

On the editing distance of graphs

Maria Axenovich* André Kézdy† Ryan Martin‡§

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Abstract

An edge-operation on a graph G is defined to be either the deletion of an existing edge or the addition of a nonexisting edge. Given a family of graphs \mathcal{G} , the editing distance from \mathcal{G} to G is the smallest number of edge-operations needed to modify G into a graph from \mathcal{G} . In this paper, we fix a graph H and consider $\text{Forb}(n, H)$, the set of all graphs on n vertices that have no induced copy of H . We provide bounds for the maximum over all n -vertex graphs G of the editing distance from $\text{Forb}(n, H)$ to G , using an invariant we call the *binary chromatic number* of the graph H . We give asymptotically tight bounds for that distance when H is self-complementary and exact results for several small graphs H .

1 Introduction

The investigation of graphs not containing subgraphs with given properties is a classical problem. For example, determining the maximum number of edges in a graph with no copy of a fixed subgraph H has been studied intensively for the last 70 years [13, 19]. Very often, though, the desired task is not to determine the extremal graph without a given fixed subgraph, but rather to start with an arbitrary graph and modify it in a small number of steps such that the resulting graph does not contain a forbidden subgraph.

The problem of modifying the given graph such that the resulting graph satisfies some *global properties* has been addressed by Erdős et al. [9, 7, 10]. They investigated the number of edge deletions sufficient to transform an arbitrary triangle-free graph into a bipartite graph, as well as the smallest number of edge additions sufficient to decrease diameter.

In this paper, we investigate the problem of transforming a given graph into a new graph having the *local property* of avoiding a fixed induced subgraph. Starting with an arbitrary graph G , we would like to calculate the minimum number of edges needed to be added to or deleted from G to obtain a graph not containing a fixed induced subgraph. Formally, an **edge-operation** is defined to be either the deletion of an existing edge or the addition of

*Department of Mathematics, Iowa State University, Ames, IA 50011, axenovic@math.iastate.edu

†Department of Mathematics, University of Louisville, Louisville, KY 40292, kezdy@louisville.edu

‡Department of Mathematics, Iowa State University, Ames, IA 50011, rymartin@iastate.edu

§Corresponding author.

a nonexisting edge. Let $\text{Dist}(G, H)$ denote the minimum number of edge-operations needed to transform the graph G into a graph isomorphic to H . In other words, it can be described as a symmetric difference as follows:

$$\text{Dist}(G, H) = \min\{|E(G) \Delta E(H')| : H' \cong H\}.$$

Clearly this parameter is defined if and only if G and H have the same number of vertices. If \mathcal{H} is a class of graphs on n vertices, we define $\text{Dist}(G, \mathcal{H}) = \min\{\text{Dist}(G, H) : H \in \mathcal{H}\}$ for a graph G on n vertices. Finally, $\text{Dist}(n, \mathcal{H}) = \max\{\text{Dist}(G, \mathcal{H}) : |V(G)| = n\}$. We call the metric $\text{Dist}(G, H)$ the **editing distance** since the operations performed can be considered as editing the edge set of a graph. Our interest here is the class \mathcal{H} of graphs on n vertices containing no copies of a given fixed graph H as an induced subgraph. We denote this class by $\text{Forb}(n, H)$ (or simply by $\text{Forb}(H)$ when it is clear from the context). Similarly $\text{Forb}'(H)$ is the family of all graphs on n vertices with no subgraph isomorphic to H .

This graph editing problem has numerous applications in computer science and bioinformatics. For example, consider a metabolic network and identify genes with vertices of a graph and pairs of interacting genes with edges of the graph. It is a fundamental question in biology (from evolutionary and practical points of view) to find how many edge-changes in such a graph must be performed to avoid an induced subgraph corresponding to a certain metabolic process. Another example involves consensus trees. It is known that two consensus trees are comparable if there is no induced path on five vertices in a corresponding bipartite graph [20, 5, 6]. In particular, finding the smallest number of edge-changes in such a graph will determine the distance between these trees.

On the other hand, the editing problem of graphs corresponds to determining the distance between $\{0, 1\}$ -matrices. If A and B are the adjacency matrices of graphs G_1 and G_2 respectively, then $\text{Dist}(G_1, G_2)$ corresponds to the number of positions where A and B differ, i.e., to the Hamming distance between A and B . Thus finding editing distance between classes of graphs provides the Hamming distance between classes of symmetric matrices with the same diagonal entries. Moreover, when the graph editing problem is restricted to bipartite graphs in which edge additions and deletions are limited to edges between partite sets, it corresponds to the problem of determining the distance between the sets of arbitrary $\{0, 1\}$ -matrices.

We define the distance $\text{Dist}'(n, \text{Forb}'(H))$ to be analogous to $\text{Dist}(n, \text{Forb}(H))$, but in this case, only permit edge-deletions. This quantity will always be equal to $\text{Dist}(K_n, \text{Forb}'(H))$, i.e., it is the minimum number of edges in the complement of an H -free graph. If $\text{ex}(n, H)$ is the maximum number of edges in a graph on n vertices with no subgraph isomorphic to H , then

$$\text{Dist}(K_n, \text{Forb}'(H)) = \binom{n}{2} - \text{ex}(n, H). \quad (1)$$

The asymptotic behavior of $\text{ex}(n, H)$ is provided by the following theorem, which was generalized by Erdős and Simonovits [11].

Theorem 1.1 (Erdős, Stone [12]) $\text{ex}(n, H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right) \binom{n}{2}$.

In particular, the distance $\text{Dist}(n, \text{Forb}'(H))$ is asymptotically determined by the chromatic number of a graph H .

Clearly, when a forbidden graph is complete or empty, finding the editing distance becomes a trivial task immediately reduced to Turán’s theorem [22]. On the other hand, perhaps the most interesting case is when the forbidden induced subgraph is self-complementary, i.e., when both operations of edge-deletions and edge-additions carry “an equal power”. In this case, we derive asymptotically tight estimates for $\text{Dist}(n, \text{Forb}(H))$. We also give general bounds for other graphs. Our main tool in providing the lower bounds is Szemerédi’s Regularity Lemma which allows us to express the bounds in terms of an invariant which we designate the **binary chromatic number**. In defining it, we use the term **coclique** in place of the term **independent set**.

Definition 1.2 *The binary chromatic number of a graph G , $\chi_B(G)$ is the least integer $k + 1$ such that, for all $c \in \{0, \dots, k + 1\}$, there exists a partition of $V(G)$ into c cliques and $k + 1 - c$ cocliques.*

This invariant was first introduced by Prömel and Steger (called τ in [18]) to express, asymptotically, the number of n -vertex graphs which fail to have an induced copy of some small, fixed graph H . In [4, 2], τ was generalized as the so-called colouring number of a hereditary property, \mathcal{P} . In particular, when $\mathcal{P} = \text{Forb}(H)$, the colouring number of \mathcal{P} is exactly $\chi_B(H) - 1$. It should be noted that χ_B is not the cochromatic number (see [17]) even though the definitions may seem to be similar at first glance. We use the term “binary chromatic number” in order to emphasize the close connection to the chromatic number and the complementary rôle that cliques and cocliques play.

The following comprise the main results of this paper.

Theorem 1.3 *If H is a graph with binary chromatic number $k + 1$, then*

$$\text{Dist}(n, \text{Forb}(H)) > (1 - o(1)) \frac{n^2}{4k}.$$

If $k = \chi_B(G) - 1$, then let c_{\min} be the least c so that G cannot be partitioned into c cliques and $k - c$ cocliques. Let c_{\max} be the greatest such number. We now have an upper bound that can be expressed in terms of the binary chromatic number of H and corresponding c_{\min} and c_{\max} .

Theorem 1.4 *Let H be a graph with binary chromatic number $k + 1$ and c_{\min} and c_{\max} be defined as above. If $c_{\min} \leq k/2 \leq c_{\max}$, then*

$$\text{Dist}(n, \text{Forb}(H)) \leq \frac{1}{2k} \binom{n}{2}. \tag{2}$$

Otherwise, let c_0 be the one of $\{c_{\max}, c_{\min}\}$ that is closest to $k/2$. Then

$$\text{Dist}(n, \text{Forb}(H)) \leq \left(\frac{1}{1 + 2\sqrt{\frac{c_0}{k} \left(1 - \frac{c_0}{k}\right)}} \right) \frac{1}{k} \binom{n}{2} \leq \frac{1}{k} \binom{n}{2}. \tag{3}$$

Corollary 1.5 *If H is a self-complementary graph with the property that $\chi_B(H) = k + 1$, then*

$$\text{Dist}(n, \text{Forb}(H)) = (1 + o(1)) \frac{n^2}{4k}.$$

In Section 2 we give preliminary definitions and results that aid in the proof of the theorems. Section 3 contains proofs of the main theorems. We investigate the properties of the binary chromatic number in Section 4. Finally, Section 5 gives several exact results.

2 Definitions and preliminary results

We denote by K_n , E_n , C_n , and P_n a complete graph, an empty graph, a cycle, and a path on n vertices, respectively. We also define K_p^q to be a complete p -partite graph with each partite set of cardinality q . We use \overline{G} to denote the complement of G . For the other definitions, we refer the reader to [23].

Definition 2.1 *For a graph $G = (V, E)$ and two disjoint subsets A and B of vertices, the density of a pair (A, B) is denoted $d(A, B)$ and is given by the formula*

$$d(A, B) = \frac{e(A, B)}{|A||B|}$$

where $e(A, B)$ is the number of edges of G with one end-point in A and another in B .

Definition 2.2 *For a graph $G = (V, E)$ and two disjoint subsets A and B of vertices, a pair (A, B) is ϵ -regular if*

$$X \subset A, Y \subset B, |X| > \epsilon|A|, |Y| > \epsilon|B|$$

imply

$$|d(X, Y) - d(A, B)| < \epsilon;$$

otherwise, (A, B) is ϵ -irregular.

The proof of Theorem 2.7 makes use of the Regularity Lemma (see [16] and [15]).

Lemma 2.3 (Regularity Lemma [21]) *For every positive ϵ and positive integer m , there are positive integers $M = M(\epsilon, m)$ and $N = N(\epsilon, m)$ with the following property: For every graph G with at least N vertices there is a partition of the vertex set into $\ell + 1$ classes (clusters)*

$$V = V_0 \dot{\cup} V_1 \dot{\cup} V_2 \dot{\cup} \cdots \dot{\cup} V_\ell$$

such that

1. $m \leq \ell \leq M$,
2. $|V_1| = |V_2| = \cdots = |V_\ell|$,

3. $|V_0| < \epsilon n$,
4. at most $\epsilon \ell^2$ of the pairs (V_i, V_j) are ϵ -irregular.

We give a name to the partition given by Lemma 2.3:

Definition 2.4 An (m, ϵ, ℓ) -**equipartition** of a vertex set V is a partition $V = V_0 \dot{\cup} V_1 \dot{\cup} \dots \dot{\cup} V_\ell$ such that the Regularity Lemma's conditions (1), (2) and (3) are satisfied (with $M = M(\epsilon, m)$ as defined by the Lemma).

In order to state our lower bound, we need to generalize the idea of an ϵ -regular pair.

Definition 2.5 An ϵ -**regular r -tuple** is an r -partite graph with partite sets V_1, \dots, V_r such that $|V_i| = |V_j|$ and (V_i, V_j) is an ϵ -regular pair for all i, j , with $1 \leq i < j \leq r$.

We say that an ϵ -regular r -tuple is **of size rL** if $|V_1| = \dots = |V_r| = L$. For $0 < \delta < 1/2$, an ϵ -regular r -tuple has **δ -bounded density** if $d(V_i, V_j) \in (\delta, 1 - \delta)$ whenever $1 \leq i < j \leq r$.

For convenience, we define an (ϵ, r, L, δ) -**configuration** to be an ϵ -regular r -tuple of size rL that has δ -bounded density.

The following Theorem is our major tool in proving the main result. We prove it in Section 3.

Theorem 2.6 Let r be a positive integer and δ and ϵ be real numbers, $0 < \delta < 1$ and $\epsilon > 0$, such that $\epsilon < \delta/(16r - 16)$. There is a graph G on n vertices and a constant $M(\epsilon)$ such that if the number of edge-deletions and edge-additions performed on G is less than $\frac{n^2}{4(r-1)}(1 - 3\delta)(1 - \epsilon)^2$ then the resulting graph contains an (ϵ, r, L, δ) -configuration with $L \geq n(1 - \epsilon)/M(\epsilon)$.

The following theorem is essentially Lemma 3.5 in [18]. We shall use it to prove Theorem 1.3.

Theorem 2.7 (Prömel, Steger [18]) Let H be a fixed graph with binary chromatic number r and δ be a real number with $0 < \delta < 1/2$. There exists an $\epsilon_0 = \epsilon_0(H, \delta, r) > 0$ such that for all ϵ , where $0 < \epsilon \leq \epsilon_0$, there exists an $n_0 = n_0(H, \delta, \epsilon, r)$ such that every graph $G = (V, E)$ on $n \geq n_0$ vertices has the following property: Let $V = V_0 \dot{\cup} V_1 \dot{\cup} \dots \dot{\cup} V_\ell$ be an (r, ϵ, ℓ) -regular partition with $|V_1| = \dots = |V_\ell| = L$ for $L \geq n/M(\epsilon, r)$ where $M(\epsilon, r)$ is the constant given in Lemma 2.3 for ϵ and r , such that (V_1, \dots, V_r) forms an (ϵ, r, L, δ) -configuration. Then, the subgraph induced by $\bigcup_{i=1}^r V_i$ contains the graph H as an induced subgraph.

For functions $f = f(n)$ and $g = g(n)$, let $f = \omega(g)$ be the usual asymptotic notation denoting that $g/f \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 2.8 For a positive real number ϵ and positive integer m , let ℓ have the property that $m \leq \ell \leq M(\epsilon, m)$, where $M(\epsilon, m)$ is the constant given in the Regularity Lemma (Lemma 2.3). Let $f = f(n) = \omega(n^{-1/2})$. Then, for n large enough, there is a graph G on n vertices so that for any (m, ϵ, ℓ) -equipartition, all pairs of clusters (V_i, V_j) , $1 \leq i < j \leq \ell$, are ϵ -regular with density in the interval $(1/2 - f, 1/2 + f)$.

The proof of this Lemma is a routine calculation that we include in the Appendix, for completeness.

3 The proofs

In order to prove Theorem 1.3, we prove Theorem 2.6, which basically asserts that a graph described in Lemma 2.8 requires many editing operations to eliminate all induced copies of H .

3.1 Proof of Theorem 2.6

Fix δ such that $0 < \delta < 1/2$. Let G be a graph on n vertices as described in Lemma 2.8 with ϵ , r and L as given in that Lemma. Let G' be a graph with no (ϵ, r, L, δ) -configuration having least distance from G .

Apply the Regularity Lemma to G' with parameters ϵ and $m = \epsilon^{-1}$, to get $\ell + 1$ clusters (we have $\ell \geq \epsilon^{-1}$) V_0, V_1, \dots, V_ℓ such that $|V_1| = \dots = |V_\ell| = L$. Furthermore, all but $\epsilon\ell^2$ pairs (V_i, V_j) , $1 \leq i < j \leq \ell$ are ϵ -regular. Recalling Definition 2.5, we say that an ϵ -regular pair is δ -**bounded** if its density is at least δ and at most $1 - \delta$; otherwise, it is δ -**unbounded**.

Since G' does not have an (ϵ, r, L, δ) -configuration, it is not possible to have a set of r clusters such that between any two clusters, there is a δ -bounded ϵ -regular pair. Thus, according to Turán's theorem, the number of pairs of clusters, (V_i, V_j) , that induce either a δ -unbounded ϵ -regular pair or an ϵ -irregular pair is at least

$$(r-1) \frac{\frac{\ell}{r-1} \left(\frac{\ell}{r-1} - 1 \right)}{2} = \frac{\ell(\ell - r + 1)}{2(r-1)}.$$

The number of ϵ -irregular pairs, (V_i, V_j) , is at most $\epsilon\ell^2$, thus the number of δ -unbounded ϵ -regular pairs is at least

$$\ell^2 \left(\frac{1}{2(r-1)} - \epsilon \right) - \frac{\ell}{2}.$$

Because G came from Lemma 2.8, if some pair (V_i, V_j) were δ -unbounded in G' , then at least $(1/2 - \delta - o(1))L^2$ edges had to be either added or deleted between V_i and V_j in order to get G' from G . Hence, the total number of edges that had to be changed is at least

$$\ell^2 L^2 \left(\frac{1}{2(r-1)} - \epsilon - \frac{1}{2\ell} \right) \left(\frac{1}{2} - \delta - o(1) \right) \geq \frac{\ell^2 L^2}{4(r-1)} (1 - 3\delta).$$

The inequality is valid as long as $8(r-1)(\epsilon + \ell^{-1}/2) < \delta$.

Since $\ell L \geq n(1 - \epsilon)$, the total number of edges that have to be altered in order to obtain G' from G is at least

$$\frac{n^2}{4(r-1)} (1 - 3\delta)(1 - \epsilon)^2.$$

■

3.2 Proof of Theorem 1.3

Choose a δ arbitrarily small, and let G be the graph guaranteed by Theorem 2.6. If fewer than $\frac{n^2}{4k}(1-3\delta)(1-\epsilon)^2$ edge-operations are performed on G to obtain a graph G' , then there are disjoint vertex sets V_1, \dots, V_{k+1} in G' that satisfy the conditions of Theorem 2.7. Theorem 2.7 then implies that G' contains an induced H . Thus the editing distance $\text{Dist}(G, \text{Forb}(H))$ is at least $\frac{n^2}{4k}(1-3\delta)(1-\epsilon)^2$. ■

3.3 Proof of Theorem 1.4

Lemma 3.1 emphasizes the importance of c in the definition of χ_B .

Lemma 3.1 *Let H be a graph with binary chromatic number $k+1$ and c be an integer, $0 \leq c \leq k$, so that H cannot be covered by exactly c cliques and $k-c$ independent sets. Let G be a graph with density $d = e(G)/\binom{n}{2}$. As long as it is not the case that both $d = 0$ and $c = k$, or both $d = 1$ and $c = 0$,*

$$\text{Dist}(G, \text{Forb}(H)) \leq \frac{d(1-d)}{dc + (1-d)(k-c)} \binom{n}{2}. \quad (4)$$

Otherwise, $\text{Dist}(G, \text{Forb}(H)) \leq \frac{1}{k} \binom{n}{2}$.

Proof. In order to prove the statement of the Lemma, we provide a probabilistic algorithm adding and deleting some edges of G such that the resulting graph has no induced copy of H . We begin by assigning colors independently to the vertices of G : $1, \dots, c$ each with probability p and $c+1, \dots, k$ each with probability q . Call such a coloring g . If $g(x) = g(y) \in \{1, \dots, c\}$ and $xy \notin E(G)$, then add an edge xy to $E(G)$. If $g(x) = g(y) \in \{c+1, \dots, k\}$, and $xy \in E(G)$, then delete xy from $E(G)$. As a result, we obtain a graph G' with the vertex set partitioned into k subsets. The first c of these subsets induce cliques and the others induce independent sets. Since the vertices of H can not be partitioned into c cliques and $k-c$ cocliques, H is not an induced subgraph of G' .

The expected number of changes is

$$f(p, q) = \left(\binom{n}{2} - e(G) \right) cp^2 + e(G)(k-c)q^2 = ((1-d)cp^2 + d(k-c)q^2) \binom{n}{2}.$$

We also have the restriction

$$cp + (k-c)q = 1. \quad (5)$$

As long as we do not have the case that both $d = 0$ and $c = k$ or the case that both $d = 1$ and $c = 0$, the method of Lagrange multipliers gives that the minimum of $f(p, q)$ restricted to (5) occurs when $p = d/(dc + (1-d)(k-c))$ and $q = (1-d)/(dc + (1-d)(k-c))$ and is equal to

$$\frac{d(1-d)}{dc + (1-d)(k-c)} \binom{n}{2}.$$

Since this is the expected number of changes, there exists a partition of the vertices of G such that the above procedure requires at most $\frac{d(1-d)}{dc+(1-d)(k-c)} \binom{n}{2}$ changes to make the graph H -free.

If both $d = 0$ and $c = k$, then perform the above procedure, but fix $p = 1$. If both $d = 1$ and $c = 0$, then perform the above procedure, but fix $q = 1$. In both cases, the expected number of changes to be performed is $\frac{1}{k} \binom{n}{2}$. ■

In order to prove inequality (2) of Theorem 1.4, we use Lemma 3.1 and find conditions when

$$\frac{d(1-d)}{dc+(1-d)(k-c)} \leq \frac{1}{2k}. \quad (6)$$

If $c \leq k/2$, then (6) holds when $d \in [0, 1/2] \cup [1 - c/k, 1]$. If $c \geq k/2$, then (6) holds when $d \in [0, 1 - c/k] \cup [1/2, 1]$. Consider a graph G of density d and an H for which $c_{\min} \leq k/2 \leq c_{\max}$. If $d \leq 1/2$, then choose c_{\min} ; otherwise, choose c_{\max} . As a result, $\text{Dist}(G, \text{Forb}(H)) \leq \frac{1}{2k} \binom{n}{2}$.

In order to prove inequality (3) of Theorem 1.4 we need to maximize expression (4) over d . The maximum value occurs when $d = \frac{k-c-\sqrt{c(k-c)}}{k-2c}$ and is

$$\frac{k-2\sqrt{c(k-c)}}{(k-2c)^2} \binom{n}{2} = \left(\frac{1}{1+2\sqrt{\frac{c}{k}(1-\frac{c}{k})}} \right) \frac{1}{k} \binom{n}{2}.$$

The expression in parentheses is at most 1. ■

3.4 Proof of Corollary 1.5

Let H be a self-complementary graph with $\chi_B(H) = k + 1$ such that c_{\min} and c_{\max} are defined as in preparation for Theorem 1.4. That is, c_{\min} is the least c so that G cannot be partitioned into c cliques and $k - c$ cocliques. The quantity c_{\max} is the greatest such c .

Because $\overline{H} = H$, H can be partitioned into c cliques and $k - c$ cocliques if and only if H can be partitioned into $k - c$ cliques and c cocliques. Hence, $c_{\max} = k - c_{\min}$ and it must be the case that $c_{\min} \leq k/2 \leq c_{\max}$. Now, from Theorem 1.3 and the first inequality of Theorem 1.4, the result follows. ■

4 Binary chromatic number

It is easy to see the following.

Fact 4.1 *Let G be a graph.*

1. $\chi_B(G) \geq \chi(G), \chi(\overline{G})$
2. $\chi_B(G) = \chi_B(\overline{G})$.

Recall that K_p^q is the complete p -partite graph with q vertices in each part.

Proposition 4.2 *Let G be a graph.*

$$\chi_B(G) \leq \chi(G) + \chi(\overline{G}) - 1.$$

This bound is tight for $G = K_p^q$.

Proof. Consider c cliques spanning a set A of vertices in G . If $c < \chi(\overline{G})$ we are done since $\chi(G - A) \leq \chi(G)$. Otherwise, $c = \chi(\overline{G})$ and it is possible to partition all vertices into c cliques. We can obtain required cocliques by considering single vertices.

We see that $\chi_B(K_p^q) \geq p + q - 1$ by observing that if we require $q - 1$ cliques in a partition of a vertex set of K_p^q into cliques and cocliques then $p - 1$ cocliques is not enough to partition the rest of the vertices. ■

Next we determine the binary chromatic number of some classes of graphs to partition the rest of the vertices.

Proposition 4.3 *Let $\chi_B(G)$ denote the binary chromatic number of a graph G .*

1. *If $n \geq 5$, then $\chi_B(C_n) = \lceil n/2 \rceil$.*
2. *If $n \geq 3$, then $\chi_B(P_n) = \lceil n/2 \rceil$.*
3. *$\chi_B(K_p^q) = p + q - 1$.*

Proof.

1. The lower bound follows from Fact 4.1(1). For the upper bound, we can construct the partition of a vertex set in at most $\lceil \frac{n}{2} \rceil$ cliques and cocliques as follows. If we need only cliques, or only cocliques, it is clear. When we need at least one clique and at least one coclique in that partition, take the largest coclique on $\lfloor \frac{n}{2} \rfloor$ vertices. The leftover graph consists of independent vertices and, if n is odd, of one edge. Take this edge (or a single vertex when n is even) as a clique of our partition. The number of leftover vertices is $\lceil \frac{n}{2} \rceil - 2$ and we are done.
2. This is quite similar to the case of C_n . We leave it to the reader.
3. This follows from Proposition 4.2, since $\chi(K_p^q) = p$ and $\chi(\overline{K_p^q}) = q$. ■

Proposition 4.4 gives the bounds on the smallest binary chromatic number among all n -vertex graphs.

Proposition 4.4 *If n is a positive integer, then*

$$\sqrt{n} \leq \min_{|V(G)|=n} \chi_B(G) \leq \sqrt{n} + (1 + o(1))n^{0.2625}.$$

Moreover there are infinitely many graphs for which the lower bound is attained.

Proof. For the lower bound, we use Fact 4.1(1) and the fact that $\chi(G)\chi(\overline{G}) \geq n$. As a result, one of $\chi(G), \chi(\overline{G})$ is larger than \sqrt{n} .

The lower bound is, in fact, attained by an infinite class of graphs on $n = k^2$ vertices where k is a prime. To realize this lower bound, consider the following construction of a graph $G = G_n = G_{k^2}$.

Let $V(G_n)$ be pairs of integers (i, j) , for $i, j = 1, \dots, k$. We create $k+1$ distinct partitions of $V(G)$ into sets of cardinalities k . Let the i^{th} partition $P_i = \{V_1^i, V_2^i, \dots, V_k^i\}$ be defined as follows for $i = 0, \dots, k$: $V_j^i = \{(j, 1), (j + i, 2), (j + 2i, 3), \dots, (j + (k - 1)i, k)\}$. Here, addition is taken modulo k . Let V_j^i induce a clique if $i < j$ and let V_j^i induce a coclique if $i \geq j$. Next we verify that G is well defined.

Note that for each pair of vertices $x, y \in V(G)$, $x, y \in V_i^j$ for some i, j . Moreover, if $x, y \in V_j^i$ then at most one vertex x or y is in $V_{j'}^{i'}$ where $i \neq i'$. Indeed, if $x, y \in V_i^j$ and $x = (x_1, x_2)$ then $y = (x_1 + li, x_2 + l)$. If $x, y \in V_{i'}^{j'}$ then $y = (x_1 + l'i', x_2 + l')$. Now, since $x_2 + l = x_2 + l'$ we have $l = l' \pmod{k}$. Thus $x_1 + li = x_1 + l'i = x_1 + l'i'$, therefore $l'i = l'i'$ and $i = i'$ if k is prime.

We see that P_i provides a vertex-partition of G into i cocliques and $k-i$ cliques, $0 \leq i \leq k$. Therefore $\chi_B(G) \leq k = \sqrt{n}$.

For arbitrary n , we find the upper bound by taking the smallest $k \geq \sqrt{n}$ such that k is a prime. Consider G_{k^2} as defined above and let G_n be a subgraph of G_{k^2} induced by a set of n vertices. As we have shown, $\chi_B(G_{k^2}) \leq k$, which implies $\chi_B(G_n) \leq k$. In a paper of Baker, Hartman and Pintz [1], for x at least some x_0 , there is a prime in the interval $[x - x^{0.525}, x]$. Thus, $\chi_B(G) \leq k \leq \sqrt{n} + (1 + o(1))n^{0.2625}$. \blacksquare

5 Better bounds for small graphs

The results stated in Section 1 are asymptotic. However, for some graphs H we are able to determine the exact value of $\text{Dist}(n, \text{Forb}(H))$.

Here we shall use the fact that the extremal graphs for forbidden induced subgraphs on three vertices as well as for induced subgraphs on 4 vertices and 3 edges are known precisely [7].

Theorem 5.1 *If $H \in \{K_3, \overline{K_3}, K_{1,2}, \overline{K_{1,2}}\}$, then $\text{Dist}(n, \text{Forb}(H)) = \binom{\lceil n/2 \rceil}{2} + \binom{\lfloor n/2 \rfloor}{2}$.*

Proof. The cases of the triangle K_3 and of the empty graph $\overline{K_3}$ follow immediately from equation (1).

Now, we consider the editing distance for $K_{1,2}$ -free graphs. Note that the graph which contains no induced $K_{1,2}$ is a disjoint union of cliques.

Let G be an arbitrary graph on n vertices. If G has minimum degree at least $\lceil n/2 \rceil$ then we add all missing edges to obtain a complete graph. In this case, at most $\binom{n}{2} - \lceil n/2 \rceil \frac{n}{2} \leq \binom{\lceil n/2 \rceil}{2} + \binom{\lfloor n/2 \rfloor}{2}$ edges were added. Otherwise, delete all edges incident to a vertex v of degree at most $\lfloor n/2 \rfloor$ and apply induction to $G \setminus v$. The total number of additions and deletions is at most $\lfloor n/2 \rfloor + \binom{\lceil (n-1)/2 \rceil}{2} + \binom{\lfloor (n-1)/2 \rfloor}{2} \leq \binom{\lceil n/2 \rceil}{2} + \binom{\lfloor n/2 \rfloor}{2}$. This provides an upper bound on f .

For the lower bound, we consider a complete bipartite graph H on n vertices with almost equal parts A, B . Let G be the disjoint union of cliques S_1, S_2, \dots, S_k on the same vertex set as H . Let $a_i = |A \cap V(S_i)|$ and $b_i = |B \cap V(S_i)|$, for $i = 1, \dots, k$. It is clear that the number of editing operations performed on H to obtain G is

$$s = \sum_{i=1}^k \left[\binom{a_i}{2} + \binom{b_i}{2} + a_i(|B| - b_i) \right].$$

This function is minimized when $a_i = b_i$ for all i , except perhaps one $i \in \{1, \dots, k\}$ such that $|a_i - b_i| = 1$. Now, $s \geq n^2/4 - n/2$ for even n and $s \geq (n-1)^2/4 - (n-1)/2 + (n-1)/2$ for odd n , and the result follows. ■

Let \mathcal{Q} be the set of graphs on n vertices with no induced subgraphs on 4 vertices and 3 edges. In [8] it was shown that any graph in \mathcal{Q} or its complement is a disjoint union of 4-cycles and trees on at most 3 vertices. Note that $G \in \mathcal{Q}$ if and only if $\overline{G} \in \mathcal{Q}$.

Theorem 5.2 $\lfloor (n^2 - 5n)/4 \rfloor \leq \text{Dist}(n, \mathcal{Q}) \leq (n^2 - n)/4$.

Proof. Let G be a graph on n vertices. Since $E_n, K_n \in \mathcal{Q}$, it is sufficient either to add all missing edges or to delete all edges to obtain a graph from \mathcal{Q} . Thus, the upper bound follows.

For the lower bound consider a graph G with $\lfloor (n^2 - n)/4 \rfloor$ edges. Assume first that the minimal number of edit operations results in a graph whose components are either 4-cycles or trees on at most 3 vertices. As a result, the total number of edges within these components is at most n . Therefore, at least $|E(G)| - n$ edges of G had to be deleted.

Similarly, if the minimum number of edit operations results in a graph such that its complement has components that are either 4-cycles or trees on at most 3 vertices. So, at least $|E(\overline{G})| - n = \binom{n}{2} - |E(G)| - n$ edges had to be added to G .

As a result, the number of edit operations is at least $\lfloor (n^2 - n)/4 \rfloor - n$. ■

Conclusions

The editing problem of graphs we consider in this paper can be reformulated in terms of complete edge-colored graphs, where the edges of the graphs correspond to edges of one color, say red, and the edges of the complement correspond to edges of another color, say blue. Our editing operations are equivalent to changing the color of some edges from red to blue or from blue to red.

It is natural to consider more than two colors. Specifically for any two colorings of $E(K_n)$ in colors from $\{1, \dots, \gamma\}$, we define the distance to be the smallest number of edge-recolorings to obtain one coloring from the other. Our results for classes of graphs with forbidden induced subgraphs can be generalized for classes of multicolored graphs with forbidden color patterns.

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6 Appendix: Random graph

Let $G(n, p)$ be a graph in which each edge from K_n is chosen independently with probability p (see [3, 14]). Lemma 2.8 follows immediately from the following:

Lemma 6.1 *Fix a constant $\epsilon > 0$ and positive integer m . Let ℓ have the property that $m \leq \ell \leq M(\epsilon, m)$ where $M(\epsilon, m)$ is the constant given in the Regularity Lemma (Lemma 2.3). Let $G = G(n, 1/2)$, $f(n) = \omega(n^{-1/2})$ and P be the probability that for each (m, ϵ, ℓ) -equipartition of the vertices of G , all pairs of clusters (V_i, V_j) , $1 \leq i < j \leq \ell$, have density in the interval $(1/2 - f(n), 1/2 + f(n))$. Then P approaches 1 as n goes to infinity.*

Proof. We just want to compute the probability that **all** pairs of disjoint sets, each of size at least $\epsilon'n$ (where $\epsilon' = \frac{1-\epsilon}{M(\epsilon)-1}$), have density in the interval $(1/2 - f, 1/2 + f)$, for any $f = f(n) = \omega(n^{-1/2})$.

$$\begin{aligned}
& \Pr \left\{ \bigvee_{\substack{S, T \subseteq V(G) \\ S \cap T = \emptyset \\ |S|, |T| \geq \epsilon' n}} (d(S, T) \notin (1/2 - f, 1/2 + f)) \right\} \\
& \leq 2^n 2^n 2 \Pr \{d(S, T) < 1/2 - f\} \\
& \leq 2 \cdot 4^n \exp(-2(f|S||T|)^2/(|S||T|)) \\
& \leq 2 \cdot 4^n \exp(-2f^2|S||T|) \\
& \leq 2 \cdot 4^n \exp(-2(\epsilon')^2 f^2 n^2) \rightarrow 0
\end{aligned} \tag{7}$$

Chernoff's bound (see [14]) is used to achieve inequality (7). ■