A GENERALIZED RAMSEY PROBLEM.

MARIA AXENOVICH

Abstract. Let $f(n)$ be the minimum number such that there is a proper edge-coloring of K_n with $f(n)$ colors with no path or cycle of 4 edges using one or two colors. It is shown that $\frac{1+\sqrt{5}}{2}n - 3 \leq f(n) \leq 2n^{1+c/\sqrt{\log n}}$ for a positive constants c. This improves the existent bounds on the variant of the Ramsey number $f(n,5,9)$ studied by Erdős and Gyárfás.

1. Introduction

The problem considered in this note is motivated by the generalized Ramsey problem studied by Erdős and Gyárfás [2]. They introduced the function $f(n,p,q)$ which is the minimal number of colors needed to color the edges of a complete graph on n vertices such that every p-clique uses at least q colors on its edges. They gave the tight bounds for the order of magnitude of f for many values of p pointing out that the "most annoying cases for small p " are $f(n,4,3)$ and $f(n,5,9)$. They showed that $(4/3)n \leq f(n,5,9) \leq O(n^{3/2})$ and $f(n,4,3) \leq c\sqrt{n}$. D. Mubayi [3] lowered the upper bound for $f(n,4,3)$ to $e^{\sqrt{c \log n}}(1+o(1))$. Here, among others, we improve both the upper bound and lower bounds for $f(n,5,9)$ by looking at a more general coloring problem.

Let $f(n)$ be the minimum number of colors in a proper edge-coloring of of K_n with no path or cycle of 4 edges using at most two colors. This function was defined by Erdős and Gyárfás [2]. We are going to determine upper and lower bounds on $f(n)$ and use them to improve the bounds on $f(n,5,9)$.

Let $f'(n,5,9)$ be the minimum number of colors in a proper edge-coloring of K_n such that every K_5 uses at least 9 colors on its edges. Notice that

$$f'(n,5,9) = f(n). \tag{1}$$

University of Illinois at Urbana-Champaign and Ames Laboratory, Iowa State University

Email: axenovic@cyclops.ameslab.gov

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Indeed, suppose we have a proper edge-coloring of K_n such that there is K_5 using at most 8 colors. Then, since the coloring is proper, there are at most two edges of this K_5 of the same color. But the number of colors is at most 8, so there is other pair of edges having the same color and thus there is a path or cycle on 4 edges using two colors. On the other hand if every K_5 uses at least 9 colors then there is no path or cycle on 4 edges using only two colors. Also, it is clear that

$$f(n,5,9) \le f'(n,5,9) = f(n). \tag{2}$$

Therefore the upper bound on $f(n)$ provides the upper bound on $f(n,5,9)$. We are going to show that our lower bound on $f(n)$ is a lower bound on $f(n,5,9)$ as well.

Theorem 1.
(i) $\frac{1+\sqrt{5}}{2}n - 3 \le f(n) \le 2n^{1+c/\sqrt{\log n}}$,
(ii) $\frac{1+\sqrt{5}}{2}n - 3 \le f(n,5,9) \le 2n^{1+c/\sqrt{\log n}}$, *where c is a positive constant.*

Proof. Let $V(K_n) = \{v_1, \ldots, v_n\}$. For the upper bound we provide the following proper coloring ϕ of $E(K_n)$. Let $0 \le x_1 < \ldots < x_n \le c'n^{1+c/\sqrt{\log n}}$ be the set of integers with no arithmetic progression of length 3. Such set exists by the result of Behrend [1]. We set $\phi(v_iv_j) = x_i + x_j$. This coloring is proper and it uses at most $2x_n$ colors. Suppose there is a cycle with 4 edges which is 2-colored, i.e., $\phi(v_iv_k) = \phi(v_jv_l), \phi(v_iv_l) = \phi(v_jv_k)$, for some i,j,k,l. Then $x_i + x_k = x_j + x_l$, $x_i + x_l = x_j + x_k$ and $x_k = x_l$ which is impossible. Now suppose there is a 2-colored path with 4 edges, i.e. $\phi(v_iv_k) = \phi(v_jv_l), \phi(v_jv_k) = \phi(v_tv_l)$ for some i,j,t,k,l. Then $x_i + x_k = x_j + x_l$, $x_j + x_k = x_t + x_l$ and $x_i - x_j = x_j - x_t$ which is also impossible. Thus the upper bound on $f(n)$ follows.

To show the lower bound on $f(n)$, consider a proper coloring ϕ of $E(K_n)$ using q colors such that there is no path or cycle on 4 edges using two colors. There is a color a with number of edges of this color at least $\binom{n}{2}/q$. Let Q be the set of vertices incident to the edges of color a. Let $v,u \in Q$, such that $\phi(uv) = a$, and let $B = \{\phi(ux) : x \in Q \setminus \{v,u\}\}$. Then for all $y \in V(K_n) \setminus \{v\}, \quad \phi(vy) \notin B$. From this we conclude that $|B| + n - 1 \le q$. Since ϕ is a proper coloring, $|B| = |Q| - 2 \ge 2\binom{n}{2}/q - 2$. Thus $n(n-1)/q - 2 + n - 1 \le q$ and the result follows by solving the inequality for q. This concludes the proof of the first part of the theorem.

The upper bound for $f(n,5,9)$ follows from the first part and (2). For the lower bound consider an edge-coloring of K_n such that every K_5 induces at least 9 colors. If the coloring is proper then by the first part and (1) we have at least $f'(n,5,9) \ge \frac{1+\sqrt{5}}{2}n - 3$ colors. Suppose that there are two incident edges xy and yz of the same color. It is easy to notice that any two edges joining $\{x, y, z\}$ to $V(K_n) \setminus \{x, y, z\}$ must have different colors. Therefore we have at least $3(n-3)+2 > \frac{1+\sqrt{5}}{2}n - 3$ colors. □

Remark. The lower bound in Theorem 1 can be improved to $2n-6$ as was noted by G. Tóth [4].

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