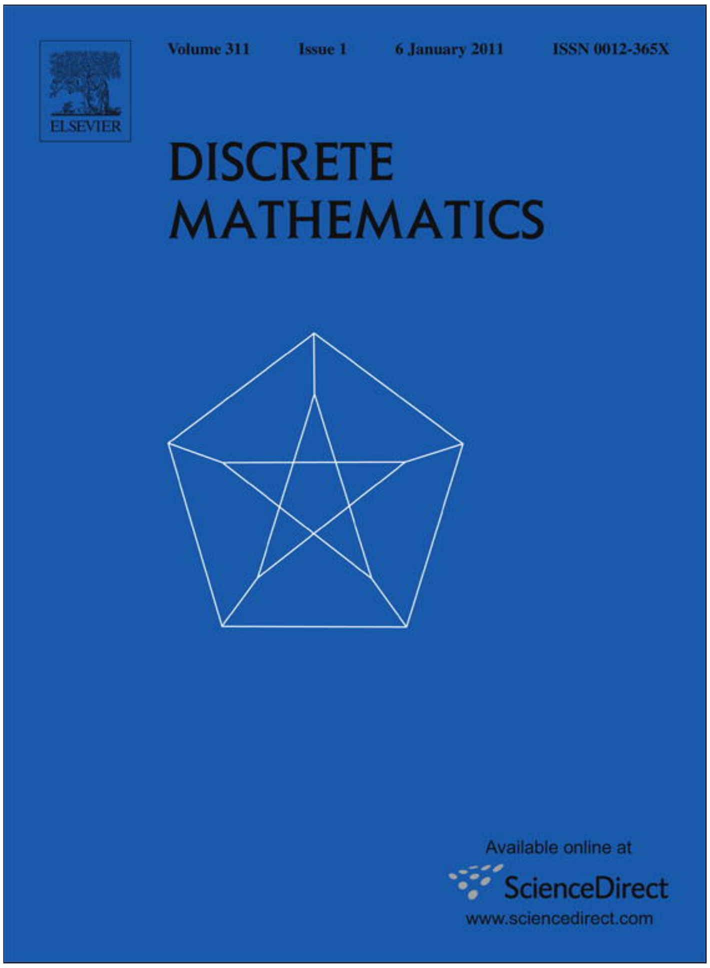

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Note

A note on the monotonicity of mixed Ramsey numbers

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ABSTRACT

For two graphs, G and H, an edge coloring of a complete graph is (G, H)-good if there is no monochromatic subgraph isomorphic to G and no rainbow subgraph isomorphic to H in this coloring. The set of numbers of colors used by (G, H)-good colorings of K_n is called a mixed Ramsey spectrum. This note addresses a fundamental question of whether the spectrum is an interval. It is shown that the answer is "yes" if G is not a star and H does not contain a pendant edge.

1. Introduction

Let G and H be two graphs on fixed numbers of vertices. An edge coloring of a complete graph K_n is called (G, H)-*good* if there is no monochromatic copy of G and no rainbow (totally multicolored) copy of H in this coloring. This, sometimes called *mixed Ramsey coloring*, is a hybrid of classical Ramsey [12] and anti-Ramsey [4] colorings. As shown by Jamison and West [10], a (G, H)-good coloring of an arbitrarily large complete graph exists unless G is a star or H is a forest.

Let $S(n; G, H)$ be the set of the numbers k, such that there is a (G, H)-good k-coloring of K_n. Here, a k-coloring is a coloring using exactly k colors. We call $S(n; G, H)$ a *spectrum*. Let $\max S(n; G, H)$ and $\min S(n; G, H)$ be the maximum number and minimum number in $S(n; G, H)$, respectively. The behaviors of these functions have been studied by [2,6,1] among others. Note that, if there were no restrictions on graphs G or H, then the spectrum would be an interval. For a graph G on at least $n + 1$ vertices, $S(n; G, H) = \left[1, AR(n, H)\right]$, where $AR(n, H)$ is the classical anti-Ramsey number; for a graph H on at least $n+1$ vertices, $S(n; G, H) = \left[k, \binom{n}{2}\right]$, where $k = k(n, G)$ is the largest number such that $n \geq r_{k-1}(G)$, the classical multicolor Ramsey number.

The main question investigated in this note is whether the same behavior continues to hold for mixed Ramsey colorings. Specifically, for a given integer n, and for graphs G and H, is $S(n; G, H)$ an interval? When G is not a star, for most graphs H, we show that $S(n; G, H)$ is an interval.

Theorem 1. *Let G be a graph that is not a star, and let H be a graph with minimum degree at least 2. Then, for any natural number n, $S(n; G, H)$ is an interval.*

The simplest connected graph H that is not a tree and that has a vertex of degree 1 is $K_3 + e$, a 4-vertex graph obtained by attaching a pendant edge to a triangle. We show that $S(n; G, K_3 + e)$ could have a gap for some graphs G and some values of n. However, when n is arbitrarily large, we do not have a single example of a graph G and a graph H for which $S(n; G, H)$ is not an interval.

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The next theorem is a collection of results on $S(n; G, K_3+e)$. Here, ℓK_2 is a matching of size ℓ, C_4 is a 4-cycle, and P_4 is a path on four vertices.

Theorem 2. • $S(n; \ell K_2, K_3) = S(n; \ell K_2, K_3+e) = \left[\lceil \frac{n-2\ell+1}{\ell-1} \rceil + 1, n-1\right]$ *for* $n \geq 4$ *and* $\ell \geq 2$,
$S(n; P_4, K_3) = S(n; P_4, K_3+e) = [n-2, n-1]$ *for* $n \geq 4$,
$S(n; C_4, K_3) = S(n; C_4, K_3+e) = [n-3, n-1]$ *for* $n \geq r_3(C_4) = 11$,
$S(n; K_3, K_3) = S(n; K_3, K_3+e) = [c \log n + c', n-1]$ *for* $n \geq r_3(K_3) = 17$, *where* c *and* c' *are constants,*
$S(n; K_{1,\ell}, K_3) = S(n; K_{1,\ell}, K_3+e) = \emptyset$ *for* $n \geq 3\ell+1$ *and* $\ell \geq 2$.
• $S(10; C_4, K_3+e) = \{3, 7, 8, 9\}$.

Corollary 3. *If* $\ell \geq 2$ *and* $n \geq \max\{17, 3\ell+1\}$, *then* $S(n; G, K_3+e)$ *is an interval for any* $G \in \{\ell K_2, K_3, P_4, C_4, K_{1,\ell}\}$. *However,* $S(n; G, K_3+e)$ *is not an interval if* $n = 10$ *and* $G = C_4$.

Open question. Are there graphs G and H such that for any natural number N there is $n > N$ so that $S(n; G, H)$ is not an interval?

2. Definitions and proofs of main results

For an edge coloring c of K_n and a vertex $x \in V(K_n)$, let $N_c(x)$ be the set of colors used only on edges incident to x, and for $X \subseteq V(K_n)$ let $c(X)$ be the set of colors used on edges induced by X. Let $|c|$ denote the number of colors used in the coloring c. Then $|c| = |N_c(x)| + |c(V(K_n) - x)|$ for any $x \in V(K_n)$.

Observation 1. If G is not a star, and A and B are color classes that are stars with the same center in a (G, H)-good k-coloring c of K_n, then replacing A and B in c with a new color class $A \cup B$ gives a (G, H)-good $(k-1)$-coloring.

Observation 2. For any graphs G and H,

$$\min S(n; G, H) \leq \min S(n+1; G, H).$$

Proof. Consider a (G, H)-good coloring of K_{n+1} with k colors. Delete one vertex to get a (G, H)-good coloring of K_n with $k' \leq k$ colors.

Observation 3. For $G \subseteq G'$ and $H \subseteq H'$,

$$S(n; G, H) \subseteq S(n; G', H) \subseteq S(n; G', H'), \quad \text{and} \quad S(n; G, H) \subseteq S(n; G, H') \subseteq S(n; G', H').$$

Proof. If there is no monochromatic G and no rainbow H in a coloring of $E(K_n)$, then there is no monochromatic G' and no rainbow H' in this coloring. □

Observation 4. If G is not a star, H has minimum degree at least 2, and $k \in S(n; G, H)$, then $k+1 \in S(n+1; G, H)$.

Proof. Consider a (G, H)-good k-coloring of K_n. Add a new vertex x, and color edges incident to x by a new color to get a (G, H)-good $(k+1)$-coloring of K_{n+1}. □

Proof of Theorem 1. We prove that $[\min S(n; G, H), \max S(n; G, H)] \subseteq S(n; G, H)$ using induction on n. When $n = 2$, every coloring uses one color.

Let $n \geq 3$. Consider the smallest k such that $[k, \max S(n; G, H)] \subseteq S(n; G, H)$. First, note that, for any (G, H)-good k-coloring c of K_n and any vertex x, we have $|N_c(x)| \leq 1$; otherwise, by applying Observation 1, we obtain a (G, H)-good $(k-1)$-coloring of K_n, which contradicts the minimality of k. Consider a (G, H)-good k-coloring of K_n and any vertex x, and delete it. We now have a (G, H)-good coloring of K_{n-1} with k or $k-1$ colors. Here, we note that $\max S(n-1; G, H) \geq k-1$. By induction hypothesis, $S(n-1; G, H)$ is an interval, i.e. $[\min S(n-1; G, H), \max S(n-1; G, H)] = S(n-1; G, H)$. Then, by Observation 4, $[\min S(n-1; G, H)+1, \max S(n-1; G, H)+1] \subseteq S(n; G, H)$. Since $\min S(n; G, H) \geq \min S(n-1; G, H)$ from Observation 2, $[\min S(n; G, H), \max S(n-1; G, H)+1] \subseteq S(n; G, H)$. Since $k \leq \max S(n-1; G, H)+1$ and $[k, \max S(n; G, H)] \subseteq S(n; G, H)$, we finally have that $[\min S(n; G, H), \max S(n; G, H)] \subseteq S(n; G, H)$. □

For the proof of Theorem 2, we shall use the function

$$f(k; G, H) := \max\{n : \text{there is a } (G, H)\text{-good coloring of } K_n \text{ using exactly } k \text{ colors}\}.$$

Note that, if $f(k; G, H) = n$, then $\min S(n; G, H) \leq k$.

Observation 5. If $f(k; G, H) = n$ and $f(\tilde{k}; G, H) < n$ for any $\tilde{k} < k$, then $\min S(n; G, H) = k$. In particular, if f is strictly increasing in k, then $f(k; G, H) = n$ implies that $\min S(n; G, H) = k$.

Proof of Theorem 2. First, observe that $\max S(n; G, H) \le AR(n, H)$, where $AR(n, H)$ is the *classical anti-Ramsey number*, the maximum number of colors in an edge coloring of K_n with no rainbow subgraphs isomorphic to H. If G is not a star, then $\max S(n; G, K_3) = AR(n, K_3) = n - 1$; see [2]. Moreover, from Observation 3, we obtain that $\max S(n; G, K_3) \le \max S(n; G, K_3 + e)$; and from [7], we know that $AR(n, K_3) = AR(n, K_3 + e)$ for $n \ge 4$. Thus, when G is not a star, $\max S(n; G, K_3) = \max S(n; G, K_3 + e) = n - 1$ for $n \ge 4$.

Therefore, for $n \ge 4$, if $\min S(n; G, K_3) = \min S(n, G, K_3+e)$ and G is not a star, then we can conclude that $S(n; G, K_3+e) = S(n; G, K_3)$, which is an interval by Theorem 1. Next, we shall analyze $\min S(n, G, K_3+e)$. We note that $f(k; G, H)+1 \le r_k(G)$, where $r_k(G)$ denotes the *classical k-color Ramsey number* for G. The equality holds if there is a k-coloring of $E(K_{r_k(G)-1})$ with no monochromatic G and no rainbow H.

Case 1. $G = \ell K_2$

From [11], we have that $r_k(\ell K_2) = (k-1)(\ell-1) + 2\ell$. The extremal coloring providing this Ramsey number can be constructed as follows. Starting with a complete graph on $2\ell - 1$ vertices, whose edges are colored entirely with color 1, add $\ell - 1$ vertices and color all edges incident to these vertices with color 2. Then add another $\ell - 1$ vertices and color all edges incident to these vertices with color 3. Repeat this process until we get a k-coloring of a complete graph on $2\ell - 1 + (k-1)(\ell-1)$ vertices with no monochromatic ℓK_2. Note that this coloring contains no rainbow cycles; thus, it contains neither a rainbow copy of K_3 nor a rainbow copy of $K_3 + e$. Hence, $f(k; \ell K_2, H) = f(k; \ell K_2, H + e) = (k-1)(\ell-1) + 2\ell - 1$ for any H that is not a forest. By Observation 5, $\min S(n; \ell K_2, H) = \min S(n; \ell K_2, H + e)$. In particular, for $\ell \ge 2$, $\min S(n; \ell K_2, K_3) = \min S(n; \ell K_2, K_3 + e) = \lceil \frac{n-2\ell+1}{\ell-1} \rceil + 1$.

Case 2. $G \in \{K_3, P_4, C_4\}$

From [3,2,8,5,6], we have that (i) $f(k; K_3, K_3) = f(k; K_3, K_3 + e) = \lambda(k)$ for $k \ge 1$ and $k \ne 3$, $f(3; K_3, K_3) = 10$, and $f(3; K_3, K_3 + e) = r_3(K_3) - 1 = 16$, where $\lambda(k) = 5^{k/2}$ if k is even, $\lambda(k) = 2 \cdot 5^{(k-1)/2}$ if k is odd; (ii) $f(k; P_4, K_3) = f(k; P_4, K_3 + e) = k + 2$ for $k \ge 1$; and (iii) $f(k; C_4, K_3) = f(k; C_4, K_3 + e) = k + 3$ for $k = 2$ or $k \ge 4$, $f(3; C_4, K_3) = 6$, and $f(3; C_4, K_3+e) = r_3(C_4) - 1 = 10$. Therefore, from Observation 5, we conclude that (i) $\min S(n; K_3, K_3) = \min S(n; K_3, K_3 + e) = c \log n + c'$ for $n \ge r_3(K_3) = 17$ and some constants c, c'; (ii) $\min S(n; P_4, K_3) = \min S(n; P_4, K_3+e) = n-2$ for $n \ge 4$; and (iii) $\min S(n; C_4, K_3) = \min S(n; C_4, K_3 + e) = n - 3$ for $n \ge r_3(C_4) = 11$. Thus, $\min S(n; G, K_3) = \min S(n; G, K_3 + e)$ for $G \in \{K_3, P_4, C_4\}$ and $n \ge r_3(G)$.

Case 3. $G = K_{1,\ell}$

In [9], it was shown that any coloring of $E(K_n)$ with no rainbow triangles has a monochromatic star $K_{1,2n/5}$. Using this fact and the pigeonhole principle, we easily see that any coloring of $E(K_n)$ with no rainbow $K_3 + e$ has a monochromatic star $K_{1,(n-1)/3}$. Namely, let c be a coloring of $E(K_n)$ with no rainbow $K_3 + e$. Since $2n/5 \ge (n-1)/3$, we may assume that there is a rainbow copy T of K_3. To avoid a rainbow $K_3 + e$ in this coloring, the edges between $V(T)$ and $V(K_n) - V(T)$ have colors only presented on the edges of T. Thus, for any vertex x in T, the number of colors used on edges incident to x is at most three. By the pigeonhole principle, we can find a monochromatic star $K_{1,s}$ with vertex x and $s \ge (n-1)/3$ vertices in $V(K_n) - x$. This is sharp, as the referee remarked (one can also find it in [6]). Consider a complete graph K_n and partition $V(K_n)$ into four subsets V_0, V_1, V_2, and V_3, with $|V_0| = 1$ and $|V_i| = (n-1)/3$ for $1 \le i \le 3$. Color all edges induced by $V_0 \cup V_i$ with color i for $1 \le i \le 3$, and all edges between V_j and V_k with color i for $\{i, j, k\} = \{1, 2, 3\}$. In this coloring, there is no monochromatic star $K_{1,s}$ with $s > (n-1)/3$. Therefore, $S(n; K_{1,\ell}, K_3) = S(n; K_{1,\ell}, K_3 + e) = \emptyset$ if $n \ge 3\ell + 1$.

Summarizing Cases 1, 2, and 3, we have that $S(n; G, K_3) = S(n; G, K_3+e)$ is an interval if G is one of $\{\ell K_2, K_3, P_4, C_4, K_{1,\ell}\}$ and $n \ge N$, where N is a constant depending only on G. This concludes the proof of the first part of Theorem 2.

Consider the case when $G = C_4$, $H = K_3 + e$, and $n = 10$. Since $r_2(C_4) = 6 < 10$, we see that there is no $(C_4, K_3 + e)$-good coloring of K_{10} in two colors. On the other hand, since $r_3(C_4) = 11$, there is a $(C_4, K_3 + e)$-good coloring of K_{10} in three colors. Thus $\min S(10; C_4, K_3 + e) = 3$. We also have that $\max S(10; C_4, K_3 + e) = AR(10, K_3) = 9$. Since $f(k; C_4, K_3 + e) = k + 3 < 10$ for $4 \le k \le 6$, there is no $(C_4, K_3 + e)$-good coloring of K_{10} with 4, 5, or 6 colors. To construct an 8-colorings and a 7-coloring of K_{10} with no rainbow $K_3 + e$ and no monochromatic C_4, consider a vertex set $\{v_1, \ldots, v_{10}\}$. Let $c(v_iv_j) = i$, $1 \le i \le 7$, $i < j$; $c(v_8v_9) = c(v_8v_{10}) = c(v_9v_{10}) = 8$. Let $c'(v_iv_j) = i$, $1 \le i \le 5$, $i < j$; $c'(v_6v_7) = c'(v_7v_8) = c'(v_8v_9) = c'(v_9v_{10}) = c'(v_{10}v_6) = 6$, and all other edges get color 7 under c'. Note that c and c' are an 8-coloring and a 7-coloring, respectively, containing no rainbow K_3 and no monochromatic C_4. Therefore, $S(10; C_4, K_3 + e) = \{3, 7, 8, 9\}$. □

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