Extremal functions for bipartite graphs

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For positive integers m, n, s, t let $z(m, n; s, t)$ be the largest number of edges in a bipartite graph G with parts X and Y, where $|X| = m, |Y| = n$ not containing $K_{s,t}$ as a subgraph with s vertices in X and t vertices in Y. This function is called Zarankiewicz' function.

Theorem 0.1 (Kővari, Sós, Turán 1954)**.** $z(m, n; s, t) \leq (s-1)^{1/t}(n-t+1)m^{1-1/t} + (t-1)m.$

Proof. Let G be a bipartite graph with parts X and Y of sizes m and n respectively, such that G contains no copy of $K_{s,t}$ with s vertices in X and t in Y. Let T be the set of all stars with t edges and center in X. Then

$$|T| = \sum_{x \in X} \binom{deg(x)}{t},$$

where $\binom{deg(x)}{t} = 0$ if $deg(x) < t$, because each vertex x in X is a center of a star with $deg(x)$ leaves and there are $\binom{d(x)}{t}$ ways to choose a star with center in x and t leaves. On the other hand, for each subset Y' of t vertices in Y there are at most $s-1$ stars from T that have Y' as their leaf-set. Indeed, otherwise s stars with centers in X and a leaf-set Y' would form a copy of $K_{s,t}$. Thus

$$|T| \leq (s-1)\binom{n}{t}.$$

Jensen's inequality applied to a convex function ϕ gives

$$\phi\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) \leq \frac{\sum_{i=1}^{n} \phi(x_i)}{n}.$$

Plugging $\phi(x) = \binom{x}{t}$, we have that

$$|T| = \sum_{x \in X} \binom{deg(x)}{t} \geq m\binom{\frac{\sum_{x \in X} deg(x)}{m}}{t} = m\binom{e/m}{t},$$

where $e = |E(G)|$.

Setting together the lower and the upper bounds on $|T|$, we have

$$(s-1)\binom{n}{t} \geq |T| \geq m\binom{e/m}{t}.$$

What remains is to solve this inequality for e.

$$\begin{aligned} (s-1)\binom{n}{t} \geq m\binom{e/m}{t} &\iff \\ \frac{(s-1)}{m} \geq \frac{\binom{e/m}{t}}{\binom{n}{t}} &\iff \\ \frac{(s-1)}{m} \geq \frac{(e/m)(e/m-1)\cdots(e/m-t+1)}{n(n-1)\cdots(n-t+1)} &\iff \\ \frac{(s-1)}{m} \geq \frac{e/m}{n}\frac{e/m-1}{n-1}\cdots\frac{e/m-t+1}{n-t+1}&. \end{aligned} \tag{1}$$

Observe that $e/m \leq n$ since $e \leq mn$. Further, observe that $\frac{e/m-i}{n-i} \geq \frac{e/m-t+1}{n-t+1}$, for $0 \leq i < t-1$. Indeed,

$$\begin{aligned}
\frac{e/m-i}{n-i} \geq \frac{e/m-t+1}{n-t+1} &\iff \\
(e/m-i)(n-t+1) \geq (e/m-t+1)(n-i) &\iff \\
en/m - et/m + e/m - in + it - i \geq en/m - nt + n - ie/m + it - i &\iff \\
-et/m + e/m - in \geq -nt + n - ie/m &\iff \\
e/m(-t+1+i) \geq n(-t+1+i) &\iff \\
e/m \leq n.
\end{aligned}$$

Returning to (1), we have

$$\frac{(s-1)}{m} \geq \frac{e/m}{n}\frac{e/m-1}{n-1}\cdots\frac{e/m-t+1}{n-t+1} \geq \left(\frac{e/m-t+1}{n-t+1}\right)^t.$$

Thus $e \leq (s-1)^{1/t}(n-t+1)m^{1-1/t} + (t-1)m$.

□

Theorem 0.2. *There are positive constants c, c' such that*

$$c'n^{2-2/t} \leq \mathrm{ex}(n, K_{t,t}) \leq cn^{2-1/t}.$$

Proof.
Upper Bound: Let G be an n-vertex graph that does not contain a copy of $K_{t,t}$. Assume first that n is even. Then each bipartite subgraph with $n/2$ vertices in each part contains at most $z(n/2, n/2; t, t)$ edges. Therefore

$$|E(G)| \leq z(n/2, n/2; t, t)\frac{1}{2}\binom{n}{n/2}\binom{n-1}{n/2-1}^{-1},$$

here $\frac{1}{2}\binom{n}{n/2}$ is the number of balanced induced bipartite subgraphs in G and $\binom{n-1}{n/2-1}$ is the number of such subgraphs to which a given edge belongs (i.e. an over-count factor). Using the upper bound on Zarankiewicz's function, we have

$$\begin{aligned}
|E(G)| &\leq z(n/2, n/2; t, t)\frac{1}{2}\binom{n}{n/2}\binom{n-1}{n/2-1}^{-1} \\
&\leq C(t)(n/2)^{2-1/t}\frac{n!}{n/2!\ n/2!}\frac{(n/2-1)!(n/2-1)!}{(n-1)!} \\
&= C_1(T)n^{2-1/t},
\end{aligned}$$

where $C(t)$ and $C_1(t)$ are constants depending on t. If n is odd, then $\mathrm{ex(n, K_{t,t}) \leq ex(n+1, K_{t,t}) \leq C_1(n+1)^{2-1/t} \leq cn^{2-1/t}}$, for a constant c.

Another way to get the Upper Bound: Let G be an n-vertex graph that does not contain a copy of $K_{t,t}$ and vertices $v_1, \ldots, v_n$. Let H be a balanced bipartite graph with partite sets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ and an edge a_ib_j in H iff $v_iv_j \in E(G)$. Then H has no copy of $K_{t,t}$ and

$$|E(G)| = 2|E(H)| \leq z(n, n; t, t) \leq cn^{2-1/t}.$$

Lower Bound. Let G be a random graph where each edge is chosen with probability p, i.e., $G \in G(n,p)$. Then $\mathrm{Exp}(\#K'_{t,t}s \text{ in } G) = p^{t^2}\binom{n}{2t}1/2\binom{2t}{t}$, where $\binom{n}{2t}$ is the number of ways to choose a $2t$-element subset of vertices and $\frac{1}{2}\binom{2t}{t}$ is the number of ways to split this set into two equal parts. Now, let G' be a graph obtained from G by deleting an edge from each copy of $K_{t,t}$. Then G' has no copy of $K_{t,t}$ and

$$\mathrm{Exp}(|E(G')|) = \mathrm{Exp}(|E(G)|) - \mathrm{Exp}(\#K'_{t,t}s \text{ in } G) = p\binom{n}{2} - p^{t^2}\binom{n}{2t}\frac{1}{2}\binom{2t}{t} \geq C(pn^2 - p^{t^2}n^{2t}),$$

for a constant C. Let $p = n^{-2/t}$, then for a positive constant C_1,

$$\text{Exp}(|E(G')|) \geq C(n^{2-2/t} - n^{-2t+2t}) \geq C_1 n^{2-2/t}.$$

Thus there is an n-vertex graph with $C_1 n^{2-2/t}$ edges and no subgraph isomorphic to $K_{t,t}$. □

Theorem 0.3. *There are positive constants c, c' such that $c'n^{3/2} \leq \text{ex}(n, C_4) \leq cn^{3/2}$.*

Proof. The upper bound follows from the general upper bound on $\text{ex}(n, K_{t,t})$ with $t = 2$. For the lower bound, assume first that $n = 2(N^2 + N + 1)$ for a prime N. Consider a finite projective plane P of order N, i.e., a pair (V, F), where V is a set of $N^2 + N + 1$ points and F is a set of $N^2 + N + 1$ subsets of V each of size $N + 1$ such that for any two sets $A, B \in F$, $|A \cap B| = 1$ and for any two vertices $a, b \in V$, there is $A \in F$, such that $a, b \in A$. Let G be the incidence graph of P, i.e. a bipartite graph with bipartitions V and F such that $a \in V$ is adjacent to $A \in F$ iff $a \in A$. Then there is no copy of C_4 in G, otherwise, such a C_4 would have vertices a, a', A, A', with $a, a' \in A, A'$, $a, a' \in V$ and $A, A' \in F$, i.e., $|A \cap A'| \geq 2$, a contradiction. Since $|A| = N + 1$ for each $A \in F$, we have that

$$\begin{aligned}
|E(G)| &= |V(G)|(N+1) \\
&= 2(N+1)(N^2+N+1) \\
&\geq 2N^3 \\
&= 2(N^2)^{3/2} \\
&\geq 2(\frac{1}{2}(N^2+N+1))^{3/2} \\
&= \frac{1}{\sqrt{2}}(N^2+N+1))^{3/2} \\
&= \frac{1}{\sqrt{2}}|V(G)|^{3/2}.
\end{aligned}$$

Now, if n is arbitrary and sufficiently large, let $n' = \sqrt{n/4}$. We know that there is a prime N, such that $n'/2 \leq N \leq n'$. Then $2(N^2 + N + 1) \leq 2(n/4 + \sqrt{n/4} + 1) \leq n$, for $n \geq 4$. From the first part of the proof, we know that there is a graph G on $2(N^2+N+1)$ vertices and $2(N^2+N+1)(N+1)$ edges with no C_4 as a subgraph. By adding $n-2(N^2+N+1)$ isolated vertices to G, we obtain a graph G' with n vertices and $2(N^2+N+1)(N+1)$ edges. Since $N \geq \sqrt{n}/4$, we have $|E(G')| = 2(N^2 + N + 1)(N + 1) \geq 2(n^2/16 + \sqrt{n}/4 + 1)(\sqrt{n}/4 + 1) \geq cn^{3/2}$, for a positive constant c.

□