The $k$-strong induced arboricity of a graph

Maria Axenovich, Daniel Gonçalves, Jonathan Rollin, Torsten Ueckerdt

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Abstract

The induced arboricity of a graph $G$ is the smallest number of induced forests covering the edges of $G$. This is a well-defined parameter bounded from above by the number of edges of $G$ when each forest in a cover consists of exactly one edge. Not all edges of a graph necessarily belong to induced forests with larger components. For $k \geq 1$, we call an edge $k$-valid if it is contained in an induced tree on $k$ edges. The $k$-strong induced arboricity of $G$, denoted by $f_k(G)$, is the smallest number of induced forests with components of sizes at least $k$ that cover all $k$-valid edges in $G$. This parameter is highly non-monotone. However, we prove that for any proper minor-closed graph class $C$, and more generally for any class of bounded expansion, and any $k \geq 1$, the maximum value of $f_k(G)$ for $G \in C$ is bounded from above by a constant depending only on $C$ and $k$.

We prove that $f_2(G) \leq 3 \left( \frac{t+1}{3} \right)$ for any graph $G$ of tree-width $t$ and that $f_k(G) \leq (2k)^d$ for any graph of tree-depth $d$. In addition, we prove that $f_2(G) \leq 310$ when $G$ is planar, which implies that the maximum adjacent closed vertex-distinguishing chromatic number of planar graphs is constant.

1 Introduction

Let $G = (V,E)$ be a simple, finite and undirected graph. An induced forest in $G$ is an acyclic induced subgraph of $G$. A cover of $X \subseteq V \cup E$, is a set of subgraphs of $G$ whose union contains every element of $X$. It is certainly one of the most classical problems in graph theory to cover the vertex set $V$ or the edge set $E$ of $G$ with as few as possible subgraphs from a specific class, such as independent sets [26], stars [2], paths [1], forests [17], planar graphs [16], interval graphs [15], or graphs of tree-width $t$ [10], just to name a few. Extensive research on graph covers has been devoted to the following two graph parameters: The vertex arboricity of $G$ is the minimum $t$ such that $V$ can be covered with $t$ induced forests [17]. The arboricity of $G$, denoted as $a(G)$, is the minimum $t$ such that $E$ can be covered with $t$ forests [17]. Nash-Williams [17] proved that the arboricity of a graph $G$ is given by $\max\{\frac{|E(H)|}{|V(H)|} - 1\}$ where the maximum is taken over all subgraphs $H$ of $G$.

Here, we define the induced arboricity $f_1(G)$ of $G$ to be the minimum $t$ such that $E$ can be covered with $t$ induced forests. Moreover, we introduce the $k$-strong induced arboricity $f_k(G)$, where we additionally require that each connected component of the induced forests has at least $k$ edges. More precisely, for $k \geq 1$, let a $k$-strong forest of $G$ be an induced forest $F$ in $G$, each of whose connected components consists of at least $k$ edges. Hence, a 1-strong forest is one that has no isolated vertices and a 2-strong forest is one that has neither $K_1$-nor $K_2$-components. As isolated vertices in a forest do not help to cover the edges of $G$, these can be easily omitted. Thus for the induced arboricity of $G$ as defined above it suffices to consider induced forests where every component has at least one edge.
However, note that for $k \geq 2$ we possibly can not cover $E(G)$ with $k$-strong forests, for example when $G$ is a clique or when $|E(G)| < k$. An edge $e \in E(G)$ is defined to be $k$-valid, $k \geq 1$, if there exists a $k$-strong forest in $G$ containing $e$. Of course, every edge is 1-valid. When $k$ is fixed and clear from context, we write valid instead of $k$-valid. By removing a leaf in an induced tree, one obtains an induced tree with exactly one edge less. Thus an edge $e$ is $k$-valid if and only if it belongs to an induced tree with exactly $k$ edges. We call such a tree a witness tree for $e$.

We define the $k$-strong induced arboricity of $G$, denoted by $f_k(G)$, as the smallest number of induced $k$-strong forests covering all $k$-valid edges of $G$. The main result of this paper shows that for well-behaved classes of graphs, such as for example minor-closed families, the parameter $f_k$ is bounded from above by a constant independent of the order of the graph. Recall that a graph class $C$ is called minor-closed if for each $G \in C$ any graph $H$ obtained from $G$ by deleting edges or vertices, or by contracting edges is contained in $C$.

To define such graph families, we define the tree-depth and tree-width and follow the notions used by Nešetřil and Ossona de Mendez [20, 21], see also [12].

Tree-width: For a positive integer $t$, a $t$-tree is a graph obtained from a union of copies $G_1, \ldots, G_q$ of $K_{t+1}$, called bags, such that for any $j = 2, \ldots, q$ we have that the set $V(G_j) \cap (V(G_1) \cup \cdots \cup V(G_{j-1}))$ has size $t$ and is contained in $V(G_i)$ for some $i \in \{1, \ldots, j-1\}$. I.e., a $t$-tree is a graph obtained by starting with $K_{t+1}$ and “adding” cliques (bags) on $t+1$ vertices one at a time, by identifying $t$ vertices of the new clique with some $t$ vertices of some previously added clique. A graph $G$ has tree-width $t$ and we write $\text{tw}(G) = t$, if $t$ is the smallest integer such that $G$ is a subgraph of a $t$-tree. Note that the graphs of tree-width 1 are the forests on at least one edge.

Tree-depth: The transitive closure of a rooted tree $T$ with a root $r$ is the graph obtained from $T$ by adding every edge $uv$ such that $v$ is on the $u$-$r$-path of $T$. A rooted tree has depth $d$ if the largest number of vertices on a path to the root is $d$. Now, a graph $G$ has tree-depth $d$, denoted by $\text{td}(G) = d$, if $d$ is the smallest integer such that each connected component of $G$ is a subgraph of the transitive closure of a rooted tree of depth $d$.

Tree-depth coloring: A $p$-tree-depth coloring of a graph $G$ is a vertex coloring such that each set of $p'$ color classes, $p' \leq p$, induces a subgraph of $G$ with tree-depth at most $p'$. So a 1-tree-depth coloring is exactly a proper coloring of $G$, while a 2-tree-depth coloring is a proper coloring of $G$ in which any two color classes induce a star forest (a graph of tree-depth at most 2). Let $\chi_p(G)$ be the minimum number of colors needed in a $p$-tree-depth coloring of $G$. Then $\chi(G) = \chi_1(G)$ and $\chi_p(G) \leq \text{td}(G)$ for any $p \geq 1$ [21].

Bounded Expansion: A class $C$ of graphs is of bounded expansion if for each positive integer $p$ there is a constant $a_p = a(p, C)$ such that for each $G \in C$ we have $\chi_p(G) \leq a_p$.

**Theorem 1.** Let $C$ be a class of graphs that is of bounded expansion. Then for each positive integer $k$ there is a constant $b_k = b(k, C)$ such that for each $G \in C$ we have $f_k(G) \leq b_k$.

Building on work of DeVos et al. [9], Nešetřil and Ossona de Mendez [20, 21] proved that several classes of graphs are of bounded expansion, such as minor-closed classes, classes of graphs with an excluded topological minor, or classes of graphs of bounded tree-width or tree-depth. This implies the following.
Corollary 2. Let $k$ be a positive integer and let $\mathcal{C}$ be one of the following classes: a minor-closed class of graphs that is not the class of all graphs, a class of graphs with no topological minor isomorphic to a given fixed graph, or a class of graphs of tree-width or tree-depth at most $t$, for some fixed $t$. Then there is a constant $c = c(k, \mathcal{C})$ such that $f_k(G) \leq c$ for any $G \in \mathcal{C}$.

Theorem 1 states that bounded expansion implies for each $k$ the existence of a constant upper bound on the parameter $f_k$. We show in Theorem 3(iii) that the converse statement is not true. While the $k$-strong induced arboricity is bounded by a constant on the families of graphs listed above, it is a highly non-monotone and unbounded parameter in general. We also show in Theorem 3(i),(ii) some relations between the parameters $f_k(G)$, $\tw(G)$, $\td(G)$, $a(G)$, and the acyclic chromatic number $\chi_{\text{acyc}}(G)$. Recall that the acyclic chromatic number of a graph $G$ is the smallest number of colors in a proper coloring of $G$ in which any two color classes induce a forest. Note that the arboricity of $G$ is at most $f_1(G)$.

Theorem 3. (i) There exists a constant $c > 0$ such that for each graph $G$ we have $c \log(\chi_{\text{acyc}}(G)) \leq f_1(G) \leq (\chi_{\text{acyc}}(G))^2$.

(ii) For any integers $k \geq 2$, $n \geq 3$, and for each item below there is graph $G$ satisfying the listed conditions:

(a) $a(G) = 2$ and $f_1(G) \geq n$,
(b) $f_k(G) \leq 3$ and $f_{k+1}(G) \geq n$,
(c) $f_k(G) \geq n$ and $f_{k+1}(G) = 0$,
(d) $G$ has an induced subgraph $H$ such that $f_k(G) = 3$ and $f_k(H) \geq k$,
(e) $\tw(G) = 2$ and $f_k(G) \geq k$,
(f) $\td(G) = 3$ and $f_k(G) \geq k - 1$.

(iii) There is a class $\mathcal{C}$ of graphs that is not of bounded expansion such that for each $G \in \mathcal{C}$ and each $k \geq 1$ we have $f_k(G) \leq 2$.

Theorem 1 provides the existence of constants bounding $f_k(G)$ for graphs $G$ from special classes. Next, we give more specific bounds on these constants. Clearly, if $\tw(G) \leq 1$, then $f_k(G) \leq 1$ for every $k$. However, already for graphs $G$ of tree-width 2 finding the largest possible value of $f_k(G)$ for $k \geq 2$ is non-trivial. We show, in particular that $f_1(G) \leq \left(\tw(G) + 1\right)^2$ for any graph $G$, which is best-possible, since for $\tw(K_{t+1}) = t$ and $f_1(K_{t+1}) = \left(\binom{t+1}{2}\right)$, and that $f_2(G) \leq 3\left(\tw(G) + 1\right)$ for any graph $G$, which is best-possible when $\tw(G) = 2$, as certified by $G$ being $K_3$ with a pendant edge at each vertex.

Theorem 4. For every graph $G$ of tree-width $t \geq 2$, we have that $f_1(G) \leq \left(\binom{t+1}{2}\right)$ and $f_2(G) \leq 3\left(\binom{t+1}{2}\right)$.

Nešetřil and Ossona de Mendez [19] prove that for each minor-closed class $\mathcal{C}$ of graphs that is not the class of all graphs there is a constant $x$ such that each graph in $\mathcal{C}$ has acyclic chromatic number at most $x$. We show how to bound $f_2$ in terms of $x$.

Theorem 5. For every minor-closed class of graphs $\mathcal{C}$ whose members have acyclic chromatic number at most $x$, we have that for every $G \in \mathcal{C}$,

$$f_2(G) \leq \begin{cases} \left(\binom{x}{2}\right)(3\left(\binom{x}{2}\right) + 1), & \text{if } x \leq 9, \\ \left(\binom{x}{2}\right)(12x + 1), & \text{if } x \geq 9. \end{cases}$$
Using Borodin’s result that each planar graph has acyclic chromatic number at most 5 [7], Theorem 5 implies the following.

**Corollary 6.** For every planar graph \( G \), \( f_2(G) \leq 310 \).

This result answers an open question about vertex-distinguishing numbers of graphs. Given a graph \( G \), an assignment of positive integers to its vertices is called distinguishing if the sum of the labels in the closed neighborhood of any vertex \( v \) differs from such this sum in the closed neighborhood of any of the neighboring vertices \( u \) of \( v \), unless \( N[u] = N[v] \). I.e., the labeling distinguishes between adjacent vertices. The smallest positive integer \( \ell \) such that there is a distinguishing labeling of \( G \) with labels in \( \{1, \ldots, \ell\} \) is called adjacent closed vertex-distinguishing number of \( G \), denoted \( \text{dis}[G] \). While for an analogous notion \( \text{dis}(G) \) with open neighborhoods considered instead of closed neighborhoods, it is known that there is a constant \( c \) such that \( \text{dis}(G) \leq c \) for any planar graph \( G \), as noted by Norine, see [4], it was not known whether \( \text{dis}[G] \) is bounded by a universal constant for all planar graphs. In [3] it was shown that if \( f_2(G) \leq x \) then \( \text{dis}[G] \) is bounded from above by some product of \( x \) pairwise co-prime numbers. Thus, Corollary 6 implies the following.

**Corollary 7.** There is an absolute constant \( c \) such that for any planar graph \( G \), the adjacent closed distinguishing number \( \text{dis}[G] \leq c \).

The tree-depth is somehow a more restrictive variant of the tree-width. We have tw\((G) \leq \text{td}(G) - 1 \) for any graph \( G \), but when \( G \) is a graph of tree-depth \( d \), the longest path in \( G \) has at most \( 2^d - 1 \) vertices. In particular, even graphs of tree-width 1 can have arbitrarily large tree-depth [21]. The next theorem shows that the parameter \( f_k \) is bounded for graphs of tree-depth \( d \) by a polynomial in \( d \) as well as a polynomial in \( k \).

**Theorem 8.** For all positive integers \( k, d \) and any graph \( G \) of tree-depth \( d \), \( f_k(G) \leq (2k)^d \). If \( d \geq k + 1 \) then \( f_k(G) \leq (2k)^{k+1}(\frac{d}{k+1})^d \). Moreover \( f_1(G) \leq (\frac{d}{2})^d \).

**Organization of the paper:** We prove Theorem 3 in Section 2. We consider graphs of bounded tree-width in Section 3 and prove Theorem 4 in that section. The proofs of the bounds on \( f_k \) in terms of the acyclic chromatic number and the proof of Theorem 5 are given in Section 4. Graphs of bounded tree-depth, and more general classes of graphs, are considered in Section 5, where we prove Theorems 8. We prove the main Theorem 1 in Section 6. Finally we summarize our results, state some open questions and discuss other variants of the strong induced arboricity in Section 7.

## 2 General inequalities

In this section we prove the general properties of the parameter \( f_k \) listed in Theorem 3.

**Proof of Theorem 3(i).** For the first inequality, consider a covering of \( E(G) \) with \( x = f_1(G) \) induced forests \( F_1, \ldots, F_x \) and for each forest a proper 2-coloring of its vertices. Let \( c_1, \ldots, c_x \) be colorings of \( V(G) \) in colors \( \{0, 1, 2\} \) such that \( c_i(v) = 0 \) if \( v \not\in V(F_i) \), \( c_i(v) = 1 \) if \( v \) is from the first color class of \( F_i \), and \( c_i(v) = 2 \) if \( v \) is from the second color class of \( F_i \). Let a coloring \( c \) of \( V(G) \) be defined as \( c(v) = (c_1(v), \ldots, c_x(v)) \), \( v \in V(G) \). To see that \( c \) is an acyclic coloring assume that two color classes \( (a_1, \ldots, a_{x_1}), (b_1, \ldots, b_{x_2}) \) induce a cycle \( C \). Let \( e \) be an edge of \( C \). It is in some \( F_i \) and hence \( \{a_i, b_i\} = \{1, 2\} \). Thus the \( i^{th} \) coordinate of \( c \) in the cycle \( C \) alternates between 1 and 2. This implies that all edges of \( C \) belong to \( F_i \), a contradiction since \( F_i \) is acyclic. For similar reasons \( c \) is proper. Thus \( c \) is an acyclic coloring.
For the second inequality, consider an acyclic proper coloring of $G$ using $\chi_{\text{acyc}}(G)$ colors. For every pair of colors $c_1$, $c_2$ the subgraph of $G$ induced by the vertices of color $c_1$ or $c_2$ is an induced forest in $G$. Moreover, every edge of $G$ is contained in exactly one such induced forest. Hence, by removing all isolated vertices from each such forest, we get $f_1(G) \leq \chi_{\text{acyc}}^2$.

**Proof of Theorem 3(ii.a).** Let $R^{-1}(t)$ denote the smallest number of colors needed to color $E(K_t)$ without monochromatic triangles. By Ramsey’s Theorem [23, 25] we have $R^{-1}(t) \to \infty$ as $t \to \infty$. Choose $t$ sufficiently large such that $R^{-1}(t) \geq n^2$. Since $n \geq 3$, clearly $t \geq 3$.

Let $G$ be the graph obtained from $K_t$ by subdividing each edge once. For an edge $e$ in $K_t$ let $e_1$ and $e_2$ denote the two corresponding edges in $G$. Split $G$ into two subgraphs $G_1$ and $G_2$ where $G_i$ contains all edges $e_i$, $e \in E(K_t)$, $i = 1, 2$. Then $E(G) = E(G_1) \cup E(G_2)$ and, for $i = 1, 2$, each component of $G_i$ is a star with center at an original vertex of $K_t$. Therefore $a(G) \leq 2$ and as $t \geq 3$, we have $a(G) = 2$.

Let $N = f_1(G)$ and consider induced forests $F_1, \ldots, F_N$ covering all edges of $G$. We consider the following edge-coloring of $K_t$. If there is an $i$, $1 \leq i \leq N$, with $e_1, e_2 \in E(F_i)$, then color the edge $e$ with color $i$ (choose an arbitrary such $i$). Otherwise there are $i$ and $j$, $1 \leq i < j \leq N$, with $e_1, e_2 \in E(F_i) \cup E(F_j)$, $i \neq j$, and we color the edge $e$ with color $(i, j)$ (choose an arbitrary such pair). This coloring uses at most $N + \left(\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right) = \binom{N+1}{2}$ colors. We claim that there are no monochromatic triangles under this coloring. Indeed there is no triangle in color $i$, $1 \leq i \leq N$, since $F_i$ contains no cycle, and there is no triangle in color $(i, j)$, $1 \leq i < j \leq N$, since $F_i$ and $F_j$ are induced. Therefore $f_1(G) = \binom{N+1}{2} \geq R^{-1}(t) \geq n^2$. This shows that $f_1(G) \geq n$, since $\binom{n}{2} < n^2$.

**Proof of Theorem 3(ii.b).** Like in the proof of part (ii.a), let $R^{-1}(t)$ denote the smallest number of colors needed to color $E(K_t)$ without monochromatic triangles. By Ramsey’s Theorem [23, 25] we have $R^{-1}(t) \to \infty$ as $t \to \infty$. Choose $t$ sufficiently large such that $R^{-1}(t) \geq n^2$ and, additionally, $t \geq 2k + 2$.

Let $G$ be obtained from $K_t$ by subdividing each edge twice and choosing for each original edge of $K_t$ one of its subdivision vertices and adding $k - 1$ pendant edges to this vertex, see Figure 1 (left part) when $k = 2$. Observe that all edges of $G$ are $k$-valid and $(k + 1)$-valid.

First we shall show that $f_k(G) \leq 3$ by finding 3 $k$-strong forests covering all edges of $G$. For an edge $e$ in $K_t$ let $e_1, e_2, e_3$ denote the subdividing edges in $G$, with $e_2$ the middle one. Let $T_1$ be the subgraph consisting of all edges $e_2$, $e \in E(K_t)$, and all edges adjacent to $e_2$ different from $e_1$ and $e_3$ (the pendant edges). Then $T_1$ is an induced forest and each component of $T_1$ is a star on $k$ edges. Since $t \geq 2k + 2$, we can choose an orientation of $K_t$ such that each vertex has out-degree and in-degree at least $k$. Indeed, if $t$ is odd we find such an orientation by following an Eulerian walk, if $t$ is even, we find such an orientation of $K_{t-1}$ as before and orient the edges incident to the remaining vertex $x$ such that at least $k$ of these edges are in-edges at $x$ and at least $k$ of them are out-edges at $x$. For each edge $e = uv$ in $K_t$ that is oriented from $u$ to $v$ put the edge in $\{e_1, e_3\}$ that is incident to $u$ into $T_2$ and the other edge from $\{e_1, e_3\}$ into $T_3$. Then $T_2$ and $T_3$ are induced forests and each component of $T_2$ and $T_3$ is a star on at least $k$ edges. Moreover each edge of $G$ is contained in $E(T_1) \cup E(T_2) \cup E(T_3)$. Therefore $f_k(G) \leq 3$.

Next, we prove that $f_{k+1}(G) \geq n$. Let $N = f_{k+1}(G)$ and consider $(k + 1)$-strong forests $F_1, \ldots, F_N$ covering all edges of $G$. For each edge $e$ of $K_t$, if $F_i$ contains $e_2$, then it contains either $e_1$ or $e_3$ as well, since each component of $F_i$ has at least $k + 1$ edges. We consider the following edge-coloring of $K_t$. If there is an $i$, $1 \leq i \leq N$, such that $e_1, e_2, e_3 \in E(F_i)$, then color the edge $e$ with $i$ (choose an arbitrary such $i$). Otherwise there are distinct $i, j$, $1 \leq i, j \leq N$, such that, without loss of generality, $e_1, e_2 \in E(F_i)$ and $e_3 \in E(F_j)$. In this
Proof of Theorem 3(ii.c).
Consider the graph $G$ formed by taking the union of a clique on $n+1$ vertices and a path of length $k-1$ that shares an endpoint with the clique. Then we see that all edges of $G$ incident to the path are $k$-valid. However, no two edges of the clique could be in the same induced forest, thus $f_k(G) \geq n$. On the other hand, since each induced tree in $G$ contains at most one edge from the clique, it could have at most $k$ edges. Thus there are no $(k+1)$-valid edges and $f_{k+1}(G) = 0$.

Proof of Theorem 3(ii.d) and (ii.e).
Consider the graph $G$ shown in Figure 2. We see from Figure 2 that $G$ is covered by three large induced trees (a bold, a solid, and a dashed path) and thus $f_k(G) \leq 3$. Let $H$ be its induced subgraph formed by the bold vertices shown in the Figure 2. We see that $H$ is formed by a path $u_1, u_2, \ldots, u_{2k}$ and independent vertices $w_1, w_2, \ldots, w_{2k-1}$ such that $w_i$ is adjacent to $u_i$ and $u_{i+1}$. Then consider the matching in $H$ formed by the edges $u_iw_i, k \leq i \leq 2k-1$, and an induced tree $T_i$ in $H$ of size $k$ containing $u_iw_i, k \leq i \leq 2k-1$. We see that the trees $T_k, \ldots, T_{2k-1}$ are distinct and their pairwise union induces a triangle in $H$. Thus no two of them can belong to the same $k$-strong forest in $H$. Hence $f_k(H) \geq k$. This proves Theorem 3(ii.d). In addition, $tw(H) = 2$. This proves Theorem 3(ii.e) (where $H$ plays the role of $G$ from the Theorem).

Proof of Theorem 3(ii.f).
Consider the graph $G$ shown in the Figure 3. Then $td(G) = 3$. Hence $f_{k+1}(H) \geq k$. This proves Theorem 3(ii.f).
Consider the complete bipartite graph $K_{n,n}$ obtained from $K_{n,n}$ by subdividing every edge once.

Figure 4: Two maximum induced trees $T_1$, $T_2$ covering all edges of the graph $G_n$ ($n = 4$) obtained from $K_{n,n}$ by subdividing every edge once.

(look at the cut vertex as a root of the underlying tree) and $f_k(G) = k - 1$.

Proof of Theorem 3(iii). Consider the complete bipartite graph $K_{n,n}$ and let $G_n$ be the graph obtained from $K_{n,n}$ by subdividing each edge once. We claim that for any positive integers $k$ and $n$ we have $f_k(G_n) \leq 2$, and moreover that the graph class $C = \{G_n \mid n \in \mathbb{N}\}$ is not of bounded expansion. For the latter, we shall show that for each integer $a$ there is an $n$ such that $\chi_3(G_n) > a$. We consider an arbitrary vertex coloring $c$ of $G_n$ with $a$ colors. Color an edge $e$ of $K_{n,n}$ with the set of colors assigned to the three corresponding vertices in $G_n$. This coloring uses at most $\binom{n}{3} \leq 2^n$ colors and hence, for sufficiently large $n$, contains a monochromatic path on at least 5 vertices in some color $C$. So the subgraph of $G_n$ induced by vertices colored with colors from $C$ contains a path on at least $9 > 2^3$ vertices. Since such a path has tree-depth greater than 3 and since $|C| \leq 3$, $c$ is not a 3-tree-depth coloring. Since $e$ was arbitrary, $\chi_3(G_n) > a$ for sufficiently large $n$. Hence $C$ is not of bounded expansion.

Next we prove that, for any integers $k, n \geq 1$ we have $f_k(G_n) \leq 2$. To this end, we construct two maximum induced trees in $G_n$ covering all edges of $G_n$. Clearly, $|V(G_n)| = 2n + n^2$ and we claim that a largest induced tree in $G_n$ contains exactly $n + 1 + n^2 = |V(G_n)| - (n - 1)$ vertices. Let $X(G_n)$ denote the smallest number of vertices in $G_n$ whose deletion makes the graph acyclic. (That is, $X(G_n)$ denotes the size of a minimum feedback vertex set [11].) We shall prove by induction on $n$ that $X(G_n) \geq n - 1$. In fact, for $n = 1$, $G_n$ is a tree itself and thus $X(G_1) = 0$. For $n \geq 2$, consider an 8-cycle in $G_n$ consisting of four original vertices $v_1, v_2, v_3, v_4$ of $K_{n,n}$, $v_1, v_2$ from one bipartition class and $v_3, v_4$ from the other, and the four subdivision vertices corresponding to the four edges $v_1v_3, v_1v_4, v_2v_3$, and $v_2v_4$ in $K_{n,n}$. At least one of these eight vertices has to be deleted to make the graph acyclic, say it is one of $v_1, v_3$, or the vertex $x$ subdividing edge $v_1v_3$. Then $G_n - \{v_1, v_2, x\}$ is isomorphic to $G_{n-1}$ and thus at least $X(G_{n-1})$ further vertices have to be deleted. Hence by induction we get $X(G_n) \geq X(G_{n-1}) + 1 \geq (n - 2) + 1 = n - 1$, as desired. Thus any induced tree in $G_n$ has at most $n^2 + 2n - (n - 1)$ vertices.

On the other hand, one obtains a maximum induced tree $T_1$ by deleting $n - 1$ original vertices of $K_{n,n}$ that belong to the same bipartition class, see Figure 4. Deleting $n - 1$ vertices from the other bipartition class gives symmetrically a maximum induced tree $T_2$. Finally, observe that $T_1$ and $T_2$ together cover all edges of $G_n$, which certifies that $f_k(G_n) \leq 2$ for $k \leq n + 1 + n^2$. For $k > n + 1 + n^2$ no edge of $G_n$ is $k$-valid and thus $f_k(G_n) = 0$.

3 Graphs of bounded tree-width

We start with a list of properties of graphs of tree-width 2. Then we shall prove that $f_2(G) \leq 3$ for any graph $G$ of tree-width 2. This is the main part of the proof. Then, we shall do an easy reduction argument to express the upper bound from Theorem 4 on $f_2(G)$ for graphs $G$ of larger tree-width.
3.1 Properties of graphs with tree-width 2 and observations

Consider any fixed graph \( G \) of tree-width 2. Firstly, \( G \) contains no subdivision of \( K_4 \) [5]. (In fact, this property characterizes tree-width 2 graphs.) Moreover, it is well-known (see for example [24]) that as long as \( |V(G)| \geq 3 \), there is a 2-tree \( H \) with \( G \subseteq H \) and \( V(H) = V(G) \).

Let us fix such a 2-tree \( H \). Every edge of \( H \) is in at least one triangle of \( H \). Consider the partition \( E(H) = E_{\text{in}}(H) \cup E_{\text{out}}(H) \) of the edges of \( H \), where \( E_{\text{out}}(H) \) consists of those edges that are contained in only one triangle of \( H \), called the outer edges of \( H \). Respectively, \( E_{\text{in}}(H) \) consists of those edges that are contained in at least two triangles of \( H \), called the inner edges of \( H \). Note, if \( H \) is outerplanar, every edge is in at most two triangles, and our definition corresponds to the usual partition into outer and inner edges of an outerplanar embedding of \( H \).

The following two statements can be easily proved by induction on \( |V(H)| \). Indeed, both statements hold with “if and only if” and are maintained in the construction sequence of the 2-tree \( H \).

\((P1)\)
If \( v \in V(H) \) is incident to two outer edges in the same triangle of \( H \), then \( \deg_H(v) = 2 \).

\((P2)\)
If \( uw \in E_{\text{in}}(H) \), then \( H - \{u, w\} \) is disconnected.

It is easy to see that for any 2-connected graph \( F \) with \( |V(F)| \geq 4 \) and for any two vertices \( u, w \in V(F) \) we have the following:

\((P3)\)
For every connected component \( K \) of \( F - \{u, w\} \) we have \( N(u) \cap V(K) \neq \emptyset \) and \( N(w) \cap V(K) \neq \emptyset \).

\((P4)\)
The graph \( F - \{u, w\} \) is connected if and only if the graph \( F' \) obtained from \( F \) by identifying \( u \) and \( w \) into a single vertex is 2-connected.

Now if \( G \) is a 2-connected graph of tree-width 2 and \( H \) is a 2-tree with \( G \subseteq H \) and \( V(H) = V(G) \), then we have the following properties.

\((P5)\)
\( E_{\text{out}}(H) \subseteq E(G) \)

\((P6)\)
For every \( e \in E_{\text{out}}(H) \) the graph \( G/e \) obtained from \( G \) by contracting edge \( e \) is 2-connected.

To see \((P5)\), consider any edge \( e = uw \) in \( E_{\text{out}}(H) \). As \( G \) is 2-connected, there exists a cycle \( C \) in \( G \) through \( u \) and \( w \). If \( e \in E(C) \) then \( e \in E(G) \) and we are done. Otherwise, in \( H \), edge \( e \) is a chord of cycle \( C \), splitting it into two cycles \( C_1 \) and \( C_2 \). As \( H \) is a chordal graph, \( C_1 \) and \( C_2 \) are triangulated, i.e., \( e \) is contained in a triangle with vertices in \( C_1 \) and another triangle with vertices in \( C_2 \). Thus \( e \in E_{\text{in}}(H) \), a contradiction to \( e \in E_{\text{out}}(H) \).

To see \((P6)\), consider any outer edge \( e = uw \) of \( H \). By \((P4)\) we have that \( G/e \) is 2-connected if \( G - \{u, w\} \) is connected. Assume for the sake of contradiction that \( G - \{u, w\} \) is disconnected and let \( K_1, K_2 \) be two connected components of \( G - \{u, w\} \). Then by \((P3)\) for \( i = 1, 2 \) we have \( N(u) \cap V(K_i) \neq \emptyset \) and \( N(w) \cap V(K_i) \neq \emptyset \). Hence we can find a cycle \( C \) in \( H \) for which \( e = uw \) is a chord by going from \( u \) to \( w \) through \( K_1 \) and from \( w \) to \( u \) through \( K_2 \). As before, it follows that \( e \in E_{\text{in}}(H) \), a contradiction to \( e \in E_{\text{out}}(H) \). Hence, \( G/e \) is 2-connected.

Finally, let us characterize the edges of \( G \) that are not 2-valid. An edge \( uw \) of \( G \) is called a twin edge if \( N[u] = N[v] \), i.e., if the closed neighborhoods of \( u \) and \( v \) coincide. Observe that twin edges are exactly the edges that are not 2-valid.
(P7) If $G$ is 2-connected, $\text{tw}(G) = 2$, and $xy$ is a twin edge in $G$, then $G$ is a 2-tree consisting of $r$ triangles, $r \geq 1$, all sharing the common edge $xy$.

To prove (P7), let $H$ be a 2-tree with $G \subseteq H$ and $V(H) = V(G)$. Consider the set $S = N(x) - y = N(y) - x$. As $G$ is 2-connected, we have $|S| \geq 1$. We claim that for each $w \in S$ the edges $xw$ and $yw$ are outer edges. Indeed, if $xw \in E_{in}(H)$, then by (P2) the graph $H - \{x, w\}$ and therefore also the graph $G - \{x, w\}$ is disconnected. Let $K$ be a connected component of $G - \{x, w\}$ which does not contain $y$. By (P3) we have $N(x) \cap V(K) \neq \emptyset$, as $G$ is 2-connected. This is a contradiction to $N(x) - y = N(y) - x$. Thus for every $w \in S$ we have $xw \in E_{out}(H)$ and symmetrically $yw \in E_{out}(H)$. It follows from (P1) that $\deg_H(w) = 2$ and hence $\deg_G(w) = 2$. Thus $V(G) = S \cup \{x, y\}$, as desired.

3.2 Special decomposition of tree-width 2 graphs

Theorem 9. Let $G = (V, E)$ be a connected non-empty graph of tree-width at most 2, different from $C_4$. Then there exists a coloring $c : V \rightarrow \{1, 2, 3\}$ such that each of the following holds:

1. For each $i \in \{1, 2, 3\}$ the set $V_i = \{v \in V \mid c(v) \neq i\}$ induces a forest $F_i$ in $G$.

2. For each $i \in \{1, 2, 3\}$ there is no $K_1$-component in $F_i$.

3. For each $i \in \{1, 2, 3\}$ every $K_2$-component of $F_i$ is a twin edge.

Proof. We call a coloring $c : V \rightarrow \{1, 2, 3\}$ good if it satisfies (1)–(3). We shall prove the existence of a good coloring by induction on $|V|$, the number of vertices in $G$. We distinguish the cases whether $G$ is 2-connected or not.

Case 1: $G$ is not 2-connected. If $G$ is a single edge $uv$, then a desired coloring is given by $c(u) = c(v) = 1$. Otherwise $G$ has at least two blocks. Consider a leaf block $B$ in the block-cutvertex-tree of $G$ and the unique cut vertex $v$ of $G$ in this block. Consider the graphs $G_1 = B$ and $G_2 = G - (B - v)$, see Figure 5. We define colorings $c_1$ and $c_2$ for $G_1$ and $G_2$, respectively, as follows. For $i \in \{1, 2\}$, if $G_i \neq C_4$, then we apply induction to $G_i$ and obtain a coloring $c_i$ of $G_i$ satisfying (1)–(3). On the other hand, if $G_i = C_4$, then we take the coloring $c_i$ shown in the left of Figure 6, in which the cut vertex $v$ is incident to the only $K_2$-component. Note that this coloring satisfies (1) and (2).

$$
\text{Figure 5: Splitting at cut vertex } v.
$$

Without loss of generality we have $c_1(v) = c_2(v) = 1$ and hence $c_1$ and $c_2$ can be combined into a coloring $c$ of $G$ by setting $c(x) = c_i(x)$ whenever $x \in V(G_i)$, $i = 1, 2$. Clearly, this coloring $c$ satisfies (1) and (2).

If $xy$ is a $K_2$-component of $F_i$ in $G$ for some $i \in \{1, 2, 3\}$, with $v \neq x, y$, then $xy$ is also a $K_2$-component of the corresponding forest in $G_1$ or $G_2$, say in $G_1$. In particular, $G_1 \neq C_4$, since $v \neq x, y$. So $c_1$ satisfies (3) and $xy$ is a twin edge in $G_1$ and thus also in $G$, as desired. On the other hand, if $xy$ is an edge of $F_i$ in $G$ for some $i \in \{1, 2, 3\}$, say with $x = v$ and $y \in V(G_1)$, then $v$ is incident to another edge of $F_i$ in $G_2$, since $c_2$ satisfies (2). Thus $xy$ is not a $K_2$-component of $F_i$.

In any case, $c$ satisfies (3) and hence $c$ is a good coloring.
Case 2: \( G \) is 2-connected. Recall that by (P5) we have \( E_{\text{out}}(H) \subseteq E(G) \), i.e., every outer edge is an edge of \( G \). An outer edge \( e \) is called contractible if \( e \) is in no triangle of \( G \). If \( G \) has a contractible edge \( e \), then we shall consider the smaller graph \( G/e \) obtained by contracting \( e \). If \( G/e \) has a twin edge, we shall give a good coloring \( c \) of \( G \) directly, otherwise we obtain a good coloring \( c \) by induction. On the other hand, if \( G \) has no contractible edges, we shall give a good coloring \( c \) directly.

Case 2A: There exists a contractible edge \( e \) in \( G \). Let \( e = uv \) be contractible. Consider the graphs \( G' = G/e \) and \( H' = H/e \) obtained from \( G \) and \( H \) by contracting edge \( e \) into a single vertex \( v \). As \( e \in E_{\text{out}}(H) \) and \( H \neq K_3 \) (otherwise \( e \) would be in a triangle of \( G \)), we have that \( H' \) is a 2-tree. In particular, \( \text{tw}(G') \leq 2 \). Moreover, by (P6) \( G' \) is also 2-connected. Finally, as \( e \) is not in a triangle in \( G \), we have that \( |E(G)| = |E(G') \cup \{e\}| \).

If \( G' = C_4 \) then \( G = C_5 \) and it is easy to check that coloring the vertices around the cycle by 1, 1, 2, 2, 3 gives a coloring \( c \) satisfying (1), (2) and (3).

If \( G' \) has a twin edge \( xy \), then by (P7) we have that \( G' = H' \) and \( G' \) consists of \( r \) triangles, \( r \geq 1 \), all sharing the common edge \( xy \). Since the contractible edge \( e \) lies in no triangle of \( G \) and \( G \neq C_4 \), we have that \( G' \) is not a triangle and thus in fact \( r \geq 2 \).

Now if \( v = y \) (the case \( v = x \) being symmetric), then \( G \) looks like in Figure 6 (middle) and a good coloring of \( G \) is given by \( c(x) = 1, c(u) = c(w) = 2 \) and \( c(z) = 3 \) for every \( z \in S \). On the other hand, if \( v \in S \), then without loss of generality \( ux \in E(G) \) and \( wy \in E(G) \), and \( G \) looks like in Figure 6 (right side). A good coloring of \( G \) is given by \( c(x) = c(w) = 1, c(y) = 2 \), and \( c(z) = 3 \) for every \( z \in (S \cup w) - v \). In both cases it is easy to check that \( c \) satisfies (1), (2) and (3).

![Figure 6: Left: A coloring of \( C_4 \) satisfying (1) and (2) and with one \( K_2 \)-component (in \( F_3 \)). Middle and right: Good colorings when the graph obtained by contracting edge \( uv \) has a twin edge.](image)

Figure 7: The case that \( G \) has a contractible edge \( e = uw \).

So finally we may assume that \( G' \neq C_4 \) and \( G' \) has no twin edge. Applying induction to \( G' \), we obtain a good coloring \( c' \) of \( V(G') = V - \{u, w\} + v \) with corresponding induced forests \( F'_1, F'_2 \) and \( F'_3 \) in \( G' \). We define a coloring \( c : V \to \{1, 2, 3\} \) by \( c(x) = c'(x) \) for each \( x \in V' - v \) and \( c(u) = c(w) = c'(v) \), see Figure 7.

Say \( c'(v) = 1 \). Now \( c \) satisfies (1) as \( F_1 = F'_1 \), and \( F'_2, F'_3 \) are obtained from \( F_2, F_3 \), respectively, by contracting the edge \( uw \), that is not in a triangle in \( G \). Thus, since \( F'_1 \) had no \( K_1 \) or \( K_2 \)-components, so does \( F_1 \). This shows that \( c \) is a good coloring.
Case 2B: There are no contractible edges in $G$. In this case we define the coloring $c : V \to \{1, 2, 3\}$ to be some proper 3-coloring of $H$. To prove that $c$ satisfies (1), assume for the sake of contradiction that there is a cycle in $F_i$ for some $i \in \{1, 2, 3\}$, i.e., a cycle using the two colors in $\{1, 2, 3\} - \{i\}$. Since $H$ is chordal, a shortest such cycle would be a 2-colored triangle, which contradicts $c$ being a proper coloring.

To prove that the coloring $c$ satisfies (2) and (3), we define for any vertex $x \in V$ a trail around $x$ to be a sequence $s_1, \ldots, s_r$ of $r$ distinct neighbors of $x$, such that $x, s_i, s_{i+1}$ form a triangle in $H$ for $i = 1, \ldots, r - 1$ and $x s_i \in E_{\text{out}}(H)$. Note that $x s_i \in E_{\text{in}}(H)$ for $i = 2, \ldots, r - 2$. Moreover, for any triangle $x, y, z$ in $H$ we can greedily construct a trail around $x$ whose first elements are $s_1 = y$ and $s_2 = z$. Indeed, having constructed $s_1, \ldots, s_i$ then either $x s_i \in E_{\text{out}}(H)$ and we are done, or $x s_i \in E_{\text{in}}(H)$ and $x, s_i$ have another common neighbor $s_{i+1}$ in $H$ different from $s_{i-1}$. Moreover, $s_{i+1} \neq s_j$ for $j = 1, \ldots, i - 2$, as otherwise the subgraph of $H$ induced by $\{x, s_j, \ldots, s_i\}$ would contain a subdivision of $K_4$, a contradiction to $\text{tw}(H) = 2$. See Figure 8.

![Figure 8: An example of a trail $s_1, \ldots, s_5$ around $x$.](image)

Now we shall show that $c$ satisfies (2) by proving that any vertex $x$ of $G$, say $c(x) = 1$, has a neighbor in $G$ of color 2 and a neighbor in $G$ of color 3. As $G$ is connected, $x$ is adjacent to some vertex $y$, say $c(y) = 2$. The edge $x y$ is in a triangle in $H$ and its third vertex $z$ has color 3. Consider a trail $s_1, \ldots, s_r$ around $x$ starting with $s_1 = y$, $s_2 = z$. Note that $c(s_i) = 2$ if $i$ is odd and $c(s_i) = 3$ if $i$ is even. Hence, if $r$ is even, then as $x s_r \in E_{\text{out}}(H) \subseteq E(G)$, we have that $s_r$ is a neighbor of $x$ of color 3, as desired.

Otherwise $r$ is odd, $r \geq 3$, and $x s_{r-1} \notin E(G)$. In particular $x s_r \in E_{\text{out}}(H)$ is in only one triangle of $H$, namely $x, s_{r-1}, s_r$, and in no triangle of $G$. Hence $x s_r$ is contractible, contradicting the assumptions of Case 2B. This shows that $c$ satisfies (2).

Finally, to show that $c$ satisfies (3), consider any edge $x y$ of $G$, say $c(x) = 1$ and $c(y) = 2$. If every trail around $x$ starting with $s_1 = y$ and every trail around $y$ starting with $s_1 = x$ has length $r = 2$, then $x y$ is a twin edge. Otherwise, consider a longer such trail, say $s_1, \ldots, s_r$ is a trail around $x$ with $s_1 = y$ and $r \geq 3$. As before, note that $x s_r \in E_{\text{out}}(H)$ is an edge in $G$ and $c(s_i) = 2$ if $i$ is odd and $c(s_i) = 3$ if $i$ is even. If $x s_i \in E(G)$ for some odd $i \geq 3$, we are done. Otherwise $r$ is even, and $x s_{r-1} \notin E(G)$. As before, it follows that $x s_r$ is contractible, contradicting the assumptions of Case 2B. Hence $c$ also satisfies (3), which completes the proof. \hfill $\Box$

### 3.3 Proof of Theorem 4

Let $G$ have tree-width $t$, then $G \subseteq H$ for some $t$-tree $H$. Then $\chi(H) = t + 1$. Consider a proper coloring of $H$ and assume that there is a cycle using two colors. Let $C$ be the shortest such cycle. Since $H$ is chordal, $C$ is a triangle. This is impossible since there are no 2-colored triangles in a proper coloring. Thus $\chi_{\text{acyc}}(H) = t + 1$ and therefore $\chi_{\text{acyc}}(G) \leq t + 1$. Theorem 3(i) immediately implies that $f_1(G) \leq \left\lfloor \frac{t + 1}{2} \right\rfloor$.

Next we shall consider $f_2(G)$, where $G$ is a graph of tree-width $t$. If $t = 2$ and $G = C_4$, we see that each edge in $G$ is 2-valid and two edge disjoint paths on 2 edges each form two induced forests covering all the edges, so $f_2(C_4) = 2$. If $t = 2$ and $G \neq C_4$ is connected then
\[ f_2(G) \leq 3 = 3^{(t+1)} \] by Theorem 9. If \( t = 2 \) and \( G \) is not connected, then each component \( G' \) of \( G \) has tree-width at most 2 and thus satisfies \( f_2(G') \leq 3 \) as argued above. Picking one 2-strong forest from each component of \( G \) and taking their union yields a 2-strong forest of \( G \) and hence \( f_2(G) \leq 3 \).

Now, let \( t \geq 3 \). Given a graph \( G \) of tree-width \( t \geq 3 \), let \( H \) be a \( t \)-tree that contains \( G \). It is well-known [9], that any proper \((t + 1)\)-coloring of \( H \) has the property that any set of \( p + 1 \) colors, \( p = 1, \ldots, t \), induces a \( p \)-tree. We hence have a \((t + 1)\)-coloring of \( G \) such that each of the \( x = \binom{t+1}{3} \) sets of 3 colors induces a graph of tree-width at most 2. Call these graphs \( G_1, \ldots, G_x \). As each 2-valid edge has a witness tree induced by 3 vertices, each witness tree is contained in \( G_i \), for some \( i \in \{1, \ldots, x \} \). So each 2-valid edge is 2-valid in some \( G_i \). Since \( \mathrm{tw}(G_i) \leq 2 \), \( f_2(G_i) \leq 3 \), and so the 2-valid edges of \( G_i \) can be covered by 3 2-strong forests, \( i = 1, \ldots, x \). Hence the 2-valid edges of all \( G_i \)'s and thus the 2-valid edges of \( G \) can be covered with \( 3x = 3^{(t+1)} \) 2-strong forests.

\[ \square \]

4 Minor-closed classes of graphs with bounded acyclic chromatic number

**Lemma 10.** Let \( F \) be a graph and let \( M \) be a matching in \( F \). Let \( F_M \) be the graph obtained by contracting the edges of \( M \).

- If \( F_M \) is a forest, then \( \mathrm{tw}(F) \leq 3 \). Moreover, if \( M \) is an induced matching, then \( \mathrm{tw}(F) \leq 2 \).
- Let \( c \) be an acyclic coloring of \( F_M \) with colors \( 1, \ldots, m \). If \( e \) is a 2-valid edge of \( F \) contained in \( M \) then \( e \) is 2-valid in some subgraph \( F_{a,b} \) of \( F \), where \( F_{a,b} \) is obtained by “uncontracting” the subgraph of \( F_M \) induced by colors \( a \) and \( b \), \( a, b \in \{1, \ldots, m\} \).

**Proof.** First assume that \( T = F_M \) is a tree. We prove the first item by induction on the number of edges in \( F_M \). If \( T \) has only one edge, then \( F \) has at most 4 vertices (at most 3 if \( M \) is induced) and it is thus a subgraph of \( K_4 \) (respectively \( K_5 \)), that is a 3-tree (respectively 2-tree). Hence \( \mathrm{tw}(F) \leq 3 \) (respectively \( \mathrm{tw}(F) \leq 2 \)).

For a vertex \( y \) of \( T \), let \( X_y \) be the inverse image of \( y \) under contraction, i.e., a set of at most two vertices in \( F \). Suppose that \( T \) has at least 2 edges and \( vw \) is an edge incident to a leaf \( v \). Let \( F' = F - X_v \), \( M' \) be the edge set of \( M - v \) and \( T' \) be the graph obtained by contracting the edges of \( M' \) in \( F' \). Then we see that \( T' = T - v \) is a tree. By induction, \( F' \) is a subgraph of a \((p + 1)\)-tree \( H' \), \( p \in \{1, 2\} \). Consider a bag \( B_v \) in \( H' \) containing \( X_v \). We have that \( |X_v|, |X_u| \in \{1, 2\} \) and the vertices of \( X_v \) are adjacent only to some vertices in \( X_u \subseteq B \). Moreover, if \( M \) is induced then at most one of \( X_u \), \( X_v \) can be of size 2.

If \( H' \) is a 3-tree, then the bags are cliques on 4 vertices. If \( |X_v| = 1 \), add a bag \( B_v \) to \( H' \) that consists of \( X_v \), the neighbors of \( X_v \) in \( F \) (there are at most 2), and at most two extra vertices from \( B \) if needed to make \( B_v \) of size 4. If \( |X_v| = 2 \), let \( X_v = \{v', v''\} \). First add a bag \( B_1 \) to \( H' \) that consists of \( v' \), the neighbors of \( X_v \) in \( B \), and at most two extra vertices from \( B \). Then add a bag \( B_2 \) to \( H' \) that consists of \( v', v'' \), the neighbors of \( v'' \) in \( B \), and at most one extra vertex from \( B \). In both cases this gives a 3-tree that contains \( F \).

1. If \( M \) is induced and \( H' \) is a 2-tree, we have that either \( X_v \) or \( X_u \) is of size 1. If \( |X_v| = 1 \), add a bag consisting of \( X_v \), the neighbors of \( X_v \) in \( F \) (there are at most 2) and an extra vertex from \( B \) if necessary. If \( |X_v| = 2 \), then \( |X_u| = 1 \), so there is only vertex \( w \) in \( F \) that is adjacent to some vertex in \( X_v \), \( w \in B \). Then let \( X_v = \{v', v''\} \). First add a bag with vertices \( w, v', v' \), where \( w' \) is in \( B \) and then add a bag with vertices \( w, v', v \). In both cases this gives a 2-tree that contains \( F \). This proves the first item of the Lemma if \( F_M \) is a tree.
If $F_M$ is a forest, then each component $T'$ of $F_M$ is obtained from some component $F'$ of $F$ by contracting the edges of $M \cap E(F')$. Moreover if $M$ is induced in $F$, then $M \cap E(F')$ is induced in $F'$. By the arguments above each component of $F$ has tree-width at most 3 (respectively 2 if $M$ is induced), and hence $\text{tw}(F) \leq 3$ (respectively $\text{tw}(F) \leq 2$ if $M$ is induced).

To see the second item of the Lemma, consider a witness tree of $e = xy$ with vertices $x, y, z$. Then $x$ and $y$ got contracted to a vertex of color, say $a$, in $F_M$ and $z$ either got contracted or stayed as it is and got some color $b$ under coloring $c$ of $F_M$. So, $x, y, z \in V(F_{a, b})$. Since $x, y$, and $z$ induce a tree in $F$, they induce a tree in $F_{a, b}$ since $F_{a, b}$ is an induced subgraph of $F$.

4.1 Proof of Theorem 5

Given a graph $G \in \mathcal{C}$, consider an acyclic coloring $c$ of $G$ with $x$ colors. For each of the $\binom{x}{2}$ many pairs of colors $\{i,j\}$, $i \neq j$, we split the forest induced by these colors into the 2-strong forest $F_{i,j}$, and the induced matching $M_{i,j}$, which respectively gather the components with at least two edges and the ones with only one edge. Each edge of $G$ belongs to either $F_{i,j}$ or $M_{i,j}$ for some $i, j$. Let $E$ be the set of edges that do not belong to any of $F_{i,j}$’s. Thus each $e \in E$ is in $M_{i,j}$, for some $i, j$. We see that the $\binom{x}{2}$ 2-strong forests $F_{i,j}$ cover all 2-valid edges of $G$ that are not in $E$. Next, we shall show two different approaches how to cover the 2-valid edges of $G$ that are in $E$.

Consider fixed $i, j$, $1 \leq i < j \leq x$, and let $M = M_{i,j}$. Let $G_M$ be the graph obtained from contracting the edges of $M$ in $G$. Then $G_M$ is again in the class $\mathcal{C}$ and thus has acyclic chromatic number at most $x$. Consider an acyclic coloring $c'$ of $G_M$ and the graph $H_{a,b}$ induced by two distinct color classes $a$ and $b$ in $G_M$. Consider $G_{a,b} = G_{a,b}(M)$, the graph obtained from $H_{a,b}$ by uncontracting $M$. Then, since $H_{a,b}$ is an induced subgraph of $G_M$, $G_{a,b}$ is an induced subgraph of $G$ and $H_{a,b}$ is obtained from $G_{a,b}$ by contracting the edges of $M$ in $G_{a,b}$. Thus, since $H_{a,b}$ is a forest and $M \cap E(G_{a,b})$ is an induced matching in $G_{a,b}$, by Lemma 10 applied with $F = G_{a,b}$, we have $\text{tw}(G_{a,b}) \leq 2$. Thus by Theorem 4, the 2-valid edges of $G_{a,b}$ are covered by three 2-strong forests. By the second item of Lemma 10, applied with $F = G$, each 2-valid edge of $G$ that is in $M$ is 2-valid in some $G_{a,b}$. Each 2-valid edge of $G$ from $E$ belongs to some matching $M = M_{i,j}$ and thus is 2-valid in some $G_{a,b}(M)$. There are altogether $\binom{x}{2}$ such $M$’s and for each at most $\binom{x}{2}$ graphs $G_{a,b}(M)$, each contributing three covering forests. We see that all 2-valid edges of $G$ from $E$ are covered by at most $3\binom{x}{2}$ 2-strong forests in $G$.

To see another way to deal with the edges in $E$ consider the subgraph $G'$ of $G$ formed by these edges. Since each vertex of color $i$ under coloring $c$ is incident to at most one vertex in each $M_{i,j}$, $1 \leq j \leq x$, $j \neq i$, the maximum degree of $G'$ is at most $x - 1$. Therefore the edge set of $G'$ can be decomposed into $x$ matchings by Vizing’s theorem. Let $M$ be one such a matching. Let $G_M$ be obtained from $G$ by contracting $M$. Again, $G_M \in \mathcal{C}$. Let $c'$ be an acyclic coloring of $G_M$ with at most $x$ colors and let $H_{a,b}$ be the induced forest formed by some color classes $a$ and $b$. Further, let $G_{a,b}$ be a graph obtained by uncontracting $M$ in $H_{a,b}$. By Lemma 10 applied with $F = G_{a,b}$, $\text{tw}(G_{a,b}) \leq 3$. Thus by Theorem 4, the 2-valid edges of $G_{a,b}$, can be covered by twelve 2-strong forests. By the second item of Lemma 10 applied with $F = G$, each 2-valid edge of $G$ that is in $M$ is 2-valid in some $G_{a,b}$. Therefore, all 2-valid edges of $G$ from $E$ are covered by at most $12\binom{x}{2}$ 2-strong forests.

So, the 2-valid edges from $E$ are covered by at most $\min\{12\binom{x}{2}, 3\binom{x}{2}\}$ 2-strong forests. Recall that the remaining 2-valid edges that are in $F_{i,j}$’s are covered by at most $\binom{x}{2}$ 2-strong forests. The theorem follows.
5 Graphs of bounded tree-depth

Recall that $G$ has tree-depth at most $d$ if and only if there exists a rooted forest $F$ of depth $d$ such that $G$ is a subgraph of the closure of $F$. When $F$ consists of only one tree and $V(G) = V(F)$ we call such a tree an underlying tree of $G$. In particular any connected graph of tree-depth at most $d$ has an underlying tree of depth at most $d$. Let $TD(d)$ denote the set of all graphs of tree-depth at most $d$ having an underlying tree of depth $d$ and let $TD^*(d) \subseteq TD(d)$ be the set of all graphs $G$ in $TD(d)$ some of whose underlying trees of depth $d$ have the root of degree 1. When we talk about a graph $G$ of tree-depth at most $d$, we usually associate a fixed underlying tree $T$ with root $r$ to it. Let $f_k(d) = \max\{f_k(G) \mid td(G) \leq d\}$. We shall inductively show that this function is well-defined and is bounded by $(2k)^d$ from above.

**Lemma 11.** Let $G \in TD^*(d)$ with underlying tree $T$ of depth at most $d$ having a root $r$ of degree 1 that is adjacent to a vertex $x$ in $T$. Then $G - r, G - x \in TD(d - 1)$.

**Proof.** It suffices to observe that $G - r, G - x$ are graphs of tree-depth at most $d$ with underlying trees obtained by removing $r$ from $T$, or removing $r$ in $T$ and renaming $x$ with $r$, respectively. The roots of these trees are $x$ and $r$, respectively. \hfill \Box

An edge $e$ is almost $k$-valid in a graph $G \in TD(d)$ with associated root $r$ if it is not $k$-valid in $G$ but there is an induced path in $G$ containing $r$ and $e$. Note that, for example, if both endpoints of $e$ are adjacent to $r$, then there is no such induced path. Let $g_k(d)$ and $g^*_k(d)$ be the the maximum number of almost $k$-valid edges in a graph $G \in TD(d)$, respectively $G \in TD^*(d)$.

**Lemma 12.** For all positive integers $k, d$, with $d \geq 2$, we have $g_k(d) \leq (2k)^{d-1} - 1$ and $g^*_k(d) \leq 2(2k)^{d-2} - 1$.

**Proof.** For a fixed $k$ we prove the claim by induction on $d$. If $d = 2$, then any $G \in TD(d)$ is a subgraph of a star. Therefore either all edges form a $k$-strong forest or $G$ has at most $k - 1$ edges and thus each edge is almost $k$-valid. Hence $g_k(2) = k - 1$ for any $k \geq 1$. $g^*_k(2) = 0$, and $g^*_k(2) = 1$ for $k \geq 2$.

Now suppose that $d \geq 3$ and that the statement of the Lemma is true for smaller values of $d$. We consider $g^*_k(d)$ first. Let $G \in TD^*(d)$, $r$ be the root of the underlying tree $T$ of $G$, and $x$ be the unique neighbor of $r$ in $T$. Let $A$ be the set of almost $k$-valid edges $e$ in $G$ such that there is an induced path in $G$ containing $e, r$, and not containing $x$. Let $B$ be the set of all remaining almost $k$-valid edges in $G$. Each edge in $A$ is almost $k$-valid in $G - x$. Similarly each edge in $B \setminus \{rx\}$ is almost $k$-valid in $G - r$ (here the underlying tree is as in Lemma 11). Note that $rx$ might or might not be an edge of $G$ and if it is an edge, it is $k$-valid or almost $k$-valid.

Since $G - r, G - x \in TD(d - 1)$ by Lemma 11, we conclude that $|A|, |B \setminus \{rx\}| \leq g_k(d - 1)$. Inductively we obtain $|A| + |B| \leq 2 \cdot ((2k)^{d-2} - 1) + 1 = 2 \cdot (2k)^{d-2} - 1$. Since $G \in TD^*(d)$ was arbitrary we have that $g^*_k(d) \leq 2 \cdot (2k)^{d-2} - 1$.

Now consider $g_k(d)$ for $d \geq 3$. Let $G \in TD(d)$ and let $r$ be the root of the underlying tree $T$ of $G$. Let $x_1, \ldots, x_t$ be the neighbors of $r$ in $T$ and let $G_i, 1 \leq i \leq t$, be the subgraph of $G$ induced by $i$th branch of $T$, i.e., by $r, x_i$, and all descendants of $x_i$ in $T$. Assume that each of $G_1, \ldots, G_s$ has an edge incident to $r$ and other $G_i$’s do not have such an edge. Then, in particular, each almost $k$-valid edge of $G$ is in some $G_i$, $i = 1, \ldots, s$.

Assume first that $s \geq k$. There is a star $S$ of size $s$ with center $r$ and edges being from distinct $G_i$’s. If $e$ is an edge contained in some induced path $P$ in $G_i$ where $r$ is an endpoint,
then there is an induced tree in $G$ formed by $P$ and all edges of $S$ except perhaps for the edge from $G_i$. So any such edge is $k$-valid and there are no almost $k$-valid edges in $G$.

Now assume that $s \leq k - 1$. Any almost $k$-valid edge $e \in E(G)$ is almost $k$-valid in $G_i$, for some $i \in [s]$. There are at most $g_k^i(d)$ almost $k$-valid edges in $E(G_i)$, $i = 1, \ldots, s$. Therefore the total number of almost $k$-valid edges in $G$ is at most $s \cdot g_k^i(d) \leq (k-1) g_k^i(d) \leq (k-1)(2^{d-1}k^{d-2} - 1) \leq 2^{d-1}k^{d-1} - 1$.

Since $G \in TD(d)$ was arbitrary we have that $g_k(d) \leq (2k)^{d-1} - 1$. \hfill \Box

5.1 Proof of Theorem 8

Let $G$ be a graph of tree-depth $d$. First of all consider the case $k = 1$. It is well-known (see [6, 21]) that any graph of tree-depth at most $d$ has tree-width at most $d - 1$. Hence if $td(G) \leq d$, then by Theorem 4 we have $f_1(G) \leq \binom{d}{2}$. On the other hand $K_d$ is of tree-depth $d$ and $f_1(K_d) = \binom{d}{2}$, so the above bound is tight.

For the rest of the proof assume that $k \geq 2$. We prove that $f_k(G) \leq (2k)^d$ for any graph of tree-depth $d$. First we prove this claim for $G \in TD(d)$ by induction on $d$, then we deduce the general case. Recall that $G \in TD(d)$ if and only if $G$ has an underlying tree of depth $d$.

If $d = 1$, then any graph in $TD(d)$ has no edges. If $d = 2$, then any $G \in TD(d)$ is a subgraph of a star. If $G$ has at least $k$ edges then $G$ is a $k$-strong forest itself. If $G$ has less than $k$ edges, there are no $k$-valid edges. Hence $f_k(G) \leq 1$.

Now suppose that $d \geq 3$ and assume that $f_k(G) \leq (2k)^d$ for any $G \in TD(d')$ and $d' < d$. Let $r$ denote the root of the underlying tree $T$ of $G$. Let $x_1, \ldots, x_t$ be the neighbors of $r$ in $T$ and let $G_i$, $1 \leq i \leq t$ be the subgraph of $G$ induced by $i$th branch of $T$, i.e., by $r$, $x_i$, and all descendants of $x_i$ in $T$. Then $G_i \in TD(d')$, where in the corresponding underlying tree $r$ is the root and $x_i$ is its unique neighbor, $i = 1, \ldots, t$. Here the underlying trees for subgraphs are defined as in Lemma 11.

Let $E$ be the set of $k$-valid edges in $G$. We shall split $E$ into sets $S_1, \ldots, S_5$ and shall show that each of these sets is covered by a desired number of $k$-strong forests, see Figure 9.

- Let $S_1 = \{rx_i : i = 1, \ldots, t\} \cap E$.
- Let $S_2$ be the set of edges from $E \setminus S_1$ that are $k$-valid in $G_i - r$ for some $i \in \{1, \ldots, t\}$.
- Let $S_3$ be the set of edges from $E \setminus \{S_1 \cup S_2\}$ that are $k$-valid in $G_i - x_i$ for some $i \in \{1, \ldots, t\}$.
- Let $S_4$ be the set of edges from $E \setminus \{S_1 \cup S_2 \cup S_3\}$ that are $k$-valid in $G_i$ for some $i \in \{1, \ldots, t\}$.
- Let $S_5 = E - \{S_1 \cup S_2 \cup S_3 \cup S_4\}$.

I.e., $S_2, S_3, \text{ and } S_4$ consist of $k$-valid edges in some $G_i$, with witness trees not containing $r$, not containing $x_i$, and containing both $r$ and $x_i$, respectively. Each edge in $S_5$ is not $k$-valid in any $G_i$, but it is almost $k$-valid in some $G_i$. In the following, we say that a family of forests is a good cover of an edge set if these covering forests are $k$-strong.

**Claim.** There exists a good cover $F_1$ of $S_1$ of size at most $k - 1$.

**Proof of Claim.** If $|S_1| < k$, for each $e \in S_1$ pick a $k$-strong forest in $G$ containing $e$ and let $F_1$ be the set of all these forests. If $|S_1| \geq k$, then let $F_1$ consist of one forest that is the induced star with edge set $S_1$. \hfill \triangle
Claim. There exists a good cover $F_2$ of $S_2$ of size at most $f_k(d-1)$.

Proof of Claim. Let $i \in \{1, \ldots, t\}$. By Lemma 11 we have that $G_i - r \in TD(d-1)$. Hence we have $f_k(G_i - r) \leq f_k(d-1)$. Let $A_i$ denote a good cover of $S_i \cap E(G_i - r)$ of size at most $f_k(d-1)$ with forests contained in $G_i - r$. We shall combine the forests from $A_1, \ldots, A_t$ into a new family $F_2$ of at most $f_k(d-1)$ $k$-strong forests of $G$. Any union $F_1 \cup \cdots \cup F_i$, where $F_i \in A_i$, is a $k$-strong forest in $G$ because none of these forests contain $r$ and thus there are no edges between $F_i$ and $F_j$ for $1 \leq i < j \leq t$. So, let each forest from $F_2$ be a union of at most one forest from each $A_i$. We see that we can form such a family of size at most max $\{|A_i| : 1 \leq i \leq t\}$. Thus $F_2$ is a family of at most $f_k(d-1)$ $k$-strong forests of $G$. Since each edge $e \in S_2$ is $k$-valid in some $G_i - r$, the set $F_2$ is a good cover of $S_2$.

Claim. There exists a good cover $F_3$ of $S_3$ of size at most $f_k(d-1)$.

Proof of Claim. Let $i \in \{1, \ldots, t\}$. By Lemma 11 we have that $G_i - x_i \in TD(d-1)$. Hence we have $f_k(G_i - x_i) \leq f_k(d-1)$. Let $A_i$ denote a good cover of $S_i \cap E(G_i - x_i)$ consisting of at most $f_k(d-1)$ forests in $G_i - x_i$. Similarly as in the claim before, any union $F_1 \cup \cdots \cup F_i$, where $F_i \in A_i$, is a $k$-strong induced forest in $G$ because all of these forests contain $r$ as $S_2 \cap S_3 = \emptyset$. Let each forest in $F_3$ be a union of at most one forest from each of $A_i$, $i = 1, \ldots, t$. It is clear that one can build such a family with at most max $\{|A_i| : 1 \leq i \leq t\}$ forests. So, $F_3$ consists of at most $f_k(d-1)$ $k$-strong forests of $G$. Since each edge $e \in S_3$ is $k$-valid in some $G_i - x_i$, the set $F_3$ is a good cover of $S_3$.

Claim. There exists a good cover $F_4$ of $S_4$ of size at most $2g_k(d-1)$.

Proof of Claim. Let $i \in \{1, \ldots, t\}$ and let $e \in S_4$. Then $e$ is $k$-valid in $G_i$. Since $e$ is not $k$-valid in $G_i - r$ and not $k$-valid in $G_i - x_i$, this means that each witness tree of $e$ in $G_i$ contains both $r$ and $x_i$. Every such witness tree contains a path containing $e$, $x_i$, and not $r$, or a path containing $e$, $r$, and not $x_i$. This path is induced and thus (as $e \notin S_i$) $e$ is almost $k$-valid in either $G_i - r$ or $G_i - x_i$, respectively. Hence, $|S_4 \cap G_i| \leq 2g_k(d-1)$ by the definition of $g_k(d-1)$, since $G_i - r, G_i - x_i \in TD(d-1)$ by Lemma 11.

For each edge $e$ in $S_4 \cap G_i$, we pick one witness tree of $e$ that is contained in $G_i$. Let $A_i$ denote the set of these at most $2g_k(d-1)$ induced $k$-strong forests. As all induced forests in $A_1, \ldots, A_t$ contain the root $r$, we can again, as in the previous claim, form a set $F_4$ of at most $2g_k(d-1)$ $k$-strong forests in $G$ covering $S_4$.

Claim. There exists a good cover $F_5$ of $S_5$ of size at most $(k-1)g_k(d)$.

Proof of Claim. Note that $S_5$ consists of those edges whose witness trees all contain edges from at least two different $G_i$’s. Without loss of generality assume that each of $G_1, \ldots, G_s$
have an edge incident to \( r \) and the other \( G_i \)'s do not have such an edge. Then each \( e \in S_i \) is in \( G_i \) for some \( i \in \{1, \ldots, s\} \) and moreover \( e \) is almost \( k \)-valid in this \( G_i \). Hence

\[
|E(G_i) \cap S_i| \leq g_k^*(d) \quad \text{for all } i, 1 \leq i \leq s, \text{ and } |E(G_i) \cap S_i| = 0 \quad \text{for all } i, s < i < t.
\]

If \( s \leq k - 1 \), then \( |S_i| \leq (k - 1) g_k^*(d) \). In this case we let each forest in \( F \) consists of one witness tree for each \( e \in S_i \).

If \( s \geq k \), then for all \( i \in \{1, \ldots, s\} \) and all \( j, 1 \leq j \leq g_k^*(d) \), we pick (not necessarily distinct) induced paths \( P_i^j \) in \( G_i \) starting with \( r \) such that each edge in \( S_i \) is contained in some path \( P_i^j \). Such a path containing \( e \in S_i \) exists, since \( e \) is almost \( k \)-valid in some \( G_i \).

As \( s \geq k \), the union of paths \( P_i^j \cup \cdots \cup P_i^{j+1} \) forms a \( k \)-strong induced forest in \( G \) for each \( j \), \( 1 \leq j \leq g_k^*(d) \). Moreover each edge in \( S_i \) is contained in one of these forests. Hence \( F_5 = \{ P_i^1 \cup \cdots \cup P_i^{j+1} | 1 \leq j \leq g_k^*(d) \} \) is a good cover of \( S_i \) as desired.

From the above claims we get that \( F = F_1 \cup F_2 \cup F_3 \cup F_4 \cup F_5 \) is a good cover of all \( k \)-valid edges in \( G \), since \( S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \) contains all \( k \)-valid edges of \( G \). Moreover, with the above claims, induction, Lemma 12 and \( k \geq 2 \) we get

\[
|F| \leq |F_1| + |F_2| + |F_3| + |F_4| + |F_5| \leq k - 1 + 2 f_k(d - 1) + 2 g_k(d - 1) + (k - 1) g_k^*(d)
\]

\[
\leq k - 1 + 2 \cdot (2k)^{-d - 1} + 2 \cdot ((2k)^{-d - 2} - 1) + (k - 1)(2(2k)^{-d - 2} - 1)
\]

\[
\leq (2k)^{-d - 2}(2k + 2(2k - 1)) = (2k)^{-d - 2} 6k \leq (2k)^{-d - 2} 4k^2 = (2k)^{-d},
\]

which proves that \( f_k(G) \leq (2k)^d \) for \( G \in T \mathcal{D}(d) \).

Now if \( G \) has tree-depth at most \( d \) then each component of \( G \) is in \( T \mathcal{D}(d) \). By the previous arguments all \( k \)-valid edges of such a component can be covered by at most \((2k)^d \) \( k \)-strong forest. A union of one such forest from each component is a \( k \)-strong forest in \( G \). Hence we can form at most \((2k)^d \) \( k \)-strong forests covering all \( k \)-valid edges of \( G \). Thus \( f_k(G) \leq (2k)^d \) for each graph \( G \) of tree-depth at most \( d \).

Finally we shall prove that \( f_k(G) \leq (2k)^{k + 1}{d \choose k+1} \), for \( d \geq k + 1 \) and a graph \( G \) with \( \text{td}(G) \leq d \). Let \( H \) be a maximal tree-depth \( d \) supergraph of \( G \) on the same set of vertices. It is known [20, 21], that there is a proper \( d \)-coloring of \( H \) (a so called \( d \)-centered coloring) such that any set of \( p \) colors, \( p \leq d \), induces a tree-depth \( p \) graph. We hence have a \( d \)-coloring of \( G \) such that each of the \({d \choose k+1}\) subsets of \( (k+1) \) colors induces a graph of tree-depth \( k + 1 \). As each \( k \)-valid edge has a witness tree induced by \( k + 1 \) vertices, each witness tree belongs to one of these \({d \choose k+1}\) graphs. So each \( k \)-valid edge of \( G \) is \( k \)-valid in (at least) one of these graphs. Thus the total number of \( k \)-strong forests covering \( k \)-valid edges of \( G \) is at most \({d \choose k+1}(2k)^{k+1} \), where the bound \((2k)^{k+1} \) comes from the first part of the theorem when \( d = k + 1 \).

\[
6 \quad \text{Proof of Theorem 1}
\]

Recall that \( \chi_p(G) \) is the smallest integer \( q \) such that \( G \) admits a vertex coloring with \( q \) colors such that for each \( p' \leq p \) each \( p' \)-set of colors induces a subgraph of \( G \) of tree-depth at most \( p' \). Since \( G \) is of bounded expansion there is a sequence of integers \( a_1, a_2, \ldots \), that for any graph \( G \in \mathcal{C} \) and any \( p \), \( \chi_p(G) \leq a_p \).

Let \( G \in \mathcal{C} \). Note that \( \chi(D, G) \leq \chi(G) \) and hence Theorem 3(i) gives \( f_1(G) \leq (\chi_d(G))^2 \). Thus we can take \( b_1 = \binom{d}{2} \). For \( k \geq 2 \), consider a \((k+1)-\text{tree-depth coloring of } G \) with \( \chi_{k+1}(G) \leq a_{k+1} \). Colors. For each of the \( \chi_{k+1}(G) \) subgraphs \( H \) induced by \( k+1 \) colors, consider a cover of the \( k \)-valid edges in \( H \) with \( f_k(H) \) \( k \)-strong forests. Note that each \( k \)-valid edge of \( G \) is \( k \)-valid in at least one of these graphs. Indeed, a witness tree of an edge
is induced by \(k + 1\) vertices, that are colored with at most \(k + 1\) different colors, hence \(e\) is \(k\)-valid in a graph \(H\) induced by these colors. Then the union \(\mathcal{F}\) of all these forests is a cover of all \(k\)-valid edges in \(G\). Finally observe that each \(H\) has tree-depth at most \(k + 1\), and thus we have \(f_k(H) \leq (2k)^{k+1}\) by Theorem 8. Hence \(|\mathcal{F}| \leq \binom{n}{k+1}(2k)^{k+1}\), and we can take \(b_k = \binom{n}{k+1}(2k)^{k+1}\).

### 7 Conclusions

In this paper we introduce the \(k\)-strong induced arboricity \(f_k(G)\) of a graph \(G\) to be the smallest number of \(k\)-strong forests covering the \(k\)-valid edges of \(G\), where a forest is \(k\)-strong if all its components have size at least \(k\) and an edge is \(k\)-valid if it belongs to an induced tree on \(k\) edges. For \(k \in \mathbb{N}\), call a class \(\mathcal{C}\) of graphs \(f_k\)-bounded, if there is a constant \(c = c(\mathcal{C}, k)\) such that \(f_k(G) \leq c\) for each \(G \in \mathcal{C}\), and let us say that \(\mathcal{C}\) is \(f\)-bounded if \(\mathcal{C}\) is \(f_k\)-bounded for each \(k \in \mathbb{N}\).

We show that this new graph parameter \(f_k\) is non-monotone as a function of \(k\) and, for \(k \geq 2\), as a function of \(G\) using induced subgraph partial ordering. Indeed, there exist classes of graphs \(\mathcal{C}_k\) and \(\mathcal{C}_k'\), \(k \geq 2\), such that \(\mathcal{C}_k\) is \(f_k\)-bounded but not \(f_{k+1}\)-bounded, while \(\mathcal{C}_k'\) is \(f_{k+1}\)-bounded but not \(f_k\)-bounded. Nevertheless, \(f_k\) behaves nicely for so-called graph classes of bounded expansion, in particular for minor-closed families. Our main result is that every class \(\mathcal{C}\) of bounded expansion is \(f\)-bounded. This implies, in particular, that the adjacent closed vertex-distinguishing number for planar graphs is bounded by a constant. Additionally, we find upper and lower bounds on \(f_1(G)\), the induced arboricity, and study the relation between \(f_1\) and the well-known notions of arboricity and acyclic chromatic number.

It remains open to improve the lower and upper bounds on \(f_k\) for a given graph class. For example, for planar graphs the maximum value for \(f_1\) is between 6 (as certified by \(K_4\)) and 10 (following from \(f_1(G) \leq \binom{n}{2}\) and Borodin’s result on the acyclic chromatic number of planar graphs [7]). For graphs \(G\) of tree-width \(t\), we provide explicit universal upper bounds on \(f_1(G)\) and \(f_2(G)\) in Theorem 4. For \(k \geq 3\) Theorem 1 states the existence of a constant upper bound. Using \(f_k(G) \leq \binom{n}{k+1}(2k)^{k+1}\) from the proof of Theorem 1 and \(\chi_p(G) \leq t^p + 1\) for \(G\) of tree-width \(t\) [20], we conclude that \(f_k(G) \leq \binom{n}{k+1}(2k)^{k+1}\) for any integer \(k\) and any graph \(G\) of tree-width \(t\). This upper bound is most likely far from the actual value, and improving its order of magnitude seems to be an interesting challenge.

A natural generalization of the \(k\)-strong induced arboricity would be the following: For a set \(S \subseteq \mathbb{N}\) of natural numbers and a graph \(G\) define \(f_S(G)\) to be the minimum number of induced forests in \(G\) such that for all \(s \in S\) each \(s\)-valid edge in \(G\) lies in a component of size at least \(s\) of some of the forests. Clearly, we have \(f_k(G) = f_{\{k\}}(G)\) and for \(T \subseteq S\) we have \(f_T(G) \leq f_S(G)\). In particular considering \(S = \{1, \ldots, k\}\) gives a parameter similar to the \(p\)-tree-depth chromatic number \(\chi_p(G)\) as defined by Nešetřil and Ossona de Mendez [21]. As before, we say that a graph class \(\mathcal{C}\) is \(f_S\)-bounded if there is a constant \(c = c(\mathcal{C}, S)\) such that \(f_S(G) \leq c\) for all \(G \in \mathcal{C}\). It follows from our results, that for any finite set \(S \subseteq \mathbb{N}\) any class \(\mathcal{C}\) of bounded expansion is \(f_S\)-bounded. On the other hand, the examples in Theorem 3 show that the class of tree-width 2 graphs is not \(f_{\{2\}}\)-bounded, even the class of tree-depth 3 graphs, and the graphs of maximum degree at most 4. It is interesting to identify non-trivial graph classes that are \(f_{\mathbb{N}}\)-bounded. For example, one can show that \(f_{\{2\}}(G) \leq 4\), whenever \(G = P_n \times P_m\) is a grid graph. In Theorem 3 (iii) we present a graph class \(\mathcal{C}\) that is not of bounded expansion, for which \(f_{\mathbb{N}}(G) \leq 2\) for each \(G \in \mathcal{C}\).

Finally, let us mention the concept of nowhere dense classes of graphs, which is defined
in terms of so-called excluded shallow minors [22], see also [8, 13, 14, 18, 27]. Each class of bounded expansion is nowhere dense, but not the other way round [22]. Similarly, each class of bounded expansion is $f$-bounded (by Theorem 1), but not the other way round (by Theorem 3 (iii)). In fact, nowhere dense classes of graphs and $f$-bounded classes of graphs are two different extensions of classes of graphs of bounded expansion. For $n \in \mathbb{N}$, let $G_n$ be a graph with girth, minimum degree and chromatic number at least $n$. Moreover let $C_1 = \{G_n \mid n \in \mathbb{N}\}$, let $C_2$ be the class of 3-subdivisions of all graphs in $C_1$, and let $C_3$ denote the class of all 1-subdivisions of $K_{n,n}$, $n \in \mathbb{N}$ (the class from the proof of Theorem 3 (iii)). Then it is easy to check that $C_1$ is nowhere dense, but not $f$-bounded (in fact not $f_k$-bounded for any $k \in \mathbb{N}$), and hence not of bounded expansion, $C_2$ is nowhere dense and $f$-bounded, but not of bounded expansion, and $C_3$ is $f$-bounded, but not nowhere dense, and hence not of bounded expansion.

References


