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The Dinitz problem

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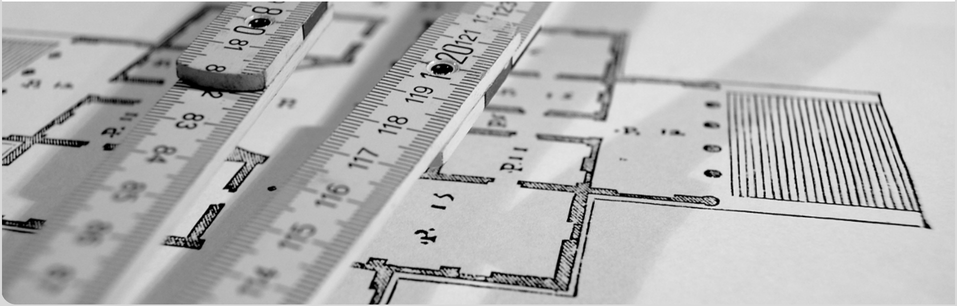
RESEARCH GROUP ON DISCRETE MATHEMATICS – PROF. DR. MARIA AXENOVICH



Introduction

The Dinitz Problem

Consider n^2 cells arranged in an *(n x n)*-square, and let (i, j) denote the cell in row i and column j. Suppose that for every cell (i, j) we are given a set *C(i,j)* of n colours.

Is it then always possible to colour the whole array by picking for each cell *(i,j)* a colour from its set *C(i,j)* such that the colours in each row and each column are distinct?

Example

Example 1

{1,2}	{2,3}
{1,3}	{2,3}

↓

{2}	{3}
{3}	{2}

Example 2

{1,3,4}	{2,8,9}	{1,2,4}
{2,4,5}	{1,4,6}	{3,5,7}
{1,2,3}	{3,4,5}	{2,3,4}

↓

{1}	{2}	{4}
{2}	{4}	{7}
{3}	{5}	{2}

Latin Square

Definition

A latin square is an *(n x n)*-square fill with the numbers 1, 2,..., n so that:

- every number appears exactly once in every row
- every number appears exactly once in every column

We speak of a ***partial latin square*** of order n if some cells of an *(n x n)*-array are filled with numbers from the set $\{1, \ldots, n\}$ such that every number appears at most once in every row and column

Latin Square

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Theorem 1 : Completing the Latin square

Any partial latin square of order n with at most *(n - 1)* filled cells can be completed to a Latin square of the same order

1	4	2	5	3

1	4	2	5	3
4	2	5	3	1
2	5	3	1	4
5	3	1	4	2
3	1	4	2	5

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Same set of colours

Consider the case of the Dinitz problem where all colour sets *C(i,j)* are the same. The following reduction is then possible:

Fill the *(n x n)*-square with the numbers 1, 2, ..., n in such a way that the numbers in any row and columnn are distinct.

One might fill the cells (1,j) correspondently with j ($1 \leq j < n$) and apply Theorem 1

$\Rightarrow$ The answer to our original question in this case is "yes"

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$\Rightarrow$ The answer to our original question in this case is ”yes”

General case trickier

Why should the general case be any harder when $\left| C := \bigcup_{i,j} C_{i,j} \right| > n$?

- Not every colour of C is available at each cell
- We cannot choose any arbitrary permutation of colours for the first row

{1,2}	{2,3}
{1,3}	{2,3}

$\Rightarrow$

1	2
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Reduction with graphs

A bit of graph theory

Before moving forward, let us first rephrase the situation in the language of graph theory.

- For our purposes, we only consider graphs $G = (V, E)$ without loops and multiple edges.

As a reminder :

- The *chromatic number* of a graph is the smallest number of colours that can be assigned to vertices such that adjacent vertices receive different colours.
- A set $A \subseteq V$ is independent if there are no edges within A

List chromatic number

Suppose in the graph *G = (V, E)* we are given a set *C(v)* of colours for each vertex *v*.

Definition: list chromatic number

A ***list colouring*** is a colouring $c : V \to \bigcup_{v \in V} C(v)$ where $c(v) \in C(v)$ for each $v \in V$.

The ***list chromatic number*** $\chi_\ell(G)$ is the smallest number *k* such for *any* list of colour sets *C(v)* with $|C(v)| = k$ for all $v \in V$ there always exists a list colouring.

- Obviously, $\chi_\ell(G) \leq |V|$ holds
- Also, $\chi(G) \leq \chi_\ell(G)$ since ordinary colouring is a special case of list colouring

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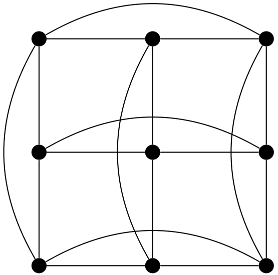
Reduction of the problem

We define S_n the graph which has as vertex set the n^2 cells of our *(n x n)*-array with two cells adjacent if and only if they are in the same row or column.

The graph S_3 for example looks like this:



Some characteristics of S_n

Note 1

$\chi(S_n) = n$ and $\chi_\ell(S_n) \geq n$

Proof :

Any colouring with n colours corresponds to a latin square. Theorem 1 $\Rightarrow \chi(S_n) = n$

Since any n cells in a row are pairwise adjacent, we need at least n colours.

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Chromatic $\neq$ List chromatic

Note 2

$\chi(G) = \chi_\ell(G)$ ***does not*** hold for every graph.

Proof :

Let us take a look at the graph $G = K_{2,4}$

- $\chi(K_{2,4}) = 2$
- A list colouring is however impossible with the colour sets on the left $\Rightarrow \chi_\ell(K_{2,4}) > 2$
- $\chi_\ell(K_{2,4}) = 3$

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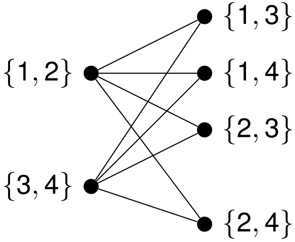

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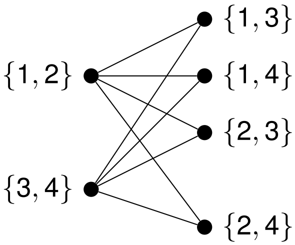

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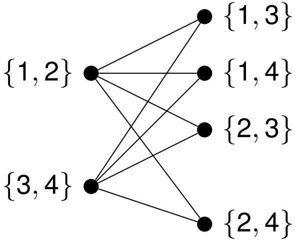

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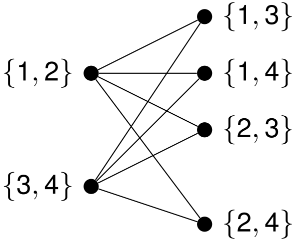

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Dinitz problem for rectangles

Dinitz problem for rectangles

Theorem 2

Let $r < n$ and let *C(i,j)* be a colour set associated to the node *(i,j)* such that $|C(i,j)| = n$ for all i, j $(1 \leq i \leq r, 1 \leq j\ n)$. Then there exists an *(r x n)* partial latin rectangle L with $L_{i,j} \in C_{i,j}$.

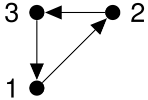
For a proper proof of this theorem, we need to prepare the field.
Our first stop for that is the definition of a graph orientation.

Graph orientation

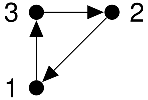
Definitions 1

Let G be a graph on an *ordered* vertex set V

- An orientation D of G is a directed graph that has the same set of vertices and edges as G
- An inverted edge is an edge $v \leftarrow w$ such that $v < w$
- An orientation is called even if the number of inverted edges of D is even
- An orientation is called odd if the number of inverted edges of D is odd



Odd orientation of K_3



Even orientation of K_3

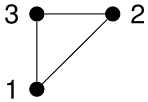
Graph orientation

Definitions 2

For any map $\delta : V \to \mathbb{Z}^+$:

- $DE_G(\delta)$ is the number of even orientations of G such that vertex v has out-degree $\delta(v)$.
- $DO_G(\delta)$ is the number of odd orientations of G such that vertex v has out-degree $\delta(v)$.

For example, $DE_G(\delta) = DO_G(\delta) = 1$ for $\delta : K_3 \to \{1\}$



K_3

Cyclic triangle

A *cyclic triangle* is a directed graph on three vertices u, v, and w with $u \to v \to w \to u$.

Lemma 2.1

Let G be the complete graph on n vertices (K_n). Then an orientation D of G contains a cyclic triangle *if and only if* there are two vertices of D that have the same out-degree.

Proof :

- D Orientation of G, $u, v \in V$ with out-degree a. WLOG u,v has direction $u \to v$. G is complete graph $\Rightarrow$ all vertices in G are contained in $n-1$ edges $\Rightarrow$ u has $n-1-a$ incoming edges and v has a outgoing edges.
  There are only $n-2$ vertices in G beside u, v. Since $(n-1-a)+a > n-2$, there must be at least one vertex w with $w \to u$ and $v \to w \Rightarrow$ D contains a cyclic triangle.

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  There are only $n-2$ vertices in G beside u, v. Since $(n-1-a)+a > n-2$, there must be at least one vertex w with $w \to u$ and $v \to w \Rightarrow$ D contains a cyclic triangle.

Proof of lemma 2.1 (rest)

- We suppose that all the out-degrees of the vertices of D are different.

G can be viewed as the rectangular graph of size $(1xn)$.

We define by *associated orientation* D^L of L the orientation of G that has $(i, j) \rightarrow (i, j')$ whenever $L_{ij} > L_{ij'}$ and $(i, j) \rightarrow (i', j)$ whenever $L_{ij} < L_{i'j}$, where L_{ij} is the entry in cell (i, j).

Since all out-degrees 0, ..., $n - 1$ occur exactly once, D is the associated orientation of a $(1xn)$ latin rectangle $\Rightarrow$ a cyclic triangle in D on vertices (1,i), (1,j) and (1,k) would imply that $i < j < k < i \Rightarrow$ contradiction

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Helpful lemmas (1/3)

The three following lemmas are given without proof.

Lemma 2.2

Let G be the rectangular graph with vertex set $V = \{(i,j)\,|1 \leq i \leq r, 1 \leq j \leq n\}$, lexicographically ordered, and let $\delta : V \to \mathbb{Z}^+$ be a map from the vertices of G to the nonnegative integers.

Then the number of even orientations of G that contain a cyclic triangle and have out-degree $\delta(v)$ at vertex v for all $v \in V$ is equal to the number of odd orientations of G with the same properties.

Helpful lemmas (2/3)

Definition

The circulant *(r x n)* latin rectangle of order n is the (*rxn*) matrix that $i + j - 2$ *mod n* as its *(i, j)*-th entry.

Lemma 2.3

Let $r < n$, and let G be the rectangular graph of size *(r x n)* with vertex set $V = \{(i, j) \,|1 \leq i \leq r, 1 \leq j \leq n\}$. Define the map $\delta : V \rightarrow \mathbb{Z}^+$ as

$$\delta((i,j)) = \begin{cases} r - 2 + j & \text{for} \quad j \leq n - r + 1 \\ n - 1 & \text{for} \quad n - r + 1 < j \leq n - i + 1 \\ r - 1 & \text{for} \quad j > n - i + 1 \end{cases}$$

Then the only orientation D of G with each vertex *(i, j)* of out-degree $\delta((i, j))$ that does not contain a cyclic triangle is the orientation associated with the circulant latin rectangle.

Helpful lemmas (3/3)

Lemma 2.4 (Alon-Tarsi)

Let G be a graph on an ordered vertex set V. Let $\varrho = \{S_v \,|v \in V\}$ be a collection of sets. If there exists a map from the vertex set of G to the nonnegative integers $\delta : V \to \mathbb{Z}^+$ such that $\delta(v) < |S_v|$ for all $v \in V$, and if

$$DE_G(\delta) \neq DO_G(\delta),$$

then G has an ϱ-legal vertex colouring.

To prove Theorem 2, one must *simply* find a map δ that satisfies the conditions of Lemma 2.4 and then invoke this latter.

Theorem 2 (Proof)

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Proof of Theorem 2

- G rectangular graph of size *(r x n)* on vertex set V, $\varrho = \{S_v | v \in V\}$ collection of sets with $|S_v| = n \, \forall v \in V$
- Order V according to the following rule: $(i, j) < (i', j')$ precisely when $i < i'$ or $i = i'$ and $j > j'$.
- $\delta : V \to \mathbb{Z}^+$ map as in Lemma 2.2 (note that $\delta(v) < n$ for all v)

- Lemma 2.2 $\Rightarrow$ the number of even orientations with out-degrees corresponding to δ that contain a cyclic triangle is equal to the number of odd orientations with the same properties
- Lemma 2.3 $\Rightarrow$ precisely one orientation of G does not contain a cyclic triangle $\Rightarrow DE_G(\delta) - DO_G(\delta) = 1$ or -1.
- Lemma 2.4 $\Rightarrow$ An ϱ-legal colouring of G exists

Back to our original problem

Back to our original problem

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Corollary

$\chi_\ell(S_n) \leq n + 1$

Proof :

- S_{ij} set of size $n+1$ for $1 \leq i, j \leq n$
- Set $S_{i,n+1} = S_{i,n}$ for $1 \leq i, j \leq n$
- Theorem 2 $\Rightarrow$ An n x $(n+1)$ latin rectanle L exists with $L_{ij} \in S_{ij}$ for all $1 \leq i \leq n, 1 \leq j \leq n+1$
- Delete the last column of L and obtain S partial latin square with desired property

Back to our original problem

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Bounds for $\chi_\ell(S_n)$

We now know that $\chi_\ell(S_n) \in \{n, n+1\}$.

To go beyond that, let us first fix some notation for a graph $G = (V, E)$.

- ***d(v)*** is the degree of v (v vertex of G)
- For a subset $A \subseteq V$ is G_A the subgraph induced by A
- $\overrightarrow{G} = (V, E)$ is an orientation of G
- For $\overrightarrow{G}$, we denote by $d^+(v)$ the outdegree and by $d^-(v)$ the indegree of v. Also $d^+(v) + d^-(v) = d(v)$

Theorem 3

$\chi_\ell(S_n) = n$

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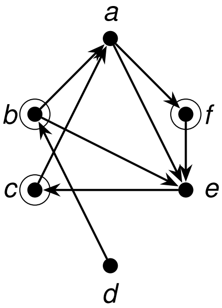
$\chi_\ell(S_n) = n$

Kernel

Definition : Kernel

Let $\overrightarrow{G} = (V, E)$ be a directed graph. A *kernel* $K \subseteq V$ is a subset of the vertices such that

1. K is independent in G
2. for every $u \notin K$ there exists a vertex $v \in K$ with an edge $u \rightarrow v$



Example

In the figure on the left, the set b, c, f constitutes a kernel. $G_{\{a,c,e\}}$ however does not have a kernel since the 3 edges cycle through the vertices.

First step to the solution

Lemma 3.1

Let $\overrightarrow{G} = (V, E)$ be a directed, and suppose that for each vertex $v \in V$ we have a colour set $C(v)$ that is larger that the outdegree $C(v) \geq d^+(v) + 1$. If every induced subgraph of $\overrightarrow{G}$ possesses a kernel, then there exists a list colouring of G with a colour from $C(v)$ for each v.

As the following example shows, the concept of kernel here is important.

- We have $max(d^+(v)) \forall v \in V = 2$.
- However $\chi(G) = 4$, where G represents the graph on the left without orientations.

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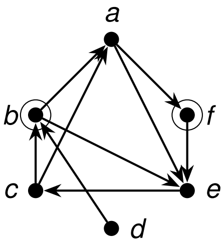

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Lemma 3.1 (Proof)

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Proof of Lemma 3.1

We proceed by induction on $|V|$.

1. For $|V| = 1$ there is nothing to prove
2. - Choose a colour $c \in C = \bigcup_{v \in V} C(v)$.
   - Set $A(c) := \{v \in V : c \in C(v)\}$.
     $G_{A(c)}$ possesses a kernel $K(c)$ (by hypothesis).
   - colour all $v \in K(c)$ with the colour c.
   - Set $C'(v) = C(v) \setminus \{c\}$ as the new list of colour sets of $G' := G_{V \setminus K(c)}$.
     Notice that for each $v \in A(c) \setminus K(c)$, $d^+(v)$ is decreased by at least $1 \Rightarrow d^+(v) + 1 \leq |C'(v)|$. The same goes for all $v \notin A(c)$ since $C(v)$ remains unchanged in that case.
   - G' contains fewer than vertices than G, and can be coloured by induction

Matchings

The point we strive to bend is now clear: we are done if we can find an orientation of S_n with outdegrees $d^+(v) \leq n - 1$ for all v and which ensures the existence of a kernel for all induced subgraphs.

For our purpose, we need the concept of “stable matchings”.

Definition : Matching

A *matching* M in a bipartite graph $G = (X \cup Y, E)$ is a set of edges such that no two edges in M have a common endvertex.

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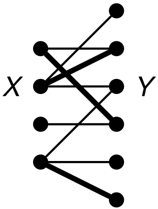

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Stable matching

To better understand stable matchings, we will use the following down-to-earth interpretation :

- X to be a set of men
- Y a set of women
- Interpret $uv \in E$ to mean that u and v might marry
- A matching is then a mass-wedding with no person committing bigamy.

In real life, every person has preferences. We assume that for every $v \in (X \cup Y)$ there is a ranking of the neighbourhood $N(v)$ of v,
$N(v) = \{z_1 > z_2 > ... > z_{d(v)}\}$.
A set of marriages is stable if it never happens that :

- u and v are not married
- u prefers v to his partner (if he has one at all)
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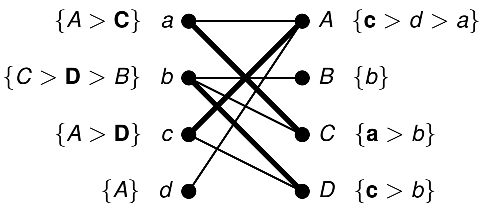
A matching M of $G = (X \cup Y, E)$ is called *stable* if the following condition holds : Whenever $uv \in E \setminus M, u \in X, v \in Y$, then either $uy \in M$ with $y > v$ in $N(u)$ or $xv \in M$ with $x > u$ in $N(v)$, or both.



Example

The bold edges constitute a stable matching. In each priority list, the choice leading to a stable matching is printed bold.

Existence of stable matchings

Lemma 3.2

A stable matching always exists.

Consider the following algorithm:

1. All men $u \in X$ propose their top choice
2. If a girl receives one proposal, keeps him on a string. If more than one, she keeps the one she likes best.
   The remaining men are rejected and form the reservoir R.
3. All men in R propose to their *next* choice
4. The women compare the proposals (with the one on the string, if there is one), pick their favorite and put hiim on the string. The rest is rejected and forms the new set R.
   A man who proposed to his last choice and was rejected again drops out from the reservoir.
5. If R is empty, the algorithm stops. Otherwise, it goes back to point 3.

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Lemma 3.2 (Proof)

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Proof of Lemma 3.2

When the algorithm described above stops, then the men on the strings together with the corresponding girls form a stable matching.

- The men on the string of a particular girl move there in increasing preference
- Hence if $uv \in E$ but $uv \notin M$, then we have 2 case scenarios:
  - u never proposed to v $\Rightarrow$ u found a *better* partner before he even got to v $\Rightarrow$ $uy \in M$ with $y > v$ in $N(u)$
  - u proposed to v but was rejected $\Rightarrow$ $xv \in M$ with $x > u$ in $N(v)$

  Both fulfill the condition of a stable matching

Final step

The Dinitz Problem

Consider n^2 cells arranged in an *(n x n)*-square, and let (i, j) denote the cell in row i and column j. Suppose that for every cell (i, j) we are given a set *C(i,j)* of n colours.

Is it then always possible to colour the whole array by picking for each cell *(i,j)* a colour from its set *C(i,j)* such that the colours in each row and each column are distinct?

Coup de grâce

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Theorem 3

We have $\chi_\ell(S_n) = n$ for all n.

Proof :

As before, we denote the vertices of S_n by (i,j), $1 \leq i,j \leq n$ and by L_{ij} the entry in cell (i,j).

We now make S_n into a directed graph $\overrightarrow{S_n}$ by

- orienting the horizontal edges $(i,j) \longrightarrow (i,j')$ **if** $L(i,j) < L(i,j')$
- orienting the vertical edges $(i,j) \longrightarrow (i',j)$ **if** $L(i,j) > L(i',j)$

In other words, we orient from the smaller to the larger horizontally, and the other way round.

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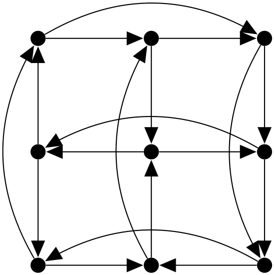
An example for $n = 3$

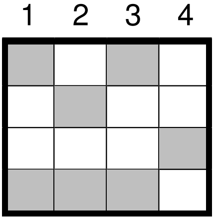
1	2	3
3	1	2
2	3	1

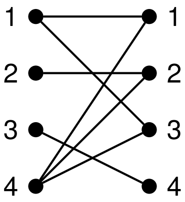


If $L_{ij} = k$, then $n - k$ cells in row i contain an entry larger than k and $k - 1$ cells in column j have an entry smaller than k $\Rightarrow d^{+}(i,j) = n - 1$ for all (i,j).

Coup de grâce

Consider a subset $A \subseteq V$ from S_n and let X be the set of rows of L, and Y the set of its columns.
Associate to A the bipartite graph $G = (X \cup Y, A)$, where every $(i, j) \in A$ is represented by the edge ij with $i \in X, j \in Y$. In the example in the margin, the cells of A are shaded.





Coup de grâce

The orientation on S_n induces a ranking on the neighbourhoods in $G = (X \cup Y, A)$:

- Set $j' > j$ in $N(i)$ if $(i, j) \longrightarrow (i, j')$ in $\overrightarrow{S}_n$
- Set $i' > i$ in $N(i)$ if $(i, j) \longrightarrow (i', j)$ in $\overrightarrow{S}_n$

Lemma 3.2 $\Rightarrow G = (X \cup Y, A)$ possesses a stable matching M.

M is a subset of A and a kernel :

- M is independent (Matching $\Rightarrow$ edges do not share an endvertex i or $j \Rightarrow$ cells are not adjacent
- If $(i, j) \in A \setminus M$, then one the two following holds (since M is stable):
  - $(i, j') \in M$ with $j' > j \Rightarrow (i, j) \longrightarrow (i, j') \in M$
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Bibliography

- M. Aigner, G. M. Ziegler *The Dinitz problem* 247-252.
- M. Aigner, G. M. Ziegler *Completing Latin Square* 238-244.
- J. C. M. Janssen *The Dinitz problem solved for rectangles state* Bulleton Amer. Math. Soc. (1993), 243-249.
- N. Alon and M. Tarsi *Colorings and Orientations of graphs* Combinatorica (1992), 125-134.