Two-color Ramsey numbers for hypergraphs

February 11, 2016

Let $R(p,q;r)$ be the smallest integer N such that for any coloring of the r-element subsets of an N-element set X in red and blue, there is either a set $X' \subseteq X$ such that $|X'| = p$ and all r-element subsets of X' are red or there is a set $X'' \subseteq X$ such that $|X''| = q$ and all r-element subsets of X'' are blue. We call r the uniformity of the set system.

To shorten the writing, we shall say that $\binom{Y}{t}$ is red (or blue) if all t-element subsets of Y are red (or blue).

Theorem 0.1. *For any integers $p, q \geq r$ and any integer $r \geq 2$, $R(p,q;r) \leq R(R(p-1,q;r), R(p,q-1;r); r-1) + 1$.*

Proof. We shall apply a double induction. First note that the statement is true for $r = 2$. Indeed $R(R(p-1,q;2), R(p,q-1;2); 1) = R(p-1,q;2) + R(p,q-1;2) - 1$ and a well-known recursive bound $R(p,q) \leq R(p-1,q) + R(p,q-1)$ holds for usual graph Ramsey numbers. Second, note that when p or q are equal to r, then $R(p,q;r) = q$ or p respectively. In addition $R(p,q;r) \geq \max\{p,q\}$, so $R(R(p-1,q;r), R(p,q-1;r); r-1) + 1 \geq R(q,p;r-1) + 1 \geq \max\{p,q\}$. Thus if p or q is equal to r then $R(p,q;r) \leq \max\{p,q\} \leq R(R(p-1,q;r), R(p,q-1;r); r-1) + 1$.

Now, we assume that the statement is true for uniformity less than r and all values of p and q and it holds for uniformity r and two first parameters $p-1, q$ or $p, q-1$.

Let
$$N = R(R(p-1,q;r),\ R(p,q-1;r);\quad r-1) + 1,$$
and
$$c : \binom{X}{r} \to \{red, blue\},$$
where $|X| = N$, let $x \in X$. We shall show that there is either a p-element subset of X with all r-element subsets red or there is a q-element subset of X with all r-element subsets blue.

Let
$$c' : \binom{X - \{x\}}{r-1} \to \{red, blue\},$$
such that $c'(E') = c(E' \cup \{x\})$.

Since $|X - \{x\}| = N - 1 = R(R(p-1,q;r), R(p,q-1;r); r-1)$, by induction applied to $\binom{X-\{x\}}{r-1}$ and c' either

(i) there is $X' \subseteq X - \{x\}$ such that $|X'| = R(p-1,q;r)$ and $\binom{X'}{r-1}$ is red under c' or

(ii) there is $X'' \subseteq X - \{x\}$ such that $|X''| = R(p,q-1;r)$ and $\binom{X''}{r-1}$ is blue under c'.

Assume first that (i) holds. Then applying induction again to $\binom{X'}{r}$ under c, we see that either
(i.1) there is $A' \subseteq X'$ such that $|A'| = p-1$ and $\binom{A'}{r}$ is red under c or
(i.2) there is $A'' \subseteq X'$ such that $|A''| = q$ and $\binom{A''}{r}$ is blue under c.

In case (i.2) we are done. In case (i.1) consider $A' \cup \{x\}$. We see that each r-element subset of $A' \cup \{x\}$ containing x is red by the definition of c'. In addition, $\binom{A'}{r}$ is red, so $\binom{A' \cup \{x\}}{r}$ is red, and we are done since $|A' \cup \{x\}| = p$.

Assume that (ii) holds. Then applying induction again to $\binom{X''}{r}$ under c, we see that either
(ii.1) there is $B' \subseteq X''$ such that $|B'| = p$ and $\binom{B'}{r}$ is red under c or
(ii.2) there is $B'' \subseteq X''$ such that $|B''| = q - 1$ and $\binom{B''}{r}$ is blue under.
In case (i.1) we are done. In case (i.2) consider $B'' \cup \{x\}$. We see that each r-element subset of $B'' \cup \{x\}$ containing x is blue by the definition of c'. In addition, $\binom{B''}{r}$ is blue, so $\binom{B'' \cup \{x\}}{r}$ is blue, and we are done since $|B'' \cup \{x\}| = q$.

□

Corollary 0.2. *For any positive integers p, q, r, $p, q \geq r$, $R(p, q; r)$ exists.*

Theorem 0.3 (Erdős-Szekerés Theorem)**.** *For any integer $m \geq 3$ there is an integer $N = N(m)$ such that for any N points in the plane with no three on a line there is a subset of m points forming the corners of a convex m-gon.*

Proof. Let $N = R(m, 5; 4)$ and let X be a set of N points in the plane in a general position. Let $c : \binom{X}{4} \to \{red, blue\}$ such that for any 4-element set $A \subseteq X$, $c(A) = red$ if the convex hull of A forms a 4-gon, and $c(A)$ is blue otherwise. Then by Ramsey Theorem either

- $\binom{Y}{4}$ is red for some $Y \subseteq X$, $|Y| = m$ or
- $\binom{Y'}{4}$ is blue for some $Y' \subseteq X$, $|Y| = 5$.

In the former case, consider a convex hull of Y. If it forms a convex m-gon, we are done. Otherwise, there is a point $x \in Y$ that is strictly inside the convex hull of Y. Triangulate the polygon formed by the corners of the convex hull of Y. Since there are no three points on a line, there is a triangle u, v, w with $u, v, w \in Y$ such that x is inside this triangle. But then the 4-element subset $\{x, u, v, w\}$ of Y is colored blue, a contradiction to the fact that all 4-element subsets of Y are red.

In the latter case, note that the convex hull of Y' is a triangle, otherwise four corners of the convex hull form a red 4-element subset of Y'. So, the convex hull of Y' has corners x, y, z and two other vertices u, v of Y' are inside the triangle. Assume without loss of generality that the line through u, v splits the plane in two half-planes, one of which containing both x and y. Then $\{x, y, u, v\}$ form a convex 4-gon, that is red. This is a contradiction to the fact that all 4-element subsets of Y' are blue.

□