# The $k$-strong induced arboricity of a graph

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**Abstract**

The induced arboricity of a graph $G$ is the smallest number of induced forests covering the edges of $G$. This is a well-defined parameter bounded from above by the number of edges of $G$ when each forest in a cover consists of exactly one edge. Not all edges of a graph necessarily belong to induced forests with larger components. For $k \geqslant 1$, we call an edge $k$-valid if it is contained in an induced tree on $k$ edges. The $k$-strong induced arboricity of $G$, denoted by $f_k(G)$, is the smallest number of induced forests with components of sizes at least $k$ that cover all $k$-valid edges in $G$. This parameter is highly non-monotone. However, we prove that for any proper minor-closed graph class $\mathcal{C}$, and more generally for any class of bounded expansion, and any $k \geqslant 1$, the maximum value of $f_k(G)$ for $G \in \mathcal{C}$ is bounded from above by a constant depending only on $\mathcal{C}$ and $k$.

We prove that $f_2(G) \leqslant 3\binom{t+1}{3}$ for any graph $G$ of tree-width $t$ and that $f_k(G) \leqslant (2k)^d$ for any graph of tree-depth $d$. In addition, we prove that $f_2(G) \leqslant 310$ when $G$ is planar, which implies that the maximum adjacent closed vertex-distinguishing chromatic number of planar graphs is constant.

## 1 Introduction

Let $G = (V, E)$ be a simple, finite and undirected graph. An *induced forest* in $G$ is an acyclic induced subgraph of $G$. A *cover* of $X \subseteq V \cup E$, is a set of subgraphs of $G$ whose union contains every element of $X$. It is certainly one of the most classical problems in graph theory to cover the vertex set $V$ or the edge set $E$ of $G$ with as few as possible subgraphs from a specific class, such as independent sets [26], stars [2], paths [1], forests [17], planar graphs [16], interval graphs [15], or graphs of tree-width $t$ [10], just to name a few. Extensive research on graph covers has been devoted to the following two graph parameters: The *vertex arboricity* of $G$ is the minimum $t$ such that $V$ can be covered with $t$ induced forests [17]. The *arboricity* of $G$, denoted as $a(G)$, is the minimum $t$ such that $E$ can be covered with $t$ forests [17]. Nash-Williams [17] proved that the arboricity of a graph $G$ is given by $\max\{|E(H)|/(|V(H)| - 1)\}$ where the maximum is taken over all subgraphs $H$ of $G$.

Here, we define the *induced arboricity* $f_1(G)$ of $G$ to be the minimum $t$ such that $E$ can be covered with $t$ induced forests. Moreover, we introduce the *$k$-strong induced arboricity* $f_k(G)$, where we additionally require that each connected component of the induced forests has at least $k$ edges. More precisely, for $k \geqslant 1$, let a *$k$-strong forest* of $G$ be an induced forest $F$ in $G$, each of whose connected components consists of at least $k$ edges. Hence, a 1-strong forest is one that has no isolated vertices and a 2-strong forest is one that has neither $K_1$- nor $K_2$-components. As isolated vertices in a forest do not help to cover the edges of $G$, these can be easily omitted. Thus for the induced arboricity of $G$ as defined above it suffices to consider induced forests where every component has at least one edge.

However, note that for $k \geqslant 2$ we possibly can not cover $E(G)$ with $k$-strong forests, for example when $G$ is a clique or when $|E(G)| < k$. An edge $e \in E(G)$ is defined to be *$k$-valid*, $k \geqslant 1$, if there exists a $k$-strong forest in $G$ containing $e$. Of course, every edge is 1-valid. When $k$ is fixed and clear from context, we write valid instead of $k$-valid. By removing a leaf in an induced tree, one obtains an induced tree with exactly one edge less. Thus an edge $e$ is $k$-valid if and only if it belongs to an induced tree with exactly $k$ edges. We call such a tree a *witness tree* for $e$.

We define the *$k$-strong induced arboricity* of $G$, denoted by $f_k(G)$, as the smallest number of induced $k$-strong forests covering all $k$-valid edges of $G$. The main result of this paper shows that for well-behaved classes of graphs, such as for example minor-closed families, the parameter $f_k$ is bounded from above by a constant independent of the order of the graph. Recall that a graph class $\mathcal{C}$ is called *minor-closed* if for each $G \in \mathcal{C}$ any graph $H$ obtained from $G$ by deleting edges or vertices, or by contracting edges is contained in $\mathcal{C}$.

To define such graph families, we define the tree-depth and tree-width and follow the notions used by Nešetřil and Ossona de Mendez [20, 21], see also [12].

Tree-width: For a positive integer $t$, a *$t$-tree* is a graph obtained from a union of copies $G_1, \ldots, G_q$ of $K_{t+1}$, called *bags*, such that for any $j = 2, \ldots, q$ we have that the set $V(G_j) \cap (V(G_1) \cup \cdots \cup V(G_{j-1}))$ has size $t$ and is contained in $V(G_i)$ for some $i \in \{1, \ldots, j-1\}$. I.e., a $t$-tree is a graph obtained by starting with $K_{t+1}$ and "adding" cliques (bags) on $t+1$ vertices one at a time, by identifying $t$ vertices of the new clique with some $t$ vertices of some previously added clique. A graph $G$ has *tree-width* $t$ and we write $\mathrm{tw}(G) = t$, if $t$ is the smallest integer such that $G$ is a subgraph of a $t$-tree. Note that the graphs of tree-width 1 are the forests on at least one edge.

Tree-depth: The *transitive closure* of a rooted tree $T$ with a root $r$ is the graph obtained from $T$ by adding every edge $uv$ such that $v$ is on the $u$-$r$-path of $T$. A rooted tree has *depth* $d$ if the largest number of vertices on a path to the root is $d$. Now, a graph $G$ has *tree-depth* $d$, denoted by $\mathrm{td}(G) = d$, if $d$ is the smallest integer such that each connected component of $G$ is a subgraph of the transitive closure of a rooted tree of depth $d$.

Tree-depth coloring: A *$p$-tree-depth coloring* of a graph $G$ is a vertex coloring such that each set of $p'$ color classes, $p' \leqslant p$, induces a subgraph of $G$ with tree-depth at most $p'$. So a 1-tree-depth coloring is exactly a proper coloring of $G$, while a 2-tree-depth coloring is a proper coloring of $G$ in which any two color classes induce a star forest (a graph of tree-depth at most 2). Let $\chi_p(G)$ be the minimum number of colors needed in a $p$-tree-depth coloring of $G$. Then $\chi(G) = \chi_1(G)$ and $\chi_p(G) \leqslant \mathrm{td}(G)$ for any $p \geqslant 1$ [21].

Bounded Expansion: A class $\mathcal{C}$ of graphs is of *bounded expansion* if for each positive integer $p$ there is a constant $a_p = a(p, \mathcal{C})$ such that for each $G \in \mathcal{C}$ we have $\chi_p(G) \leqslant a_p$.

**Theorem 1.** *Let $\mathcal{C}$ be a class of graphs that is of bounded expansion. Then for each positive integer $k$ there is a constant $b_k = b(k, \mathcal{C})$ such that for each $G \in \mathcal{C}$ we have $f_k(G) \leqslant b_k$.*

Building on work of DeVos *et al.* [9], Nešetřil and Ossona de Mendez [20, 21] proved that several classes of graphs are of bounded expansion, such as minor-closed classes, classes of graphs with an excluded topological minor, or classes of graphs of bounded tree-width or tree-depth. This implies the following.

**Corollary 2.** *Let $k$ be a positive integer and let $\mathcal{C}$ be one of the following classes: a minor-closed class of graphs that is not the class of all graphs, a class of graphs with no topological minor isomorphic to a given fixed graph, or a class of graphs of tree-width or tree-depth at most $t$, for some fixed $t$. Then there is a constant $c = c(k, \mathcal{C})$ such that $f_k(G) \leqslant c$ for any $G \in \mathcal{C}$.*

Theorem 1 states that bounded expansion implies for each $k$ the existence of a constant upper bound on the parameter $f_k$. We show in Theorem 3(iii) that the converse statement is not true. While the $k$-strong induced arboricity is bounded by a constant on the families of graphs listed above, it is a highly non-monotone and unbounded parameter in general. We also show in Theorem 3(i),(ii) some relations between the parameters $f_k(G)$, $\mathrm{tw}(G)$, $\mathrm{td}(G)$, $a(G)$, and the acyclic chromatic number $\chi_{\mathrm{acyc}}(G)$. Recall that the acyclic chromatic number of a graph $G$ is the smallest number of colors in a proper coloring of $G$ in which any two color classes induce a forest. Note that the arboricity of $G$ is at most $f_1(G)$.

**Theorem 3.** *(i) There exists a constant $c > 0$ such that for each graph $G$ we have $c \log(\chi_{\mathrm{acyc}}(G)) \leqslant f_1(G) \leqslant \binom{\chi_{\mathrm{acyc}}(G)}{2}$.*

*(ii) For any integers $k \geqslant 2$, $n \geqslant 3$, and for each item below there is graph $G$ satisfying the listed conditions:*

- *(a) $a(G) = 2$ and $f_1(G) \geqslant n$,*
- *(b) $f_k(G) \leqslant 3$ and $f_{k+1}(G) \geqslant n$,*
- *(c) $f_k(G) \geqslant n$ and $f_{k+1}(G) = 0$,*
- *(d) $G$ has an induced subgraph $H$ such that $f_k(G) = 3$ and $f_k(H) \geqslant k$,*
- *(e) $\mathrm{tw}(G) = 2$ and $f_k(G) \geqslant k$,*
- *(f) $\mathrm{td}(G) = 3$ and $f_k(G) \geqslant k - 1$.*

*(iii) There is a class $\mathcal{C}$ of graphs that is not of bounded expansion such that for each $G \in \mathcal{C}$ and each $k \geqslant 1$ we have $f_k(G) \leqslant 2$.*

Theorem 1 provides the existence of constants bounding $f_k(G)$ for graphs $G$ from special classes. Next, we give more specific bounds on these constants. Clearly, if $\mathrm{tw}(G) \leqslant 1$, then $f_k(G) \leqslant 1$ for every $k$. However, already for graphs $G$ of tree-width 2 finding the largest possible value of $f_k(G)$ for $k \geqslant 2$ is non-trivial. We show, in particular that $f_1(G) \leqslant \binom{\mathrm{tw}(G)+1}{2}$ for any graph $G$, which is best-possible, since since $\mathrm{tw}(K_{t+1}) = t$ and $f_1(K_{t+1}) = \binom{t+1}{2}$, and that $f_2(G) \leqslant 3\binom{\mathrm{tw}(G)+1}{2}$ for any graph $G$, which is best-possible when $\mathrm{tw}(G) = 2$, as certified by $G$ being $K_3$ with a pendant edge at each vertex.

**Theorem 4.** *For every graph $G$ of tree-width $t \geqslant 2$, we have that $f_1(G) \leqslant \binom{t+1}{2}$ and $f_2(G) \leqslant 3\binom{t+1}{3}$.*

Nešetřil and Ossona de Mendez [19] prove that for each minor-closed class $\mathcal{C}$ of graphs that is not the class of all graphs there is a constant $x$ such that each graph in $\mathcal{C}$ has acyclic chromatic number at most $x$. We show how to bound $f_2$ in terms of $x$.

**Theorem 5.** *For every minor-closed class of graphs $\mathcal{C}$ whose members have acyclic chromatic number at most $x$, we have that for every $G \in \mathcal{C}$,*

$$f_2(G) \leqslant \begin{cases} \binom{x}{2}(3\binom{x}{2} + 1), & \text{if } x \leqslant 9, \\ \binom{x}{2}(12x + 1), & \text{if } x \geqslant 9. \end{cases}$$

Using Borodin's result that each planar graph has acyclic chromatic number at most 5 [7], Theorem 5 implies the following.

**Corollary 6.** *For every planar graph $G$, $f_2(G) \leqslant 310$.*

This result answers an open question about vertex-distinguishing numbers of graphs. Given a graph $G$, an assignment of positive integers to its vertices is called distinguishing if the sum of the labels in the closed neighborhood of any vertex $v$ differs from such this sum in the closed neighborhood of any of the neighboring vertices $u$ of $v$, unless $N[u] = N[v]$. I.e., the labeling distinguishes between adjacent vertices. The smallest positive integer $\ell$ such that there is a distinguishing labeling of $G$ with labels in $\{1, \dots, \ell\}$ is called *adjacent closed vertex-distinguishing number* of $G$, denoted $\text{dis}[G]$. While for an analogous notion $\text{dis}(G)$ with open neighborhoods considered instead of closed neighborhoods, it is known that there is a constant $c$ such that $\text{dis}(G) \leqslant c$ for any planar graph $G$, as noted by Norine, see [4], it was not known whether $\text{dis}[G]$ is bounded by a universal constant for all planar graphs. In [3] it was shown that if $f_2(G) \leqslant x$ then $\text{dis}[G]$ is bounded from above by some product of $x$ pairwise co-prime numbers. Thus, Corollary 6 implies the following.

**Corollary 7.** *There is an absolute constant $c$ such that for any planar graph $G$, the adjacent closed distinguishing number* $\text{dis}[G] \leqslant c$.

The tree-depth is somehow a more restrictive variant of the tree-width. We have $\text{tw}(G) \leqslant \text{td}(G) - 1$ for any graph $G$, but when $G$ is a graph of tree-depth $d$, the longest path in $G$ has at most $2^d - 1$ vertices. In particular, even graphs of tree-width 1 can have arbitrarily large tree-depth [21]. The next theorem shows that the parameter $f_k$ is bounded for graphs of tree-depth $d$ by a polynomial in $d$ as well as a polynomial in $k$.

**Theorem 8.** *For all positive integers $k$, $d$ and any graph $G$ of tree-depth $d$, $f_k(G) \leqslant (2k)^d$. If $d \geqslant k+1$ then $f_k(G) \leqslant (2k)^{k+1}\binom{d}{k+1}$. Moreover $f_1(G) \leqslant \binom{d}{2}$.*

**Organization of the paper:** We prove Theorem 3 in Section 2. We consider graphs of bounded tree-width in Section 3 and prove Theorem 4 in that section. The proofs of the bounds on $f_k$ in terms of the acyclic chromatic number and the proof of Theorem 5 are given in Section 4. Graphs of bounded tree-depth, and more general classes of graphs, are considered in Section 5, where we prove Theorems 8. We prove the main Theorem 1 in Section 6. Finally we summarize our results, state some open questions and discuss other variants of the strong induced arboricity in Section 7.

## 2 General inequalities

In this section we prove the general properties of the parameter $f_k$ listed in Theorem 3.

*Proof of Theorem 3(i).* For the first inequality, consider a covering of $E(G)$ with $x = f_1(G)$ induced forests $F_1, \dots, F_x$ and for each forest a proper 2-coloring of its vertices. Let $c_1, \dots, c_x$ be colorings of $V(G)$ in colors $\{0, 1, 2\}$ such that $c_i(v) = 0$ if $v \notin V(F_i)$, $c_i(v) = 1$ if $v$ is from the first color class of $F_i$, and $c_i(v) = 2$ if $v$ is from the second color class of $F_i$. Let a coloring $c$ of $V(G)$ be defined as $c(v) = (c_1(v), \dots, c_x(v))$, $v \in V(G)$. To see that $c$ is an acyclic coloring assume that two color classes $(a_1, \dots, a_x)$, $(b_1, \dots, b_x)$ induce a cycle $C$. Let $e$ be an edge of $C$. It is in some $F_i$ and hence $\{a_i, b_i\} = \{1, 2\}$. Thus the $i^{\text{th}}$ coordinate of $c$ in the cycle $C$ alternates between 1 and 2. This implies that all edges of $C$ belong to $F_i$, a contradiction since $F_i$ is acyclic. For similar reasons $c$ is proper. Thus $c$ is an acyclic coloring.

For the second inequality, consider an acyclic proper coloring of $G$ using $\chi_{\rm acyc}(G)$ colors. For every pairs of colors $c_1$, $c_2$ the subgraph of $G$ induced by the vertices of color $c_1$ or $c_2$ is an induced forest in $G$. Moreover, every edge of $G$ is contained in exactly one such induced forest. Hence, by removing all isolated vertices from each such forest, we get $f_1(G) \leqslant \binom{\chi_{\rm acyc}}{2}$. □

*Proof of Theorem 3(ii.a).* Let $R^{-1}(t)$ denote the smallest number of colors needed to color $E(K_t)$ without monochromatic triangles. By Ramsey's Theorem [23, 25] we have $R^{-1}(t) \to \infty$ as $t \to \infty$. Choose $t$ sufficiently large such that $R^{-1}(t) \geqslant n^2$. Since $n \geqslant 3$, clearly $t \geqslant 3$.

Let $G$ be the graph obtained from $K_t$ by subdividing each edge once. For an edge $e$ in $K_t$ let $e_1$ and $e_2$ denote the two corresponding edges in $G$. Split $G$ into two subgraphs $G_1$ and $G_2$ where $G_i$ contains all edges $e_i$, $e \in E(K_t)$, $i = 1, 2$. Then $E(G) = E(G_1) \cup E(G_2)$ and, for $i = 1$, 2, each component of $G_i$ is a star with center at an original vertex of $K_t$. Therefore $a(G) \leqslant 2$ and as $t \geqslant 3$, we have $a(G) = 2$.

Let $N = f_1(G)$ and consider induced forests $F_1, \ldots, F_N$ covering all edges of $G$. We consider the following edge-coloring of $K_t$. If there is an $i$, $1 \leqslant i \leqslant N$, with $e_1$, $e_2 \in E(F_i)$, then color the edge $e$ with color $i$ (choose an arbitrary such $i$). Otherwise there are $i$ and $j$, $1 \leqslant i < j \leqslant N$, with $e_1$, $e_2 \in E(F_i) \cup E(F_j)$, $i \neq j$, and we color the edge $e$ with color $\{i, j\}$ (choose an arbitrary such pair). This coloring uses at most $N + \binom{N}{2} = \binom{N+1}{2}$ colors. We claim that there are no monochromatic triangles under this coloring. Indeed there is no triangle in color $i$, $1 \leqslant i \leqslant N$, since $F_i$ contains no cycle, and there is no triangle in color $\{i, j\}$, $1 \leqslant i < j \leqslant N$, since $F_i$ and $F_j$ are induced. Therefore $\binom{f_1(G)+1}{2} = \binom{N+1}{2} \geqslant R^{-1}(t) \geqslant n^2$. This shows that $f_1(G) \geqslant n$, since $\binom{n}{2} < n^2$. □

*Proof of Theorem 3(ii.b).* Like in the proof of part (ii.a), let $R^{-1}(t)$ denote the smallest number of colors needed to color $E(K_t)$ without monochromatic triangles. By Ramsey's Theorem [23, 25] we have $R^{-1}(t) \to \infty$ as $t \to \infty$. Choose $t$ sufficiently large such that $R^{-1}(t) \geqslant n^2$ and, additionally, $t \geqslant 2k + 2$.

Let $G$ be obtained from $K_t$ by subdividing each edge twice and choosing for each original edge of $K_t$ one of its subdivision vertices and adding $k - 1$ pendant edges to this vertex, see Figure 1 (left part) when $k = 2$. Observe that all edges of $G$ are $k$-valid and $(k + 1)$-valid.

First we shall show that $f_k(G) \leqslant 3$ by finding 3 $k$-strong forests covering all edges of $G$. For an edge $e$ in $K_t$ let $e_1$, $e_2$, $e_3$ denote the subdividing edges in $G$, with $e_2$ the middle one. Let $T_1$ be the subgraph consisting of all edges $e_2$, $e \in E(K_t)$, and all edges adjacent to $e_2$ different from $e_1$ and $e_3$ (the pendant edges). Then $T_1$ is an induced forest and each component of $T_1$ is a star on $k$ edges. Since $t \geqslant 2k + 2$, we can choose an orientation of $K_t$ such that each vertex has out-degree and in-degree at least $k$. Indeed, if $t$ is odd we find such an orientation by following an Eulerian walk, if $t$ is even, we find such an orientation of $K_{t-1}$ as before and orient the edges incident to the remaining vertex $x$ such that at least $k$ of these edges are in-edges at $x$ and at least $k$ of them are out-edges at $x$. For each edge $e = uv$ in $K_t$ that is oriented from $u$ to $v$ put the edge in $\{e_1, e_3\}$ that is incident to $u$ into $T_2$ and the other edge from $\{e_1, e_3\}$ into $T_3$. Then $T_2$ and $T_3$ are induced forests and each component of $T_2$ and $T_3$ is a star on at least $k$ edges. Moreover each edge of $G$ is contained in $E(T_1) \cup E(T_2) \cup E(T_3)$. Therefore $f_k(G) \leqslant 3$.

Next, we prove that $f_{k+1}(G) \geqslant n$. Let $N = f_{k+1}(G)$ and consider $(k + 1)$-strong forests $F_1, \ldots, F_N$ covering all edges of $G$. For each edge $e$ of $K_t$, if $F_i$ contains $e_2$, then it contains either $e_1$ or $e_3$ as well, since each component of $F_i$ has at least $k + 1$ edges. We consider the following edge-coloring of $K_t$. If there is an $i$, $1 \leqslant i \leqslant N$, such that $e_1$, $e_2$, $e_3 \in E(F_i)$, then color the edge $e$ with $i$ (choose an arbitrary such $i$). Otherwise there are distinct $i$, $j$, $1 \leqslant i, j \leqslant N$, such that, without loss of generality, $e_1$, $e_2 \in E(F_i)$ and $e_3 \in E(F_j)$. In this

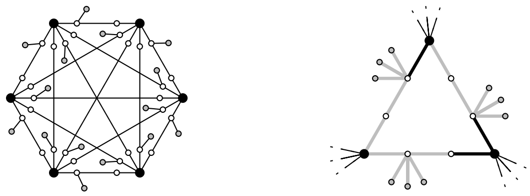

Figure 1: Illustration of the proof of Theorem 3(ii.c). The subgraph on the right corresponds to a monochromatic triangle in $K_t$. Note that the gray forest is not induced.

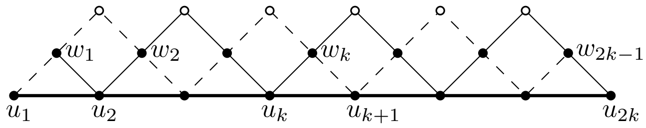


Figure 2: A graph $G$ illustrating the proof of Theorem 3(ii.d) and (ii.e) in case $k = 4$ with the subgraph $H$ induced by bold vertices. The marked paths (one bold, one solid, one dashed) form three induced forests covering all the edges of $G$.

case color the edge $e$ with the pair $(i, j)$ (choose an arbitrary such pair). This coloring uses at most $N + N(N-1) = N^2$ colors. We claim that there are no monochromatic triangles under this coloring. Indeed, for any $i$ and $j$ there is no triangle in color $i$, $1 \leqslant i \leqslant N$, since $F_i$ contains no cycle, and there is no triangle in color $(i, j)$, $1 \leqslant i, j \leqslant N$, since $F_i$ and $F_j$ are induced. See Figure 1 (right part) in case $k = 4$. Therefore the number of colors is at least $R^{-1}(t)$ and at most $N^2 = f_{k+1}^2(G)$. Thus $f_{k+1}(G) \geqslant \sqrt{R^{-1}(t)} \geqslant n$. □

*Proof of Theorem 3(ii.c).* Consider the graph $G$ formed by taking the union of a clique on $n+1$ vertices and a path of length $k-1$ that shares an endpoint with the clique. Then we see that all edges of $G$ incident to the path are $k$-valid. However, no two edges of the clique could be in the same induced forest, thus $f_k(G) \geqslant n$. On the other hand, since each induced tree in $G$ contains at most one edge from the clique, it could have at most $k$ edges. Thus there are no $(k+1)$-valid edges and $f_{k+1}(G) = 0$. □

*Proof of Theorem 3(ii.d) and (ii.e).* Consider the graph $G$ shown in Figure 2. We see from Figure 2 that $G$ is covered by three large induced trees (a bold, a solid, and a dashed path) and thus $f_k(G) \leqslant 3$. Let $H$ be its induced subgraph formed by the bold vertices shown in the Figure 2. We see that $H$ is formed by a path $u_1, u_2, \ldots, u_{2k}$ and independent vertices $w_1, w_2, \ldots, w_{2k-1}$ such that $w_i$ is adjacent to $u_i$ and $u_{i+1}$. Then consider the matching in $H$ formed by the edges $u_i w_i$, $k \leqslant i \leqslant 2k-1$, and an induced tree $T_i$ in $H$ of size $k$ containing $u_i w_i$, $k \leqslant i \leqslant 2k-1$. We see that the trees $T_k, \ldots, T_{2k-1}$ are distinct and their pairwise union induces a triangle in $H$. Thus no two of them can belong to the same $k$-strong forest in $H$. Hence $f_k(H) \geqslant k$. This proves Theorem 3(ii.d). In addition, $\text{tw}(H) = 2$. This proves Theorem 3(ii.e) (where $H$ plays the role of $G$ from the Theorem). □

*Proof of Theorem 3(ii.f).* Consider the graph $G$ shown in the Figure 3. Then $\text{td}(G) = 3$

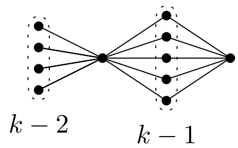


Figure 3: A graph $G$ illustrating the proof of Theorem 3(ii.f).

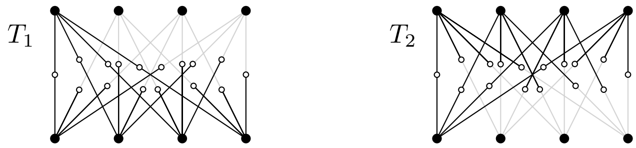


Figure 4: Two maximum induced trees $T_1$, $T_2$ covering all edges of the graph $G_n$ ($n = 4$) obtained from $K_{n,n}$ by subdividing every edge once.

(look at the cut vertex as a root of the underlying tree) and $f_k(G) = k - 1$. □

*Proof of Theorem 3(iii).* Consider the complete bipartite graph $K_{n,n}$ and let $G_n$ be the graph obtained from $K_{n,n}$ by subdividing each edge once. We claim that for any positive integers $k$ and $n$ we have $f_k(G_n) \leqslant 2$, and moreover that the graph class $\mathcal{C} = \{G_n \mid n \in \mathbb{N}\}$ is not of bounded expansion. For the latter, we shall show that for each integer $a$ there is an $n$ such that $\chi_3(G_n) > a$. So consider an arbitrary vertex coloring $c$ of $G_n$ with $a$ colors. Color an edge $e$ of $K_{n,n}$ with the set of colors assigned to the three corresponding vertices in $G_n$. This coloring uses at most $\binom{a}{3} \leqslant 2^a$ colors and hence, for sufficiently large $n$, contains a monochromatic path on at least 5 vertices in some color $C$. So the subgraph of $G_n$ induced by vertices colored with colors from $C$ contains a path on at least $9 > 2^3$ vertices. Since such a path has tree-depth greater than 3 and since $|C| \leqslant 3$, $c$ is not a 3-tree-depth coloring. Since $c$ was arbitrary, $\chi_3(G_n) > a$ for sufficiently large $n$. Hence $\mathcal{C}$ is not of bounded expansion.

Next we prove that, for any integers $k, n \geqslant 1$ we have $f_k(G_n) \leqslant 2$. To this end, we construct two maximum induced trees in $G_n$ covering all edges of $G_n$. Clearly, $|V(G_n)| = 2n + n^2$ and we claim that a largest induced tree in $G_n$ contains exactly $n + 1 + n^2 = |V(G_n)| - (n - 1)$ vertices. Let $X(G_n)$ denote the smallest number of vertices in $G_n$ whose deletion makes the graph acyclic. (That is, $X(G_n)$ denotes the size of a minimum feedback vertex set [11].) We shall prove by induction on $n$ that $X(G_n) \geqslant n - 1$. In fact, for $n = 1$, $G_n$ is a tree itself and thus $X(G_1) = 0$. For $n \geqslant 2$, consider an 8-cycle in $G_n$ consisting of four original vertices $v_1, v_2, v_3, v_4$ of $K_{n,n}$, $v_1$, $v_2$ from one bipartition class and $v_3$, $v_4$ from the other, and the four subdivision vertices corresponding to the four edges $v_1v_3$, $v_1v_4$, $v_2v_3$ and $v_2v_4$ in $K_{n,n}$. At least one of these eight vertices has to be deleted to make the graph acyclic, say it is one of $v_1$, $v_3$, or the vertex $x$ subdividing edge $v_1v_3$. Then $G_n - \{v_1, v_3, x\}$ is isomorphic to $G_{n-1}$ and thus at least $X(G_{n-1})$ further vertices have to be deleted. Hence by induction we get $X(G_n) \geqslant X(G_{n-1}) + 1 \geqslant (n - 2) + 1 = n - 1$, as desired. Thus any induced tree in $G_n$ has at most $n^2 + 2n - (n - 1)$ vertices.

On the other hand, one obtains a maximum induced tree $T_1$ by deleting $n - 1$ original vertices of $K_{n,n}$ that belong to the same bipartition class, see Figure 4. Deleting $n-1$ vertices from the other bipartition class gives symmetrically a maximum induced tree $T_2$. Finally, observe that $T_1$ and $T_2$ together cover all edges of $G_n$, which certifies that $f_k(G_n) \leqslant 2$ for $k \leqslant n + 1 + n^2$. For $k > n + 1 + n^2$ no edge of $G_n$ is $k$-valid and thus $f_k(G_n) = 0$. □

# 3 Graphs of bounded tree-width

We start with a list of properties of graphs of tree-width 2. Then we shall prove that $f_2(G) \leqslant 3$ for any graph $G$ of tree-width 2. This is the main part of the proof. Then, we shall do an easy reduction argument to express the upper bound from Theorem 4 on $f_2(G)$ for graphs $G$ of larger tree-width.

### 3.1 Properties of graphs with tree-width 2 and observations

Consider any fixed graph $G$ of tree-width 2. Firstly, $G$ contains no subdivision of $K_4$ [5]. (In fact, this property characterizes tree-width 2 graphs.) Moreover, it is well-known (see for example [24]) that as long as $|V(G)| \geqslant 3$, there is a 2-tree $H$ with $G \subseteq H$ and $V(H) = V(G)$. Let us fix such a 2-tree $H$. Every edge of $H$ is in at least one triangle of $H$. Consider the partition $E(H) = E_{\text{in}}(H) \dot\cup E_{\text{out}}(H)$ of the edges of $H$, where $E_{\text{out}}(H)$ consists of those edges that are contained in only one triangle of $H$, called the *outer edges of $H$*. Respectively, $E_{\text{in}}(H)$ consists of those edges that are contained in at least two triangles of $H$, called the *inner edges of $H$*. Note, if $H$ is outerplanar, every edge is in at most two triangles, and our definition corresponds to the usual partition into outer and inner edges of an outerplanar embedding of $H$.

The following two statements can be easily proved by induction on $|V(H)|$. Indeed, both statements hold with "if and only if" and are maintained in the construction sequence of the 2-tree $H$.

**(P1)** If $v \in V(H)$ is incident to two outer edges in the same triangle of $H$, then $\deg_H(v) = 2$.

**(P2)** If $uw \in E_{\text{in}}(H)$, then $H - \{u, w\}$ is disconnected.

It is easy to see that for any 2-connected graph $F$ with $|V(F)| \geqslant 4$ and for any two vertices $u, w \in V(F)$ we have the following:

**(P3)** For every connected component $K$ of $F - \{u, w\}$ we have $N(u) \cap V(K) \neq \emptyset$ and $N(w) \cap V(K) \neq \emptyset$.

**(P4)** The graph $F - \{u, w\}$ is connected if and only if the graph $F'$ obtained from $F$ by identifying $u$ and $w$ into a single vertex is 2-connected.

Now if $G$ is a 2-connected graph of tree-width 2 and $H$ is a 2-tree with $G \subseteq H$ and $V(H) = V(G)$, then we have the following properties.

**(P5)** $E_{\text{out}}(H) \subseteq E(G)$

**(P6)** For every $e \in E_{\text{out}}(H)$ the graph $G/e$ obtained from $G$ by contracting edge $e$ is 2-connected.

To see **(P5)**, consider any edge $e = uw$ in $E_{\text{out}}(H)$. As $G$ is 2-connected, there exists a cycle $C$ in $G$ through $u$ and $w$. If $e \in E(C)$ then $e \in E(G)$ and we are done. Otherwise, in $H$, edge $e$ is a chord of cycle $C$, splitting it into two cycles $C_1$ and $C_2$. As $H$ is a chordal graph, $C_1$ and $C_2$ are triangulated, i.e., $e$ is contained in a triangle with vertices in $C_1$ and another triangle with vertices in $C_2$. Thus $e \in E_{\text{in}}(H)$, a contradiction to $e \in E_{\text{out}}(H)$.

To see **(P6)**, consider any outer edge $e = uw$ of $H$. By **(P4)** we have that $G/e$ is 2-connected if $G - \{u, w\}$ is connected. Assume for the sake of contradiction that $G - \{u, w\}$ is disconnected and let $K_1$, $K_2$ be two connected components of $G - \{u, w\}$. Then by **(P3)** for $i = 1, 2$ we have $N(u) \cap V(K_i) \neq \emptyset$ and $N(w) \cap V(K_i) \neq \emptyset$. Hence we can find a cycle $C$ in $H$ for which $e = uw$ is a chord by going from $u$ to $w$ through $K_1$ and from $w$ to $u$ through $K_2$. As before, it follows that $e \in E_{\text{in}}(H)$, a contradiction to $e \in E_{\text{out}}(H)$. Hence, $G/e$ is 2-connected.

Finally, let us characterize the edges of $G$ that are not 2-valid. An edge $uv$ of $G$ is called a *twin edge* if $N[u] = N[v]$, i.e., if the closed neighborhoods of $u$ and $v$ coincide. Observe that twin edges are exactly the edges that are not 2-valid.

**(P7)** If $G$ is 2-connected, $\mathrm{tw}(G) = 2$, and $xy$ is a twin edge in $G$, then $G$ is a 2-tree consisting of $r$ triangles, $r \geqslant 1$, all sharing the common edge $xy$.

To prove **(P7)**, let $H$ be a 2-tree with $G \subseteq H$ and $V(H) = V(G)$. Consider the set $S = N(x) - y = N(y) - x$. As $G$ is 2-connected, we have $|S| \geqslant 1$. We claim that for each $w \in S$ the edges $xw$ and $yw$ are outer edges. Indeed, if $xw \in E_{\mathrm{in}}(H)$, then by **(P2)** the graph $H - \{x, w\}$ and therefore also the graph $G - \{x, w\}$ is disconnected. Let $K$ be a connected component of $G - \{x, w\}$ which does not contain $y$. By **(P3)** we have $N(x) \cap V(K) \neq \emptyset$, as $G$ is 2-connected. This is a contradiction to $N(x) - y = N(y) - x$. Thus for every $w \in S$ we have $xw \in E_{\mathrm{out}}(H)$ and symmetrically $yw \in E_{\mathrm{out}}(H)$. It follows from **(P1)** that $\deg_H(w) = 2$ and hence $\deg_G(w) = 2$. Thus $V(G) = S \cup \{x, y\}$, as desired.

### 3.2 Special decomposition of tree-width 2 graphs

**Theorem 9.** *Let $G = (V, E)$ be a connected non-empty graph of tree-width at most 2, different from $C_4$. Then there exists a coloring $c : V \to \{1, 2, 3\}$ such that each of the following holds:*

*(1) For each $i \in \{1, 2, 3\}$ the set $V_i = \{v \in V \mid c(v) \neq i\}$ induces a forest $F_i$ in $G$.*

*(2) For each $i \in \{1, 2, 3\}$ there is no $K_1$-component in $F_i$.*

*(3) For each $i \in \{1, 2, 3\}$ every $K_2$-component of $F_i$ is a twin edge.*

*Proof.* We call a coloring $c : V \to \{1, 2, 3\}$ *good* if it satisfies **(1)**–**(3)** We shall prove the existence of a good coloring by induction on $|V|$, the number of vertices in $G$. We distinguish the cases whether $G$ is 2-connected or not.

**Case 1: $G$ is not 2-connected.** If $G$ is a single edge $uv$, then a desired coloring is given by $c(u) = c(v) = 1$. Otherwise $G$ has at least two blocks. Consider a leaf block $B$ in the block-cutvertex-tree of $G$ and the unique cut vertex $v$ of $G$ in this block. Consider the graphs $G_1 = B$ and $G_2 = G - (B - v)$, see Figure 5. We define colorings $c_1$ and $c_2$ for $G_1$ and $G_2$, respectively, as follows. For $i \in \{1, 2\}$, if $G_i \neq C_4$, then we apply induction to $G_i$ and obtain a coloring $c_i$ of $G_i$ satisfying **(1)**–**(3)**. On the other hand, if $G_i = C_4$, then we take the coloring $c_i$ shown in the left of Figure 6, in which the cut vertex $v$ is incident to the only $K_2$-component. Note that this coloring satisfies **(1)** and **(2)**.

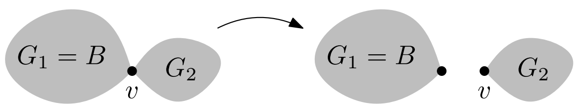


Figure 5: Splitting at cut vertex $v$.

Without loss of generality we have $c_1(v) = c_2(v) = 1$ and hence $c_1$ and $c_2$ can be combined into a coloring $c$ of $G$ by setting $c(x) = c_i(x)$ whenever $x \in V(G_i)$, $i = 1, 2$. Clearly, this coloring $c$ satisfies **(1)** and **(2)**.

If $xy$ is a $K_2$-component of $F_i$ in $G$ for some $i \in \{1, 2, 3\}$, with $v \neq x, y$, then $xy$ is also a $K_2$-component of the corresponding forest in $G_1$ or $G_2$, say in $G_1$. In particular, $G_1 \neq C_4$, since $v \neq x, y$. So $c_1$ satisfies **(3)** and $xy$ is a twin edge in $G_1$ and thus also in $G$, as desired. On the other hand, if $xy$ is an edge of $F_i$ in $G$ for some $i \in \{1, 2, 3\}$, say with $x = v$ and $y \in V(G_1)$, then $v$ is incident to another edge of $F_i$ in $G_2$, since $c_2$ satisfies **(2)**. Thus $xy$ is not a $K_2$-component of $F_i$.

In any case, $c$ satisfies **(3)** and hence $c$ is a good coloring.

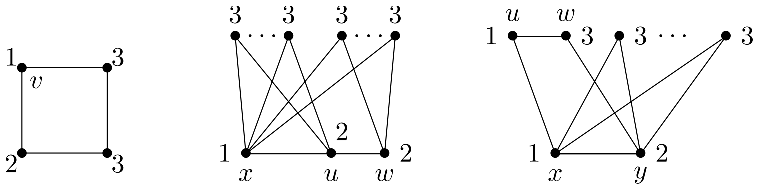


Figure 6: Left: A coloring of $C_4$ satisfying **(1)** and **(2)** and with one $K_2$-component (in $F_3$). Middle and right: Good colorings when the graph obtained by contracting edge $uw$ has a twin edge.

**Case 2: $G$ is 2-connected.** Recall that by **(P5)** we have $E_{\text{out}}(H) \subseteq E(G)$, i.e., every outer edge is an edge of $G$. An outer edge $e$ is called *contractible* if $e$ is in no triangle of $G$. If $G$ has a contractible edge $e$, then we shall consider the smaller graph $G/e$ obtained by contracting $e$. If $G/e$ has a twin edge, we shall give a good coloring $c$ of $G$ directly, otherwise we obtain a good coloring $c$ by induction. On the other hand, if $G$ has no contractible edges, we shall give a good coloring $c$ directly.

**Case 2A: There exists a contractible edge $e$ in $G$.** Let $e = uw$ be contractible. Consider the graphs $G' = G/e$ and $H' = H/e$ obtained from $G$ and $H$ by contracting edge $e$ into a single vertex $v$. As $e \in E_{\text{out}}(H)$ and $H \neq K_3$ (otherwise $e$ would be in a triangle of $G$), we have that $H'$ is a 2-tree. In particular, $\text{tw}(G') \leqslant 2$. Moreover, by **(P6)** $G'$ is also 2-connected. Finally, as $e$ is not in a triangle in $G$, we have that $|E(G)| = |E(G') \cup \{e\}|$.

If $G' = C_4$ then $G = C_5$ and it is easy to check that coloring the vertices around the cycle by $1, 1, 2, 2, 3$ gives a coloring $c$ satisfying **(1)**, **(2)** and **(3)**.

If $G'$ has a twin edge $xy$, then by **(P7)** we have that $G' = H'$ and $G'$ consists of $r$ triangles, $r \geqslant 1$, all sharing the common edge $xy$. Since the contractible edge $e$ lies in no triangle of $G$ and $G \neq C_4$, we have that $G'$ is not a triangle and thus in fact $r \geqslant 2$.

Now if $v = y$ (the case $v = x$ being symmetric), then $G$ looks like in Figure 6 (middle) and a good coloring of $G$ is given by $c(x) = 1$, $c(u) = c(w) = 2$ and $c(z) = 3$ for every $z \in S$. On the other hand, if $v \in S$, then without loss of generality $ux \in E(G)$ and $wy \in E(G)$, and $G$ looks like in Figure 6 (right side). A good coloring of $G$ is given by $c(x) = c(u) = 1$, $c(y) = 2$, and $c(z) = 3$ for every $z \in (S \cup w) - v$. In both cases it is easy to check that $c$ satisfies **(1)**, **(2)** and **(3)**.

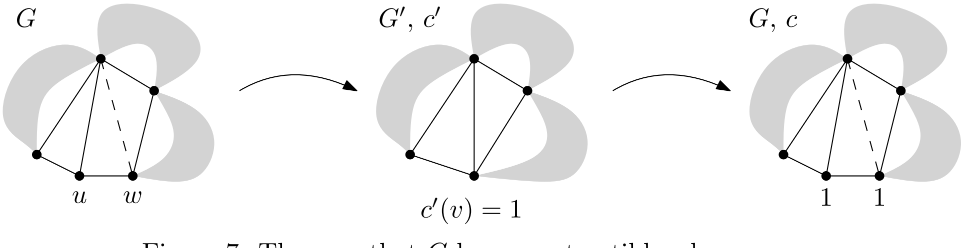


Figure 7: The case that $G$ has a contractible edge $e = uw$.

So finally we may assume that $G' \neq C_4$ and $G'$ has no twin edge. Applying induction to $G'$, we obtain a good coloring $c'$ of $V(G') = V - \{u, w\} + v$ with corresponding induced forests $F_1'$, $F_2'$ and $F_3'$ in $G'$. We define a coloring $c : V \to \{1, 2, 3\}$ by $c(x) = c'(x)$ for each $x \in V' - v$ and $c(u) = c(w) = c'(v)$, see Figure 7.

Say $c'(v) = 1$. Now $c$ satisfies **(1)** as $F_1 = F_1'$, and $F_2'$, $F_3'$ are obtained from $F_2$, $F_3$, respectively, by contracting the edge $uw$, that is not in a triangle in $G$. Thus, since $F_i'$ had no $K_1$ or $K_2$-components, so does $F_i$. This shows that $c$ is a good coloring.

**Case 2B: There are no contractible edges in $G$.** In this case we define the coloring $c : V \to \{1, 2, 3\}$ to be some proper 3-coloring of $H$. To prove that $c$ satisfies **(1)**, assume for the sake of contradiction that there is a cycle in $F_i$ for some $i \in \{1, 2, 3\}$, i.e., a cycle using the two colors in $\{1, 2, 3\} - \{i\}$. Since $H$ is chordal, a shortest such cycle would be a 2-colored triangle, which contradicts $c$ being a proper coloring.

To prove that the coloring $c$ satisfies **(2)** and **(3)**, we define for any vertex $x \in V$ a *trail around* $x$ to be a sequence $s_1, \dots, s_r$ of $r$ distinct neighbors of $x$, $r \geqslant 2$, such that $x, s_i, s_{i+1}$ form a triangle in $H$ for $i = 1, \dots, r-1$ and $xs_r \in E_{\text{out}}(H)$. Note that $xs_i \in E_{\text{in}}(H)$ for $i = 2, \dots, r-2$. Moreover, for any triangle $x, y, z$ in $H$ we can greedily construct a trail around $x$ whose first elements are $s_1 = y$ and $s_2 = z$. Indeed, having constructed $s_1, \dots, s_i$ then either $xs_i \in E_{\text{out}}(H)$ and we are done, or $xs_i \in E_{\text{in}}(H)$ and $x$, $s_i$ have another common neighbor $s_{i+1}$ in $H$ different from $s_{i-1}$. Moreover, $s_{i+1} \neq s_j$ for $j = 1, \dots, i-2$, as otherwise the subgraph of $H$ induced by $\{x, s_j, \dots, s_i\}$ would contain a subdivision of $K_4$, a contradiction to $\text{tw}(H) = 2$. See Figure 8.

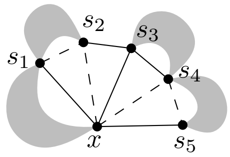


Figure 8: An example of a trail $s_1, \dots, s_5$ around $x$.

Now we shall show that $c$ satisfies **(2)** by proving that any vertex $x$ of $G$, say $c(x) = 1$, has a neighbor in $G$ of color 2 and a neighbor in $G$ of color 3. As $G$ is connected, $x$ is adjacent to some vertex $y$, say $c(y) = 2$. The edge $xy$ is in a triangle in $H$ and its third vertex $z$ has color 3. Consider a trail $s_1, \dots, s_r$ around $x$ starting with $s_1 = y$, $s_2 = z$. Note that $c(s_i) = 2$ if $i$ is odd and $c(s_i) = 3$ if $i$ is even. Hence, if $r$ is even, then as $xs_r \in E_{\text{out}}(H) \subseteq E(G)$, we have that $s_r$ is a neighbor of $x$ of color 3, as desired.

Otherwise $r$ is odd, $r \geqslant 3$, and $xs_{r-1} \notin E(G)$. In particular $xs_r \in E_{\text{out}}(H)$ is in only one triangle of $H$, namely $x, s_{r-1}, s_r$, and in no triangle of $G$. Hence $xs_r$ is contractible, contradicting the assumptions of **Case 2B**. This shows that $c$ satisfies **(2)**.

Finally, to show that $c$ satisfies **(3)**, consider any edge $xy$ of $G$, say $c(x) = 1$ and $c(y) = 2$. If every trail around $x$ starting with $s_1 = y$ and every trail around $y$ starting with $s_1 = x$ has length $r = 2$, then $xy$ is a twin edge. Otherwise, consider a longer such trail, say $s_1, \dots, s_r$ is a trail around $x$ with $s_1 = y$ and $r \geqslant 3$. As before, note that $xs_r \in E_{\text{out}}(H)$ is an edge in $G$ and $c(s_i) = 2$ if $i$ is odd and $c(s_i) = 3$ if $i$ is even. If $xs_i \in E(G)$ for some odd $i \geqslant 3$, we are done. Otherwise $r$ is even, and $xs_{r-1} \notin E(G)$. As before, it follows that $xs_r$ is contractible, contradicting the assumptions of **Case 2B**. Hence $c$ also satisfies **(3)**, which completes the proof. □

### 3.3 Proof of Theorem 4

Let $G$ have tree-width $t$, then $G \subseteq H$ for some $t$-tree $H$. Then $\chi(H) = t + 1$. Consider a proper coloring of $H$ and assume that there is a cycle using two colors. Let $C$ be the shortest such cycle. Since $H$ is chordal, $C$ is a triangle. This is impossible since there are no 2-colored triangles in a proper coloring. Thus $\chi_{\text{acyc}}(H) = t+1$ and therefore $\chi_{\text{acyc}}(G) \leqslant t+1$. Theorem 3(i) immediately implies that $f_1(G) \leqslant \binom{t+1}{2}$.

Next we shall consider $f_2(G)$, where $G$ is a graph of tree-width $t$. If $t = 2$ and $G = C_4$, we see that each edge in $G$ is 2-valid and two edge disjoint paths on 2 edges each form two induced forests covering all the edges, so $f_2(C_4) = 2$. If $t = 2$ and $G \neq C_4$ is connected then

$f_2(G) \leqslant 3 = 3\binom{t+1}{3}$ by Theorem 9. If $t = 2$ and $G$ is not connected, then each component $G'$ of $G$ has tree-width at most 2 and thus satisfies $f_2(G') \leqslant 3$ as argued above. Picking one 2-strong forest from each component of $G$ and taking their union yields a 2-strong forest of $G$ and hence $f_2(G) \leqslant 3$.

Now, let $t \geqslant 3$. Given a graph $G$ of tree-width $t \geqslant 3$, let $H$ be a $t$-tree that contains $G$. It is well-known [9], that any proper $(t+1)$-coloring of $H$ has the property that any set of $p+1$ colors, $p = 1, \dots, t$, induces a $p$-tree. We hence have a $(t+1)$-coloring of $G$ such that each of the $x = \binom{t+1}{3}$ sets of 3 colors induces a graph of tree-width at most 2. Call these graphs $G_1, \dots, G_x$. As each 2-valid edge has a witness tree induced by 3 vertices, each witness tree is contained in $G_i$, for some $i \in \{1, \dots, x\}$. So each 2-valid edge is 2-valid in some $G_i$. Since $\mathrm{tw}(G_i) \leqslant 2$, $f_2(G_i) \leqslant 3$, and so the 2-valid edges of $G_i$ can be covered by 3 2-strong forests, $i = 1, \dots, x$. Hence the 2-valid edges of all $G_i$'s and thus the 2-valid edges of $G$ can be covered with $3x = 3\binom{t+1}{3}$ 2-strong forests. □

# 4 Minor-closed classes of graphs with bounded acyclic chromatic number

**Lemma 10.** *Let $F$ be a graph and let $M$ be a matching in $F$. Let $F_M$ be the graph obtained by contracting the edges of $M$.*

- *If $F_M$ is a forest, then $\mathrm{tw}(F) \leqslant 3$. Moreover, if $M$ is an induced matching, then $\mathrm{tw}(F) \leqslant 2$.*
- *Let $c$ be an acyclic coloring of $F_M$ with colors $1, \dots, m$. If $e$ is a 2-valid edge of $F$ contained in $M$ then $e$ is 2-valid in some subgraph $F_{a,b}$ of $F$, where $F_{a,b}$ is obtained by "uncontracting" the subgraph of $F_M$ induced by colors $a$ and $b$, $a, b \in \{1, \dots, m\}$.*

*Proof.* First assume that $T = F_M$ is a tree. We prove the first item by induction on the number of edges in $F_M$. If $T$ has only one edge, then $F$ has at most 4 vertices (at most 3 if $M$ is induced) and it is thus a subgraph of $K_4$ (respectively $K_3$), that is a 3-tree (respectively 2-tree). Hence $\mathrm{tw}(F) \leqslant 3$ (respectively $\mathrm{tw}(F) \leqslant 2$).

For a vertex $y$ of $T$ let $X_y$ be the inverse image of $y$ under contraction, i.e., a set of at most two vertices in $F$. Suppose that $T$ has at least 2 edges and $uv$ is an edge incident to a leaf $v$. Let $F' = F - X_v$, $M'$ be the edge set of $M - v$ and $T'$ be the graph obtained by contracting the edges of $M'$ in $F'$. Then we see that $T' = T - v$ is a tree. By induction, $F'$ is a subgraph of a $(p+1)$-tree $H'$, $p \in \{1, 2\}$. Consider a bag $B_u$ in $H'$ containing $X_u$. We have that $|X_u|, |X_v| \in \{1, 2\}$ and the vertices of $X_v$ are adjacent only to some vertices in $X_u \subseteq B$. Moreover, if $M$ is induced then at most one of $X_u$, $X_v$ can be of size 2.

If $H'$ is a 3-tree, then the bags are cliques on 4 vertices. If $|X_v| = 1$, add a bag $B_v$ to $H'$ that consists of $X_v$, the neighbors of $X_v$ in $F$ (there are at most 2), and at most two extra vertices from $B$ if needed to make $B_v$ of size 4. If $|X_v| = 2$, let $X_v = \{v', v''\}$. First add a bag $B_1$ to $H'$ that consists of $v'$, the neighbors of $X_v$ in $B$, and at most two extra vertices from $B$. Then add a bag $B_2$ to $H'$ that consists of $v'$, $v''$, the neighbors of $v''$ in $B$, and at most one extra vertex from $B$. In both cases this gives a 3-tree that contains $F$.

If $M$ is induced and $H'$ is a 2-tree, we have that either $X_v$ or $X_u$ is of size 1. If $|X_v| = 1$, add a bag consisting of $X_v$, the neighbors of $X_v$ in $F$ (there are at most 2) and an extra vertex from $B$ if necessary. If $|X_v| = 2$, then $|X_u| = 1$, so there is only vertex $w$ in $F$ that is adjacent to some vertex in $X_v$, $w \in B$. Then let $X_v = \{v', v''\}$. First add a bag with vertices $w, w', v'$, where $w'$ is in $B$ and then add a bag with vertices $w, v', v$. In both cases this gives a 2-tree that contains $F$. This proves the first item of the Lemma if $F_M$ is a tree.

If $F_M$ is a forest, then each component $T'$ of $F_M$ is obtained from some component $F'$ of $F$ by contracting the edges of $M \cap E(F')$. Moreover if $M$ is induced in $F$, then $M \cap E(F')$ is induced in $F'$. By the arguments above each component of $F$ has tree-width at most 3 (respectively 2 if $M$ is induced), and hence $\mathrm{tw}(F) \leqslant 3$ (respectively $\mathrm{tw}(F) \leqslant 2$ if $M$ is induced).

To see the second item of the Lemma, consider a witness tree of $e = xy$ with vertices $x, y, z$. Then $x$ and $y$ got contracted to a vertex of color, say $a$, in $F_M$ and $z$ either got contracted or stayed as it is and got some color $b$ under coloring $c$ of $F_M$.So, $x, y, z \in V(F_{a,b})$. Since $x, y$, and $z$ induce a tree in $F$, they induce a tree in $F_{a,b}$ since $F_{a,b}$ is an induced subgraph of $F$. □

## 4.1 Proof of Theorem 5

Given a graph $G \in \mathcal{C}$, consider an acyclic coloring $c$ of $G$ with $x$ colors. For each of the $\binom{x}{2}$ many pairs of colors $\{i, j\}$, $i \neq j$, we split the forest induced by these colors into the 2-strong forest $F_{i,j}$, and the induced matching $M_{i,j}$, which respectively gather the components with at least two edges and the ones with only one edge. Each edge of $G$ belongs to either $F_{i,j}$ or $M_{i,j}$ for some $i$, $j$. Let $E$ be the set of edges that do not belong to any of $F_{i,j}$'s. Thus each $e \in E$ is in $M_{i,j}$, for some $i, j$. We see that the $\binom{x}{2}$ 2-strong forests $F_{i,j}$ cover all 2-valid edges of $G$ that are not in $E$. Next, we shall show two different approaches how to cover the 2-valid edges of $G$ that are in $E$.

Consider fixed $i$, $j$, $1 \leqslant i < j \leqslant x$, and let $M = M_{i,j}$. Let $G_M$ be the graph obtained from contracting the edges of $M$ in $G$. Then $G_M$ is again in the class $\mathcal{C}$ and thus has acyclic chromatic number at most $x$. Consider an acyclic coloring $c'$ of $G_M$ and the graph $H_{a,b}$ induced by two distinct color classes $a$ and $b$ in $G_M$. Consider $G_{a,b} = G_{a,b}(M)$, the graph obtained from $H_{a,b}$ by uncontracting $M$. Then, since $H_{a,b}$ is an induced subgraph of $G_M$, $G_{a,b}$ is an induced subgraph of $G$ and $H_{a,b}$ is obtained from $G_{a,b}$ by contracting the edges of $M$ in $G_{a,b}$. Thus, since $H_{a,b}$ is a forest and $M \cap E(G_{a,b})$ is an induced matching in $G_{a,b}$, by Lemma 10 applied with $F = G_{a,b}$, we have $\mathrm{tw}(G_{a,b}) \leqslant 2$. Thus by Theorem 4, the 2-valid edges of $G_{a,b}$ are covered by three 2-strong forests. By the second item of Lemma 10, applied with $F = G$, each 2-valid edge of $G$ that is in $M$ is 2-valid in some $G_{a,b}$. Each 2-valid edge of $G$ from $E$ belongs to some matching $M = M_{i,j}$ and thus is 2-valid in some $G_{a,b}(M)$. There are altogether $\binom{x}{2}$ such $M$'s and for each at most $\binom{x}{2}$ graphs $G_{a,b}(M)$, each contributing three covering forests. We see that all 2-valid edges of $G$ from $E$ are covered by at most $3\binom{x}{2}\binom{x}{2}$ 2-strong forests in $G$.

To see another way to deal with the edges in $E$ consider the subgraph $G'$ of $G$ formed by these edges. Since each vertex of color $i$ under coloring $c$ is incident to at most one vertex in each $M_{i,j}$, $1 \leqslant j \leqslant x$, $j \neq i$, the maximum degree of $G'$ is at most $x - 1$. Therefore the edge set of $G'$ can be decomposed into $x$ matchings by Vizing's theorem. Let $M$ be one such a matching. Let $G_M$ be obtained from $G$ by contracting $M$. Again, $G_M \in \mathcal{C}$. Let $c'$ be an acyclic coloring of $G_M$ with at most $x$ colors and let $H_{a,b}$ be the induced forest formed by some color classes $a$ and $b$. Further, let $G_{a,b}$ be a graph obtained by uncontracting $M$ in $H_{a,b}$. By Lemma 10 applied with $F = G_{a,b}$, $\mathrm{tw}(G_{a,b}) \leqslant 3$. Thus by Theorem 4, the 2-valid edges of $G_{a,b}$, can be covered by twelve 2-strong forests. By the second item of Lemma 10 applied with $F = G$, each 2-valid edge of $G$ that is in $M$ is 2-valid in some $G_{a,b}$. Therefore, all 2-valid edges of $G$ from $E$ are covered by at most $12x\binom{x}{2}$ 2-strong forests.

So, the 2-valid edges from $E$ are covered by at most $\min\{12x\binom{x}{2}, \binom{x}{2}3\binom{x}{2}\}$ 2-strong forests. Recall that the remaining 2-valid edges that are in $F_{i,j}$s are covered by at most $\binom{x}{2}$ 2-strong forests. The theorem follows. □

# 5 Graphs of bounded tree-depth

Recall that $G$ has tree-depth at most $d$ if and only if there exists a rooted forest $F$ of depth $d$ such that $G$ is a subgraph of the closure of $F$. When $F$ consists of only one tree and $V(G) = V(F)$ we call such a tree an *underlying tree* of $G$. In particular any connected graph of tree-depth at most $d$ has an underlying tree of depth at most $d$. Let $\mathcal{TD}(d)$ denote the set of all graphs of tree-depth at most $d$ having an underlying tree of depth $d$ and let $\mathcal{TD}^\star(d) \subseteq \mathcal{TD}(d)$ be the set of all graphs $G$ in $\mathcal{TD}(d)$ some of whose underlying trees of depth $d$ have the root of degree 1. When we talk about a graph $G$ of tree-depth at most $d$, we usually associate a fixed underlying tree $T$ with root $r$ to it. Let $f_k(d) = \max\{f_k(G) \mid \mathrm{td}(G) \leqslant d\}$. We shall inductively show that this function is well-defined and is bounded by $(2k)^d$ from above.

**Lemma 11.** *Let $G \in \mathcal{TD}^\star(d)$ with underlying tree $T$ of depth at most $d$ having a root $r$ of degree 1 that is adjacent to a vertex $x$ in $T$. Then $G - r, G - x \in \mathcal{TD}(d-1)$.*

*Proof.* It suffices to observe that $G - r$, $G - x$ are graphs of tree-depth at most $d - 1$ with underlying trees obtained by removing $r$ from $T$, or removing $r$ in $T$ and renaming $x$ with $r$, respectively. The roots of these trees are $x$ and $r$, respectively. □

An edge $e$ is *almost $k$-valid* in a graph $G \in \mathcal{TD}(d)$ with associated root $r$ if it is not $k$-valid in $G$ but there is an induced path in $G$ containing $r$ and $e$. Note that, for example, if both endpoints of $e$ are adjacent to $r$, then there is no such induced path. Let $g_k(d)$ and $g_k^\star(d)$ be the the maximum number of almost $k$-valid edges in a graph $G \in \mathcal{TD}(d)$, respectively $G \in \mathcal{TD}^\star(d)$.

**Lemma 12.** *For all positive integers $k$, $d$, with $d \geqslant 2$, we have $g_k(d) \leqslant (2k)^{d-1} - 1$ and $g_k^\star(d) \leqslant 2(2k)^{d-2} - 1$.*

*Proof.* For a fixed $k$ we prove the claim by induction on $d$. If $d = 2$, then any $G \in \mathcal{TD}(d)$ is a subgraph of a star. Therefore either all edges form a $k$-strong forest or $G$ has at most $k-1$ edges and thus each edge is almost $k$-valid. Hence $g_k(2) = k - 1$ for any $k \geqslant 1$, $g_1^\star(2) = 0$, and $g_k^\star(2) = 1$ for $k \geqslant 2$.

Now suppose that $d \geqslant 3$ and that the statement of the Lemma is true for smaller values of $d$. We consider $g_k^\star(d)$ first. Let $G \in \mathcal{TD}^\star(d)$, $r$ be the root of the underlying tree $T$ of $G$, and $x$ be the unique neighbor of $r$ in $T$. Let $A$ be the set of almost $k$-valid edges $e$ in $G$ such that there is an induced path in $G$ containing $e$, $r$, and not containing $x$. Let $B$ be the set of all remaining almost $k$-valid edges in $G$. Each edge in $A$ is almost $k$-valid in $G-x$. Similarly each edge in $B \setminus \{rx\}$ is almost $k$-valid in $G-r$ (here the underlying tree is as in Lemma 11). Note that $rx$ might or might not be an edge of $G$ and if it is an edge, it is $k$-valid or almost $k$-valid. Since $G-r, G-x \in \mathcal{TD}(d-1)$ by Lemma 11, we conclude that $|A|, |B \setminus \{rx\}| \leqslant g_k(d-1)$. Inductively we obtain $|A| + |B| \leqslant 2 \cdot ((2k)^{d-2} - 1) + 1 = 2 \cdot (2k)^{d-2} - 1$. Since $G \in \mathcal{TD}^\star(d)$ was arbitrary we have that $g_k^\star(d) \leqslant 2 \cdot (2k)^{d-2} - 1$.

Now consider $g_k(d)$ for $d \geqslant 3$. Let $G \in \mathcal{TD}(d)$ and let $r$ be the root of the underlying tree $T$ of $G$. Let $x_1, \ldots, x_t$ be the neighbors of $r$ in $T$ and let $G_i$, $1 \leqslant i \leqslant t$, be the subgraph of $G$ induced by $i^{\text{th}}$ branch of $T$, i.e., by $r$, $x_i$, and all descendants of $x_i$ in $T$. Assume that each of $G_1, \ldots, G_s$ has an edge incident to $r$ and other $G_i$'s do not have such an edge. Then, in particular, each almost $k$-valid edge of $G$ is in some $G_i$, $i = 1, \ldots, s$.

Assume first that $s \geqslant k$. There is a star $S$ of size $s$ with center $r$ and edges being from distinct $G_i$'s. If $e$ is an edge contained in some induced path $P$ in $G_i$ where $r$ is an endpoint, then there is an induced tree in $G$ formed by $P$ and all edges of $S$ except perhaps for the edge from $G_i$. So any such edge is $k$-valid and there are no almost $k$-valid edges in $G$.

Now assume that $s \leqslant k-1$. Any almost $k$-valid edge $e \in E(G)$ is almost $k$-valid in $G_i$, for some $i \in [s]$. There are at most $g_k^\star(d)$ almost $k$-valid edges in $E(G_i)$, $i = 1, \ldots, s$. Therefore the total number of almost $k$-valid edges in $G$ is at most $s \cdot g_k^\star(d) \leqslant (k-1)\, g_k^\star(d) \leqslant (k-1)\,(2^{d-1}k^{d-2}-1) \leqslant 2^{d-1}k^{d-1}-1$.

Since $G \in \mathcal{TD}(d)$ was arbitrary we have that $g_k(d) \leqslant (2k)^{d-1} - 1$. □

## 5.1 Proof of Theorem 8

Let $G$ be a graph of tree-depth $d$. First of all consider the case $k = 1$. It is well-known (see [6, 21]) that any graph of tree-depth at most $d$ has tree-width at most $d-1$. Hence if $\mathrm{td}(G) \leqslant d$, then by Theorem 4 we have $f_1(G) \leqslant \binom{d}{2}$. On the other hand $K_d$ is of tree-depth $d$ and $f_1(K_d) = \binom{d}{2}$, so the above bound is tight.

For the rest of the proof assume that $k \geqslant 2$. We prove that $f_k(G) \leqslant (2k)^d$ for any graph of tree-depth $d$. First we prove this claim for $G \in \mathcal{TD}(d)$ by induction on $d$, then we deduce the general case. Recall that $G \in \mathcal{TD}(d)$ if and only if $G$ has an underlying tree of depth $d$.

If $d = 1$, then any graph in $\mathcal{TD}(d)$ has no edges. If $d = 2$, then any $G \in \mathcal{TD}(d)$ is a subgraph of a star. If $G$ has at least $k$ edges then $G$ is a $k$-strong forest itself. If $G$ has less than $k$ edges, there are no $k$-valid edges. Hence $f_k(G) \leqslant 1$.

Now suppose that $d \geqslant 3$ and assume that $f_k(G) \leqslant (2k)^{d'}$ for any $G \in \mathcal{TD}(d')$ and $d' < d$. Let $r$ denote the root of the underlying tree $T$ of $G$. Let $x_1, \ldots, x_t$ be the neighbors of $r$ in $T$ and let $G_i$, $1 \leqslant i \leqslant t$ be the subgraph of $G$ induced by $i^{\text{th}}$ branch of $T$, i.e., by $r$, $x_i$, and all descendants of $x_i$ in $T$. Then $G_i \in \mathcal{TD}^\star(d)$, where in the corresponding underlying tree $r$ is the root and $x_i$ is its unique neighbor, $i = 1, \ldots, t$. Here the underlying trees for subgraphs are defined as in Lemma 11.

Let $E$ be the set of $k$-valid edges in $G$. We shall split $E$ into sets $S_1, \ldots, S_5$ and shall show that each of these sets is covered by a desired number of $k$-strong forests, see Figure 9.

- Let $S_1 = \{rx_i :\ i = 1, \ldots, t\} \cap E$.
- Let $S_2$ be the set of edges from $E \setminus S_1$ that are $k$-valid in $G_i - r$ for some $i \in \{1, \ldots, t\}$.
- Let $S_3$ be the set of edges from $E \setminus (S_1 \cup S_2)$ that are $k$-valid in $G_i - x_i$ for some $i \in \{1, \ldots, t\}$.
- Let $S_4$ be the set of edges from $E \setminus (S_1 \cup S_2 \cup S_3)$ that are $k$-valid in $G_i$ for some $i \in \{1, \ldots, t\}$.
- Let $S_5 = E - (S_1 \cup S_2 \cup S_3 \cup S_4)$.

I.e., $S_2, S_3$, and $S_4$ consist of $k$-valid edges in some $G_i$, with witness trees not containing $r$, not containing $x_i$, and containing both $r$ and $x_i$, respectively. Each edge in $S_5$ is not $k$-valid in any $G_i$, but it is almost $k$-valid in some $G_i$. In the following, we say that a family of forests is a *good cover* of an edge set if these covering forests are $k$-strong.

**Claim.** *There exists a good cover $\mathcal{F}_1$ of $S_1$ of size at most $k-1$.*

*Proof of Claim.* If $|S_1| < k$, for each $e \in S_1$ pick a $k$-strong forest in $G$ containing $e$ and let $\mathcal{F}_1$ be the set of all these forests. If $|S_1| \geqslant k$, then let $\mathcal{F}_1$ consist of one forest that is the induced star with edge set $S_1$. △

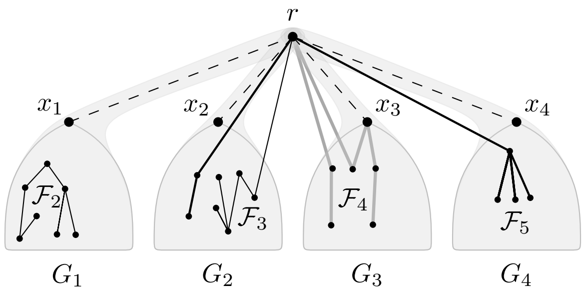

Figure 9: Illustration of the proof of Theorem 8.

**Claim.** *There exists a good cover $\mathcal{F}_2$ of $S_2$ of size at most $f_k(d-1)$.*

*Proof of Claim.* Let $i \in \{1,\ldots,t\}$. By Lemma 11 we have that $G_i - r \in \mathcal{TD}(d-1)$. Hence we have $f_k(G_i - r) \leqslant f_k(d-1)$. Let $\mathcal{A}_i$ denote a good cover of $S_2 \cap E(G_i - r)$ of size at most $f_k(d-1)$ with forests contained in $G_i - r$. We shall combine the forests from $\mathcal{A}_1,\ldots,\mathcal{A}_t$ into a new family $\mathcal{F}_2$ of at most $f_k(d-1)$ $k$-strong forests of $G$. Any union $F_1 \cup \cdots \cup F_t$, where $F_i \in \mathcal{A}_i$, is a $k$-strong forest in $G$ because none of these forests contain $r$ and thus there are no edges between $F_i$ and $F_j$ for $1 \leqslant i < j \leqslant t$. So, let each forest from $\mathcal{F}_2$ be a union of at most one forest from each $\mathcal{A}_i$. We see that we can form such a family of size at most $\max\{|\mathcal{A}_i| : 1 \leqslant i \leqslant t\}$. Thus $\mathcal{F}_2$ is a family of at most $f_k(d-1)$ $k$-strong forests of $G$. Since each edge $e \in S_2$ is $k$-valid in some $G_i - r$, the set $\mathcal{F}_2$ is a good cover of $S_2$. △

**Claim.** *There exists a good cover $\mathcal{F}_3$ of $S_3$ of size at most $f_k(d-1)$.*

*Proof of Claim.* Let $i \in \{1,\ldots,t\}$. By Lemma 11 we have that $G_i - x_i \in \mathcal{TD}(d-1)$. Hence we have $f_k(G_i - x_i) \leqslant f_k(d-1)$. Let $\mathcal{A}_i$ denote a good cover of $S_3 \cap E(G_i - x_i)$ consisting of at most $f_k(d-1)$ forests in $G_i - x_i$. Similarly as in the claim before, any union $F_1 \cup \cdots \cup F_t$, where $F_i \in \mathcal{A}_i$, is a $k$-strong induced forest in $G$ because all of these forests contain $r$ as $S_2 \cap S_3 = \emptyset$. Let each forest in $\mathcal{F}_3$ be a union of at most one forest from each of $\mathcal{A}_i$, $i = 1,\ldots,t$. It is clear that one can build such a family with at most $\max\{|\mathcal{A}_i| : 1 \leqslant i \leqslant t\}$ forests. So, $\mathcal{F}_3$ consists of at most $f_k(d-1)$ $k$-strong forests of $G$. Since each edge $e \in S_3$ is $k$-valid in some $G_i - x_i$, the set $\mathcal{F}_3$ is a good cover of $S_3$. △

**Claim.** *There exists a good cover $\mathcal{F}_4$ of $S_4$ of size at most $2g_k(d-1)$.*

*Proof of Claim.* Let $i \in \{1,\ldots,t\}$ and let $e \in S_4$. Then $e$ is $k$-valid in $G_i$. Since $e$ is not $k$-valid in $G_i - r$ and not $k$-valid in $G_i - x_i$, this means that each witness tree of $e$ in $G_i$ contains both $r$ and $x_i$. Every such witness tree contains a path containing $e$, $x_i$ and not $r$, or a path containing $e$, $r$ and not $x_i$. This path is induced and thus (as $e \notin S_1$) $e$ is almost $k$-valid in either $G_i - r$ or $G_i - x_i$, respectively. Hence, $|S_4 \cap G_i| \leqslant 2g_k(d-1)$ by the definition of $g_k(d-1)$, since $G_i - r, G_i - x_i \in \mathcal{TD}(d-1)$ by Lemma 11.

For each edge $e$ in $S_4 \cap G_i$ we pick one witness tree of $e$ that is contained in $G_i$. Let $\mathcal{A}_i$ denote the set of these at most $2g_k(d-1)$ induced $k$-strong forests. As all induced forests in $\mathcal{A}_1,\ldots,\mathcal{A}_t$ contain the root $r$, we can again, as in the previous claim, form a set $\mathcal{F}_4$ of at most $2g_k(d-1)$ $k$-strong forests in $G$ covering $S_4$. △

**Claim.** *There exists a good cover $\mathcal{F}_5$ of $S_5$ of size at most $(k-1)\, g_k^\star(d)$.*

*Proof of Claim.* Note that $S_5$ consists of those edges whose witness trees all contain edges from at least two different $G_i$'s. Without loss of generality assume that each of $G_1,\ldots,G_s$

have an edge incident to $r$ and the other $G_i$'s do not have such an edge. Then each $e \in S_5$ is in $G_i$ for some $i \in \{1, \ldots, s\}$ and moreover $e$ is almost $k$-valid in this $G_i$. Hence $|E(G_i) \cap S_5| \leqslant g_k^\star(d)$ for all $i$, $1 \leqslant i \leqslant s$, and $|E(G_i) \cap S_5| = 0$ for all $i$, $s < i \leqslant t$.

If $s \leqslant k-1$, then $|S_5| \leqslant (k-1)\, g_k^\star(d)$. In this case we let each forest in $\mathcal{F}_5$ consists of one witness tree for each $e \in S_5$.

If $s \geqslant k$, then for all $i \in \{1, \ldots, s\}$ and all $j$, $1 \leqslant j \leqslant g_k^\star(d)$, we pick (not necessarily distinct) induced paths $P_i^j$ in $G_i$ starting with $r$ such that each edge in $S_5$ is contained in some path $P_i^j$. Such a path containing $e \in S_5$ exists, since $e$ is almost $k$-valid in some $G_i$. As $s \geqslant k$, the union of paths $P_1^j \cup \cdots \cup P_s^j$ forms a $k$-strong induced forest in $G$ for each $j$, $1 \leqslant j \leqslant g_k^\star(d)$. Moreover each edge in $S_5$ is contained in one of these forests. Hence $\mathcal{F}_5 = \{P_1^j \cup \cdots \cup P_s^j \mid 1 \leqslant j \leqslant g_k^\star(d)\}$ is a good cover of $S_5$ as desired. △

From the above claims we get that $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5$ is a good cover of all $k$-valid edges in $G$, since $S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$ contains all $k$-valid edges of $G$. Moreover, with the above claims, induction, Lemma 12 and $k \geqslant 2$ we get

$$\begin{aligned}|\mathcal{F}| &\leqslant |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| + |\mathcal{F}_4| + |\mathcal{F}_5| \leqslant k - 1 + 2\, f_k(d-1) + 2\, g_k(d-1) + (k-1)\, g_k^\star(d)\\ &\leqslant k - 1 + 2 \cdot (2k)^{d-1} + 2 \cdot ((2k)^{d-2} - 1) + (k-1)(2(2k)^{d-2} - 1)\\ &\leqslant (2k)^{d-2}(2 \cdot 2k + 2 + 2(k-1)) = (2k)^{d-2} 6k \leqslant (2k)^{d-2} 4k^2 = (2k)^d,\end{aligned}$$

which proves that $f_k(G) \leqslant (2k)^d$ for $G \in \mathcal{TD}(d)$.

Now if $G$ has tree-depth at most $d$ then each component of $G$ is in $\mathcal{TD}(d)$. By the previous arguments all $k$-valid edges of such a component can be covered by at most $(2k)^d$ $k$-strong forest. A union of one such forest from each component is a $k$-strong forest in $G$. Hence we can form at most $(2k)^d$ $k$-strong forests of $G$ that cover all $k$-valid edges of $G$. Thus $f_k(G) \leqslant (2k)^d$ for each graph $G$ of tree-depth at most $d$.

Finally we shall prove that $f_k(G) \leqslant (2k)^{k+1}\binom{d}{k+1}$, for $d \geqslant k+1$ and a graph $G$ with $\mathrm{td}(G) \leqslant d$. Let $H$ be a maximal tree-depth $d$ supergraph of $G$ on the same set of vertices. It is known [20, 21], that there is a proper $d$-coloring of $H$ (a so called $d$-centered coloring) such that any set of $p$ colors, $p \leqslant d$, induces a tree-depth $p$ graph. We hence have a $d$-coloring of $G$ such that each of the $\binom{d}{k+1}$ subsets of $(k+1)$ colors induces a graph of tree-depth $k+1$. As each $k$-valid edge has a witness tree induced by $k+1$ vertices, each witness tree belongs to one of these $\binom{d}{k+1}$ graphs. So each $k$-valid edge of $G$ is $k$-valid in (at least) one of these graphs. Thus the total number of $k$-strong forests covering $k$-valid edges of $G$ is at most $\binom{d}{k+1}(2k)^{k+1}$, where the bound $(2k)^{k+1}$ comes from the first part of the theorem when $d = k+1$. □

# 6 Proof of Theorem 1

Recall that $\chi_p(G)$ is the smallest integer $q$ such that $G$ admits a vertex coloring with $q$ colors such that for each $p' \leqslant p$ each $p'$-set of colors induces a subgraph of $G$ of tree-depth at most $p'$. Since $\mathcal{C}$ is of bounded expansion there is a sequence of integers $a_1, a_2, \ldots,$ that for any graph $G \in \mathcal{C}$ and any $p$, $\chi_p(G) \leqslant a_p$.

Let $G \in \mathcal{C}$. Note that $\chi_{\mathrm{acyc}}(G) \leqslant \chi_2(G)$ and hence Theorem 3(i) gives $f_1(G) \leqslant \binom{\chi_2(G)}{2}$. Thus we can take $b_1 = \binom{a_2}{2}$. For $k \geqslant 2$, consider a $(k+1)$-tree-depth coloring of $G$ with $\chi_{k+1}(G) \leqslant a_{k+1}$ colors. For each of the $\binom{\chi_{k+1}(G)}{k+1}$ subgraphs $H$ induced by $k+1$ colors, consider a cover of the $k$-valid edges in $H$ with $f_k(H)$ $k$-strong forests. Note that each $k$-valid edge of $G$ is $k$-valid in at least one of these graphs. Indeed, a witness tree of an edge

$e$ is induced by $k+1$ vertices, that are colored with at most $k+1$ different colors, hence $e$ is $k$-valid in a graph $H$ induced by these colors. Then the union $\mathcal{F}$ of all these forests is a cover of all $k$-valid edges in $G$. Finally observe that each $H$ has tree-depth at most $k+1$, and thus we have $f_k(H) \leqslant (2k)^{k+1}$ by Theorem 8. Hence $|\mathcal{F}| \leqslant \binom{a_{k+1}}{k+1}(2k)^{k+1}$, and we can take $b_k = \binom{a_{k+1}}{k+1}(2k)^{k+1}$.

# 7 Conclusions

In this paper we introduce the $k$-strong induced arboricity $f_k(G)$ of a graph $G$ to be the smallest number of $k$-strong forests covering the $k$-valid edges of $G$, where a forest is $k$-strong if all its components have size at least $k$ and an edge is $k$-valid if it belongs to an induced tree on $k$ edges. For $k \in \mathbb{N}$, call a class $\mathcal{C}$ of graphs *$f_k$-bounded*, if there is a constant $c = c(\mathcal{C}, k)$ such that $f_k(G) \leqslant c$ for each $G \in \mathcal{C}$, and let us say that $\mathcal{C}$ is *$f$-bounded* if $\mathcal{C}$ is $f_k$-bounded for each $k \in \mathbb{N}$.

We show that this new graph parameter $f_k$ is non-monotone as a function of $k$ and, for $k \geqslant 2$, as a function of $G$ using induced subgraph partial ordering. Indeed, there exist classes of graphs $\mathcal{C}_k$ and $\mathcal{C}'_k$, $k \geqslant 2$, such that $\mathcal{C}_k$ is $f_k$-bounded but not $f_{k+1}$-bounded, while $\mathcal{C}'_k$ is $f_{k+1}$-bounded but not $f_k$-bounded. Nevertheless, $f_k$ behaves nicely for so-called graph classes of bounded expansion, in particular for minor-closed families. Our main result is that every class $\mathcal{C}$ of bounded expansion is $f$-bounded. This implies, in particular, that the adjacent closed vertex-distinguishing number for planar graphs is bounded by a constant. Additionally, we find upper and lower bounds on $f_1(G)$, the induced arboricity, and study the relation between $f_1$ and the well-known notions of arboricity and acyclic chromatic number.

It remains open to improve the lower and upper bounds on $f_k$ for a given graph class. For example, for planar graphs the maximum value for $f_1$ is between 6 (as certified by $K_4$) and 10 (following from $f_1(G) \leqslant \binom{X_{\text{acyc}}(G)}{2}$ and Borodin's result on the acyclic chromatic number of planar graphs [7]). For graphs $G$ of tree-width $t$, we provide explicit universal upper bounds on $f_1(G)$ and $f_2(G)$ in Theorem 4. For $k \geqslant 3$ Theorem 1 states the existence of a constant upper bound. Using $f_k(G) \leqslant \binom{\chi_{k+1}(G)}{k+1}(2k)^{k+1}$ from the proof of Theorem 1 and $\chi_p(G) \leqslant t^p + 1$ for $G$ of tree-width $t$ [20], we conclude that $f_k(G) \leqslant \binom{t^{k+1}+1}{k+1}(2k)^{k+1}$ for any integer $k$ and any graph $G$ of tree-width $t$. This upper bound is most likely far from the actual value, and improving its order of magnitude seems to be an interesting challenge.

A natural generalization of the $k$-strong induced arboricity would be the following: For a set $S \subseteq \mathbb{N}$ of natural numbers and a graph $G$ define $f_S(G)$ to be the minimum number of induced forests in $G$ such that for all $s \in S$ each $s$-valid edge in $G$ lies in a component of size at least $s$ of some of the forests. Clearly, we have $f_k(G) = f_{\{k\}}(G)$ and for $T \subset S$ we have $f_T(G) \leqslant f_S(G)$. In particular considering $S = \{1, \ldots, k\}$ gives a parameter similar to the $p$-tree-depth chromatic number $\chi_p(G)$ as defined by Nešetřil and Ossona de Mendez [21]. As before, we say that a graph class $\mathcal{C}$ is $f_S$-bounded if there is a constant $c = c(\mathcal{C}, S)$ such that $f_S(G) \leqslant c$ for all $G \in \mathcal{C}$. It follows from our results, that for any *finite* set $S \subset \mathbb{N}$ any class $\mathcal{C}$ of bounded expansion is $f_S$-bounded. On the other hand, the examples in Theorem 3 show that the class of tree-width 2 graphs is not $f_{\mathbb{N}}$-bounded, even the class of tree-depth 3 graphs, and the graphs of maximum degree at most 4. It is interesting to identify non-trivial graph classes that are $f_{\mathbb{N}}$-bounded. For example, one can show that $f_{\mathbb{N}}(G) \leqslant 4$, whenever $G = P_n \times P_m$ is a grid graph. In Theorem 3 (iii) we present a graph class $\mathcal{C}$ that is not of bounded expansion, for which $f_{\mathbb{N}}(G) \leqslant 2$ for each $G \in \mathcal{C}$.

Finally, let us mention the concept of *nowhere dense classes of graphs*, which is defined in terms of so-called excluded shallow minors [22], see also [8, 13, 14, 18, 27]. Each class of bounded expansion is nowhere dense, but not the other way round [22]. Similarly, each class of bounded expansion is $f$-bounded (by Theorem 1), but not the other way round (by Theorem 3 (iii)). In fact, nowhere dense classes of graphs and $f$-bounded classes of graphs are two different extensions of classes of graphs of bounded expansion. For $n \in \mathbb{N}$, let $G_n$ be a graph with girth, minimum degree and chromatic number at least $n$. Moreover let $\mathcal{C}_1 = \{G_n \mid n \in \mathbb{N}\}$, let $\mathcal{C}_2$ be the class of 3-subdivisions of all graphs in $\mathcal{C}_1$, and let $\mathcal{C}_3$ denote the class of all 1-subdivisions of $K_{n,n}$, $n \in \mathbb{N}$ (the class from the proof of Theorem 3 (iii)). Then it is easy to check that $\mathcal{C}_1$ is nowhere dense, but not $f$-bounded (in fact not $f_k$-bounded for any $k \in \mathbb{N}$), and hence not of bounded expansion, $\mathcal{C}_2$ is nowhere dense and $f$-bounded, but not of bounded expansion, and $\mathcal{C}_3$ is $f$-bounded, but not nowhere dense, and hence not of bounded expansion.

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