exists a $c \in \mathbb{N}$ such that $R(H, H) \leq c|V(H)|$ for every graph H with $\Delta(H) = \Delta$.

Corollary. For every n-vertex graph H with maximum degree 3 we have $R(H, H) \leq cn$ for some constant $c > 0$. This number grows much slower than $R(K_n, K_n) \geq \sqrt{2}^n$.

Theorem 55 (Anti-Ramsey theorem, Erdős-Simonvits-Sós 1973)**.** For all $n, r \in \mathbb{N}$ we have $AR(n, K_r) = \binom{n}{2}(1 - 1/(r-2))(1 - o(1))$.

8 Flows

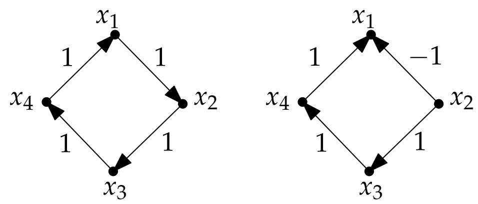
Definition. Let H be an Abelian semigroup and $\tilde{E} := \{(x,y) : xy \in E\}$.

- For $f\colon \tilde{E} \to H$ and $X, Y \subseteq V$ we define $f(X,Y) := \sum_{(x,y)\in(X\times Y)\cap\tilde{E}} f(x,y)$.
- A function $f\colon \tilde{E} \to H$ is a *circulation on* G if

  (C_1) $f(x,y) = -f(y,x)$ for all $xy \in E$ and

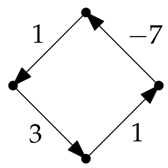
  (C_2) $f(v,V) = 0$ for all $v \in V$.

circulation



- If H is an Abelian group, then a circulation f is also called a *H-flow on* G. If $f(x,y) \neq 0$ for all $xy \in E$, then f is a *nowhere-zero flow*.

H-flow
nowhere-zero



A nowhere-zero $\mathbb{Z}_2$-flow.

- For $k \in \mathbb{N}$ a *k-flow* is a $\mathbb{Z}$-flow f such that $0 < |f(x,y)| < k$ for all $xy \in E$. The *flow number* $\varphi(G)$ *of* G is the smallest k such that G has a k-flow.

k-flow
flow number, $\varphi(G)$

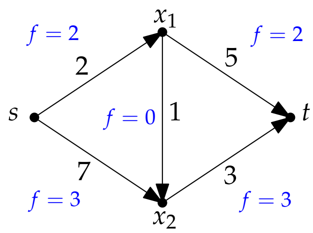
- Let $s \in V$ be a *source*, $v \in V$ be a *sink* and $c\colon \tilde{E} \to \mathbb{Z}_{\geq 0}$ be a *capacity function*. Then a *network flow on the network* (G,s,t,c) is $f\colon \tilde{E} \to \mathbb{R}$ with the following properties for all $x, y \in V$:

  (F_1) $f(x,y) = -f(y,x)$

  (F_2) $f(x,V) = 0$ if $x \notin \{s,t\}$

  (F_3) $f(x,y) \leq c(x,y)$

source, sink,
capacity,
network flow



For any $S \subseteq V$ with $s \in S$ and $t \notin S$ the pair $(S, V \setminus S)$ is called a *cut*. Its capacity is $c(S, V \setminus S)$. cut

The value $f(s, V)$ is also called the *value of* f and is denoted by $|f|$. value, $|f|$

Lemma.

- For any circulation f and $X \subseteq V$ we have $f(X, X) = 0$, $f(X, V) = 0$ and $f(X, V \setminus X) = 0$
- For any network flow f and cut $(S, \bar{S})$ we have $f(S, \bar{S}) = f(s, V)$.

Theorem 56 (Ford-Fulkerson theorem 1965, 6.2.2)**.** In any network the maximum value of a flow is the same as the minimum capacity of a cut and there is an integral flow $f \colon \tilde{E} \to \mathbb{Z}_{\geq 0}$ with this maximum flow value.

Theorem 57 (Tutte 1954, 6.3.1)**.** For every multigraph G there is a polynomial $P \in \mathbb{Z}[X]$ such that for any finite Abelian group H the number of nowhere-zero H-flows on G is $P\big(|H| - 1\big)$.

Corollary. If a H-flow on G exists for some finite Abelian group H, then there is also a $\tilde{H}$-flow on G for all finite Abelian groups $\tilde{H}$ with $|\tilde{H}| = |H|$. For example, if a $\mathbb{Z}_4$-flow exists, then a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow also exists.

Theorem 58 (Tutte 1950, 6.3.3)**.** A multigraph admits a k-flow if and only if it admits a $\mathbb{Z}_k$-flow.

Theorem 59 (Tutte 1954, 6.5.3)**.** For a planar graph G and its dual G^* we have $\chi(G) = \varphi(G^*)$.

Lemma. A graph has a 2-flow if and only if all of its degrees are even.

Lemma. A cubic (3-regular) graph has a 3-flow if and only if it is bipartite.

Conjecture (Tutte's flow conjecture)**.** Every bridgeless multigraph has flow number at most 5. Seymore proved $\varphi(G) \leq 6$ for bridgeless graphs in 1981.

9 Random graphs

In this section we deal with randomly chosen graphs. We will often use the "probabilistic method", a proof method for showing existence: By proving that an object with some desired properties can be chosen randomly (in some probability space) with non-zero probability, we also show that such an object exists.

Definition.

- $\mathcal{G}(n,p)$ is the probability space on all n-vertex graphs that results from independently deciding whether to include each of the $\binom{n}{2}$ possible edges with fixed probability $p \in [0,1]$. This model is called the *Erdős–Rényi model* of random graphs. Erdős–Rényi
- A *property* $\mathcal{P}$ is a set of graphs, e.g. $\mathcal{P} = \{G : G \text{ is } k\text{-connected}\}$. property

  Let $(p_n) \in [0,1]^{\mathbb{N}}$ be a sequence. We say that $G \in \mathcal{G}(n, p_n)$ *almost always* has property $\mathcal{P}$ if $\mathrm{Prob}\big(G \in \mathcal{G}(n,p_n) \cap \mathcal{P}\big) \to 1$ for $n \to \infty$. almost always

  A function $f(n) \colon \mathbb{N} \to [0,1]$ is a *threshold function* for property $\mathcal{P}$ if: threshold function

  - For all $(p_n) \in [0,1]^{\mathbb{N}}$ with $p_n/f(n) \overset{n\to\infty}{\longrightarrow} 0$ the graph $G \in \mathcal{G}(n,p_n)$ almost always does not have property $\mathcal{P}$.
  - For all $(p_n) \in [0,1]^{\mathbb{N}}$ with $p_n/f(n) \overset{n\to\infty}{\longrightarrow} \infty$ the graph $G \in \mathcal{G}(n,p_n)$ almost always has property $\mathcal{P}$.

Lemma.

- For a given graph G on n vertices and m edges we have

$$\mathrm{Prob}\big(G = \mathcal{G}(n,p)\big) = p^m (1-p)^{\binom{n}{2}-m}$$

- For all integers $n \geq k \geq 2$ we have

$$\mathrm{Prob}\big(G \in \mathcal{G}(n,p), \alpha(G) \geq k\big) \leq \binom{n}{k} (1-p)^{\binom{k}{2}}$$

  and

$$\mathrm{Prob}\big(G \in \mathcal{G}(n,p), \omega(G) \geq k\big) \leq \binom{n}{k} p^{\binom{k}{2}}$$

Theorem 60 (Erdős 1947)**.** Erdős proved the lower bound $R(k,k) \geq 2^{k/2}$ on Ramsey numbers by applying the probabilistic method to the Erdős–Rényi model.

Lemma. We have

$$E\big(\#k\text{-cycles in } G \in \mathcal{G}(n,p)\big) = \frac{n^{\underline{k}}}{2k} \cdot p^k$$

where $n^{\underline{k}} = n \cdot (n-1) \cdots (n-k+1)$.

Theorem 61 (Erdős 1959, 11.2.2)**.** For every $k \in \mathbb{N}$ there is a graph H with $g(H) \geq k$ and $\chi(H) \geq k$.

Lemma. For all $p \in (0,1)$ and graphs H almost all graphs in $\mathcal{G}(n,p)$ contain H as an induced subgraph.

Lemma. For all $p \in (0,1)$ and $\epsilon > 0$ almost all graphs in $\mathcal{G}(n,p)$ fulfil

$$\chi(G) > \frac{\log(1/(1-p))}{2+\epsilon} \cdot \frac{n}{\log n}$$

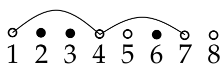
Remark. Asymptotic behaviour of $\mathcal{G}(n,p)$ for some properties:

- $p_n = \sqrt{2}/n^2 \Rightarrow G$ almost always has a component with > 2 vertices
- $p_n = 1/n \Rightarrow G$ almost always has a cycle
- $p_n = \log n/n \Rightarrow G$ is almost always connected
- $p_n = (1+\epsilon)\log n/n \Rightarrow G$ almost always has a Hamiltonian cycle
- $p_n = n^{-2/(k-1)}$ is the threshold function for containing K_k

Lemma 62 (Lovász local lemma)**.** Let $A_1, \ldots, A_n$ be events in some probabilistic space. If $\text{Prob}(A_i) \leq p \in (0,1)$, each A_i is independent from all but at most $d \in \mathbb{N}$ of the other A_i and $np(d+1) \leq 1$, then

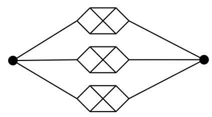
$$\bigcap_{i=1}^{n} \overline{A_i} > 0.$$

Lemma. *Van-der-Waerden's number* $W(k)$ is the smallest n such that any 2-coloring of $[n]$ contains a monochromatic arithmetic progression of length k. We can prove $W(k) \geq 2^{k-1}/(ek^2)$ with the Lovász local lemma.



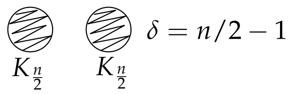
10 Hamiltonian cycles

Lemma 63 (Necessary condition for Hamiltonian cycle)**.** If G has a Hamiltonian cycle, then for every non-empty $S \subseteq V$ the graph $G - S$ cannot have more than $|S|$ components.



Non-hamiltonian graph.

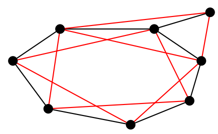
Theorem 64 (Dirac 1952, 10.1.1)**.** Every graph with $n \geq 3$ vertices and minimum degree at least $n/2$ has a Hamiltonian cycle.



Theorem. Every graph on $n \geq 3$ vertices with $\alpha(G) \leq \kappa(G)$ is Hamiltonian.

Theorem 65 (Tutte 1956, 10.1.4)**.** Every 4-connected planar graph is Hamiltonian.

Theorem 66 (Fleischner's theorem 1974, 10.3.1)**.** If G is 2-connected, then $G^2 := (V, E')$ with $E' := \{uv : u, v \in V, d_G(u, v) \leq 2\}$ is Hamiltonian.



Theorem 67 (Chvátal 1972, 10.2.1)**.** Let $0 \leq a_1 \leq \cdots \leq a_n < n$ be an integer sequence with $n \geq 3$. All graphs with the degree sequence $a_1, \ldots, a_n$ are Hamiltonian if and only if $a_i \leq i$ implies $a_{n-i} \geq n - i$ for all $i < n/2$.

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Named theorems

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