Multicolor Ramsey numbers for triple systems

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Abstract

Given an r-uniform hypergraph H, the multicolor Ramsey number $r_k(H)$ is the minimum n such that every k-coloring of the edges of the complete r-uniform hypergraph K_n^r yields a monochromatic copy of H. We investigate $r_k(H)$ when k grows and H is fixed. For nontrivial 3-uniform hypergraphs H, the function $r_k(H)$ ranges from $\sqrt{6k}(1+o(1))$ to double exponential in k.

We observe that $r_k(H)$ is polynomial in k when H is r-partite and at least single-exponential in k otherwise. Erdős, Hajnal and Rado gave bounds for large cliques K_s^r with $s \geq s_0(r)$, showing its correct exponential tower growth. We give a proof for cliques of all sizes, $s > r$, using a slight modification of the celebrated stepping-up lemma of Erdős and Hajnal.

For 3-uniform hypergraphs, we give an infinite family with sub-double-exponential upper bound and show connections between graph and hypergraph Ramsey numbers. Specifically, we prove that

$$r_k(K_3) \leq r_{4k}(K_4^3 - e) \leq r_{4k}(K_3) + 1,$$

where $K_4^3 - e$ is obtained from K_4^3 by deleting an edge.

We provide some other bounds, including single-exponential bounds for $F_5 = \{abe, abd, cde\}$ as well as asymptotic or exact values of $r_k(H)$ when H is the bow $\{abc, ade\}$, kite $\{abc, abd\}$, tight path $\{abc, bcd, cde\}$ or the windmill $\{abc, bde, cef, bce\}$. We also determine many new "small" Ramsey numbers and show their relations to designs. For example, the lower bound for $r_6(kite) = 8$ is demonstrated by decomposing the triples of [7] into six partial STS (two of them are Fano planes).

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1 Introduction, results

An r-uniform hypergraph H is a pair (V, E) where V is a vertex set and $E \subseteq \binom{V}{r}$ is the set of edges. Let K_n^r be the complete r-uniform hypergraph containing all r-subsets of vertices as edges. For an edge $\{v_1, v_2, \ldots, v_r\}$ we often write $v_1 v_2 \ldots v_r$. When $r = 2$, denote K_n^r by K_n. We shall also use the notation $\binom{[n]}{r}$ or $\binom{V}{r}$ for the edge set of K_n^r. An r-uniform hypergraph H is ℓ-partite if its vertex set can be partitioned into ℓ parts (called partite sets) such that each edge contains at most one vertex from each part; H is a complete r-partite hypergraph if each choice of r vertices from distinct partite sets forms an edge, and H is balanced if its partite sets differ in size by at most one. A matching is a hypergraph consisting of disjoint edges. A hypergraph $H = (V, E)$ is a subhypergraph of $F = (V', E')$ if $V \subseteq V'$ and $E \subseteq E'$. Denote by $\mathrm{ex}(n, H)$ the maximum number of edges in an n-vertex r-uniform hypergraph containing no copy of H as subhypergraph. The density of an r-uniform hypergraph $H = (V, E)$ on n vertices is $d(H) = |E|/\binom{n}{r}$.

The **multicolor Ramsey number** for an r-uniform hypergraph H, denoted by $r_k(H)$, is the minimum n such that no matter how the edges of K_n^r are colored with k colors, there is a monochromatic copy of H. While there is a number of results in the literature about $r_k(H)$ when k is a small fixed number (see [4]), the case when H is fixed and k grows appears not to have been extensively studied. The following three results are among the few results known in this area:

Theorem 1 (Lazebnik and Mubayi [25]). *Fix integers $r, s, t \geq 2$. Let $H^r(s,t)$ be the complete r-partite r-uniform hypergraph with $r-2$ parts of size 1, one part of size s and one part of size t. Then*

(i) $r_k(H^r(2, t+1)) = tk^2 + O(k)$;

(ii) $r_k(H^r(s,t)) = \Theta(k^s)$, for fixed $t, s \geq 2, t > (s-1)!$;

(iii) $r_k(H^r(3,3)) = (1 + o(1))k^3$.

Let M be a matching with two r-tuples. Notice that an edge-coloring of K_n^r without monochromatic copies of M corresponds to a proper vertex-coloring of Kneser graph $K(n,r)$, that is, the graph with vertex set $\binom{[n]}{r}$ and two r-sets are adjacent if and only if they are disjoint. Lovász proved that the chromatic number of $K(n,r)$ is equal to $n - 2r + 2$. Reformulating his result, we obtain the following.

Theorem 2 (Lovász [28]). *If M is a matching with two r-tuples, then $r_k(M) = k + 2r - 1$.*

Gyárfás and Raeisi observed that results of Csákány and Kahn [6] and the standard coloring of the Kneser graph imply the following.

Proposition 3 (Gyárfás and Raeisi [14]). *If C_3^3 is the hypergraph with edge set $\{abc, cde, efa\}$, then $k + 5 \leq r_k(C_3^3) \leq 3k + 1$.*

In this paper, we start a systematic investigation on the growth rate of $r_k(H)$ for some fixed H as k grows. Our first result shows that $r_k(H)$ is polynomial in k if and only if H is r-partite.

Proposition 4. *Let $r \geq 2$ be fixed and H be a connected r-uniform hypergraph. Then $r_k(H)$ is polynomial in k if and only if H is r-partite. In particular, there are positive constants c and c', such that*

(i) If H is r-partite, then $r_k(H) = O(k^c)$

(ii) If H is not r-partite, then $r_k(H) \geq 2^{c'k}$.

Determining the growth rate of $r_k(H)$ in general is known to be a very hard problem. For example, the best known bounds even for the smallest nontrivial graph case are $c^k < r_k(K_3) < c'k!$ for some positive constants c and c' (see Chung [5] and Erdős, Szekeres [11]). Define the tower function as follows: $t_1(n) = n$ and $t_{i+1}(n) = 2^{t_i(n)}$ for all $i \geq 1$. Erdős, Hajnal and Rado gave an upper bound for all cliques and a lower bound for only large cliques.

Theorem 5 (Erdős and Rado [10], Erdős et al. [9])**.** *Let $s > r \geq 2$. There are positive integers $c = c(s,r) \leq 3(s-r)$, $s_0(r)$, and $c' = c'(s,r)$ such that*

$$t_r(c'k) < r_k(K_s^r) < t_r(ck \log k)$$

where the lower bound holds for $s \geq s_0(r)$.

It is worth noting that the lower bound in [9] was stated for the case when the number of colors, k, is fixed while r grows and the bound was only for large cliques. But the proof in [9] applies naturally to our case as well, when k grows and the other parameters are fixed. Recently, an improved stepping-up lemma was proved by Conlon et al [3]. Their main result implies a lower bound for cliques of smaller sizes, but still only for $s \geq 2r-1$. Duffus, Lefmann and Rödl [7] took another approach, using shift graphs, and proved a lower bound for cliques of all sizes $s > r$, but require k being fixed and $r \gg k$. Our next result gives a proof for cliques of all sizes using a slight modification of the stepping-up lemma, due to Erdős and Hajnal (see Chapter 4.7 in [13]).

Theorem 6. *For any $s > r \geq 2$ and $k > r2^r$ we have*

$$r_k(K_s^r) > t_r\left(\frac{k}{2^r}\right).$$

Our remaining results are all for 3-uniform hypergraphs and we will address the question of determining $r_k(H)$ for most interesting H's with 6 or fewer vertices. Let $K_4^3 - e$ be a hypergraph obtained from K_4^3 by removing one edge. Our next theorem gives bounds on $r_k(K_4^3 - e)$ in terms of $r_k(K_3)$, showing that compared to the double-exponential bounds for K_4^3 from Theorems 5 and 6, the correct order of magnitude for $r_k(K_4^3 - e)$ is single-exponential.

Theorem 7. *For any $k \geq 2$,*

$$r_k(K_3) \leq r_{4k}(K_4^3 - e) \quad \text{and} \quad r_k(K_4^3 - e) \leq r_k(K_3) + 1.$$

Moreover $r_2(K_4^3 - e) = r_2(K_3) + 1 = 7$.

Denote by F_5 the hypergraph with edges $\{abc, abd, cde\}$. We show that $r_k(F_5)$ behaves similarly to $r_k(K_3)$.

Theorem 8. *There is a positive constant c such that, for $k \geq 4$,*

$$2^{ck} \leq r_k(F_5) \leq k!,$$

and $r_2(F_5) = 6, r_3(F_5) = 7$.

The simplest non-trivial triple systems have just two edges. The **kite** is a 3-uniform hypergraph with two edges sharing two vertices. The **bow** is a 3-uniform hypergraph with two edges sharing a single vertex.

Theorem 9. *Let $r_k = r_k(bow)$. Then*

$$r_k = (1 + o(1))\sqrt{6k}.$$

If $k = \frac{\binom{n}{3}}{n}$ and $n \equiv 4, 8 \pmod{12}$, then $r_k = n + 1$. Moreover, $r_2 = 5, r_3 = r_4 = r_5 = 6$, $r_6 = 7$, $r_7 = r_8 = r_9 = r_{10} = 9$, $9 \leq r_{11} \leq r_{12} \leq r_{13} \leq r_{14} \leq 10, r_{15} = 11$.

Remark. Note that $r_k(bow)$ is the smallest multicolor Ramsey number among nontrivial 3-uniform hypergraphs since $r_k(H) \geq \min\{r_k(bow), r_k(kite), r_k(M)\}$, where M is a matching with 2 triples. Indeed, each nontrivial 3 uniform hypergraph contains at least two edges that form one of *bow*, *kite* or M, and Theorem 2 gives $r_k(M) = k + 5$.

Theorem 10. *Let $r_k = r_k(kite)$. Then*

$$r_k = \begin{cases} k+1, & \text{if } k \equiv 3 \pmod 6 \\ k+1 \text{ or } k+2, & \text{if } k \equiv 4 \pmod 6 \\ k+2, & \text{if } k \equiv 0, 2 \pmod 6 \\ k+3, & \text{if } k \equiv 1, 5 \pmod 6, k \neq 5 \\ 6 & \text{if } k = 5, \\ 5 & \text{if } k = 4 \end{cases}$$

Let a, b be positive integers. Denote by $F(a, b)$ the 3-uniform hypergraph with vertex set $V = A \cup B$, $A \cap B = \emptyset$, $|A| = a, |B| = b$ and edge set consisting of all triples with one vertex in A and two vertices in B (for example, $F(2, 2)$ is the kite).

Proposition 11. *For any $a \geq 2$, we have*

$$k(a-1) < r_k(F(a,2)) \leq k(a-1)+3.$$

In general, $r_k(F(a,b))$ grows slower than double exponential in k and possibly faster than exponential in k. (Recall that Theorems 5 and 6 give double-exponential bounds.)

Theorem 12. *Given $3 \leq a \leq b$, we have, for positive constants $c = c(a,b)$ and $c' = c(a,b)$*

$$2^{c'k} < r_k(F(a,b)) < r_t(K_b) + m < 2^{ck^{a+1}\log k},$$

where $m = (a-1)k+1$, and $t = k\binom{m}{a}$.

The **windmill** W with *center edge abc* is the hypergraph with six vertices and edge set $\{abc, abd, bce, acf\}$.

Theorem 13.

$$(1-o(1))3k \leq r_k(W) \leq 3k+3.$$

It is interesting to compare Theorem 13 with Proposition 3. In fact, the upper bounds in both cases come from the corresponding Turán-type results. Indeed, $\mathrm{ex}(n, C_3^3) = \binom{n-1}{2}$ (Frankl-Füredi [12] for large n, Csákány-Kahn [6] for $n \geq 6$) while $\mathrm{ex}(n, W) \leq \binom{n}{2}$ ([12]).

The ideas giving the asymptotic of $r_k(W)$ can be also used for the **tight path** $P_3^3 = \{abc, bcd, cde\}$.

Theorem 14. $2k(1-o(1)) \leq r_k(P_3^3) \leq 2k+3.$

The rest of the paper will be organized as follows. In Section 2, we give some auxiliary results and prove Proposition 4. Theorems 6 - 14 will be proved in Sections 3-6. Section 7 is devoted to exact values of Ramsey numbers for small number of colors and Section 8 contains remarks, conjectures and problems.

In some later sections we give lower bounds on Ramsey numbers based on block designs. A $t-(v,k,\lambda)$ **design** is a subset of $\binom{[v]}{k}$, called blocks, such that each t element subset of $[v]$ is contained in exactly λ blocks.

2 General bounds and auxiliary results

In this section we prove some general bounds on $r_k(H)$ and obtain some consequences including Proposition 4. Recall that the density of an r-uniform hypergraph F with n vertices and e edges is $d(F) = \frac{e}{\binom{n}{r}}$.

Lemma 15. *Let H be a fixed r-uniform hypergraph and F be an r-uniform hypergraphs with n vertices, density $d(F) = d$, and not containing copies of H as a subhypergraph. Then*

(i) $r_k(H) \leq 1 + \max\{n : \lceil \binom{n}{r} / \mathrm{ex}(n, H) \rceil \leq k\}$,
(ii) If $\binom{n}{r}(1-d)^k < 1$ *then* $r_k(H) \geq n$.

Proof. (i) Consider a coloring of K_n^r with k colors and no monochromatic copy of H. Then each color class has at most $\mathrm{ex}(n, H)$ edges.

(ii) Consider k copies of hypergraph F obtained by mapping its vertices randomly to a given set V of n vertices. Here, we choose vertex permutations uniformly. Assign the edges of the ith copy of F color i, $i = 1, \ldots, k$. If an edge belongs to several copies of F, assign the smallest available label. We claim that with positive probability, each edge of $K = \binom{V}{r}$ belongs to some copy of F. Indeed, the probability that a given edge of K uncovered is $(1-d)^k$. Thus, the probability that there is an uncovered edge of K is at most $\binom{n}{r}(1-d)^k < 1$. Therefore, with positive probability, all edges are covered and the resulting coloring of K contains no monochromatic copy of H. □

Proof of Proposition 4. (i) The proposition follows from Lemma 15(i) by using the fact that $\mathrm{ex}(n, H) < n^{r-c}$ for some positive constant $c = c(H)$, when H is r-partite, see [8]. So, $k \geq \binom{n}{r}/\mathrm{ex}(n, F) \geq Cn^r/n^{r-c} = Cn^c$, for a constant $C = C(r)$. Thus $n \leq C^{-1/c}k^{1/c}$.

(ii) Let H be non-r-partite. Apply Lemma 15(ii) with F being a complete r-uniform r-partite balanced hypergraph on $n = 2^{c'k}$ vertices (and $r|n$). Clearly H is not contained in F as a subgraph. Moreover, $d(F) \geq \frac{(n/r)^r}{\binom{n}{r}} > \frac{(n/r)^r}{(en/r)^r} = e^{-r}$. Hence for $k = c \log n$ and $c > e^r(r+1)$,

$$\binom{n}{r}(1-d)^k = \binom{n}{r}(1-d)^{c \log n} < n^r e^{-cd \log n} = e^{(r-cd)\log n} < 1. \qquad \square$$

The *trace* of a 3-uniform hypergraph H at vertex v is the graph on vertex set $V(H) - \{v\}$ and with edge set $\{e - \{v\} : \ e \in H, \ v \in e\}$. A transversal of a hypergraph is a set of vertices non-trivially intersecting each edge.

Lemma 16. *Let H be a 3-uniform hypergraph with a single-vertex transversal $\{v\}$. Let G be a trace of H with respect to v. Then $r_k(H) \leq r_k(G) + 1$.*

Proof. Given a k-coloring c of $\binom{[n]}{3}$ with no monochromatic H, let c' be the k-coloring of $\binom{[n-1]}{2}$ defined by $c'(ij) = c(ijn)$. Then c' has no monochromatic G and consequently $r_k(G) \geq r_k(H) - 1$ as required. □

3 K_s^r for $s > r \geq 2$

In this section we prove Theorem 6 using a variant of the stepping-up lemma of Erdős and Hajnal.

Proof of Theorem 6. It suffices to prove the result for $s = r+1$ since $r_k(K_s^r) \geq r_k(K_{r+1}^r)$ for any $s > r$. We use induction on r to show that $r_k(K_{r+1}^r) \geq t_r(k/2^{r-2} - 2r)$ for all $k \geq r2^r$. Since $k \geq r2^r$, we have $k/2^{r-2} - 2r \geq k/2^r$ and the result follows.

The base case $r = 2$ is given by $r_k(K_3) > 2^k > 2^{k-4} = t_2(k-4)$. Assume the result holds for some $r \geq 2$ and let $n = r_k(K_{r+1}^r) - 1$. By the inductive hypothesis

$$n \geq t_r(k/2^{r-2} - 2r) - 1.$$

Let $\phi : \binom{[n]}{r} \to [k]$ be a coloring with no monochromatic K_{r+1}^r. We will construct a coloring $\psi : \binom{[2^n]}{r+1} \to [2k + 2r - 4]$ with no monochromatic K_{r+2}^{r+1}. This shows that

$$r_{2k+2r-4}(K_{r+2}^{r+1}) \geq 1 + 2^n \geq 1 + \frac{1}{2} t_{r+1}(k/2^{r-2} - 2r).$$

Now suppose we are given $k' \geq (r+1)2^{r+1}$. If $k' - 2r + 4$ is odd, then let $k'' = k' - 1$ and if $k' - 2r + 4$ is even then let $k'' = k'$. Set $k = (k'' - 2r + 4)/2$ (which is an integer) and observe that $k \geq r2^r$ and $k'' = 2k + 2r - 4$. Then

$$k/2^{r-2} - 2r \geq k''/2^{r-1} - 2(r+1) + 1$$

and $r_{k'}(K_{r+2}^{r+1})$ is at least

$$r_{k''}(K_{r+2}^{r+1}) \geq 1 + \frac{1}{2} t_{r+1}(k/2^{r-2} - 2r) \geq 1 + \frac{1}{2} t_{r+1}(k''/2^{r-1} - 2(r+1) + 1) > t_{r+1}(k'/2^{r-1} - 2(r+1)).$$

Now we shall construct a coloring ψ of $\binom{[2^n]}{r+1}$ using the coloring ϕ of $\binom{[n]}{r}$ that has no monochromatic K_{r+1}^r. Represent the elements of $[2^n]$ with 0-1-sequences on n coordinates. For a vertex u and integer i, we denote $u(i)$ the ith coordinate of u in this representation. Given two vertices $u, v \in [2^n]$, say that $u < v$ if $u(i) < v(i)$ and $u(j) = v(j)$ for $j < i$. Denote such an i by $f(uv)$. Given any $u_1 < \cdots < u_{r+1}$, let $f_i := f(u_i u_{i+1})$, for every $1 \leq i \leq r$. Observe crucially that

(1) $f_i \neq f_{i+1}$, for every $1 \leq i \leq r-1$;

(2) $f(u_1 u_{r+1}) = \min_{1 \leq i \leq r} \{f_i\}$ and the minimum is reached by a unique i.

We define coloring ψ as follows:

$$\psi(u_1 \ldots u_{r+1}) = \begin{cases} (\phi(f_1, \ldots, f_r), 1) & \text{if } (f_1, \ldots, f_r) \text{ is an increasing sequence,} \\ (\phi(f_1, \ldots, f_r), 2) & \text{if } (f_1, \ldots, f_r) \text{ is a decreasing sequence,} \\ (i, 3) & \text{if } f_1 < f_2 < \cdots < f_i > f_{i+1}, 2 \leq i \leq r-1, \text{ for } r \geq 3, \\ (i, 4) & \text{if } f_1 > f_2 > \cdots > f_i < f_{i+1}, 2 \leq i \leq r-1, \text{ for } r \geq 3. \end{cases}$$

Suppose to the contrary that there is a monochromatic copy of K_{r+2}^{r+1} under ψ on vertex set $U = \{u_1, \ldots, u_{r+2}\}$ with $u_1 < \cdots < u_{r+2}$. Without loss of generality, we distinguish two cases.

Case 1: The second coordinate of ψ on each $(r+1)$-tuple is 1. First notice that the second coordinate of ψ on $u_1, ..., u_{r+1}$ and $u_2, ..., u_{r+2}$ being 1 implies $f_1 < f_2 < \cdots < f_r < f_{r+1}$ and together with (2), we have $f(u_1u_i) = f(u_1u_2) = f_1$ for all $3 \le i \le r+2$. Similarly from $u_2, ..., u_{r+2}$, we have that for every $2 \le p < q \le r+2$, $f(u_pu_q) = f_p$. Recall that the color of the $(r+1)$-set $\{u_1, ..., u_{r+2}\} - \{u_i\}$ under ψ is determined by the color of the r-set $\{f_1, ..., f_{r+1}\} - \{f_i\}$ under ϕ. Let $F := \{f_1, ..., f_{r+1}\}$ and $U = \{u_1, \ldots, u_{r+2}\}$. Let us denote the above implication by

$$U \setminus \{u_i\} \Rightarrow F \setminus \{f_i\}.$$

Thus a monochromatic K_{r+2}^{r+1} on U under ψ yields a monochromatics K_{r+1}^{r} on F under ϕ, a contradiction.

Case 2: Each $(r+1)$-tuple gets color $(i, 3)$ for some i with $2 \le i \le r-1$. Then $\psi(u_1, ..., u_{r+1}) = (i, 3)$ implies $f_i > f_{i+1}$. On the other hand, $\psi(u_2, ..., u_{r+2}) = (i, 3)$ implies $f_i < f_{i+1}$, a contradiction.

If the second coordinate is 2 or 4 the arguments are almost identical to those in Case 1 or 2. □

4 $K_4^3 - e$ and F_5

Notice that in contrast to the double-exponential growth for K_4^3, $r_k(K_4^3 - e)$ is single-exponential in the number of colors k. Indeed, since $K_4^3 - e$ is not 3-partite, Proposition 4 yields $r_k(K_4^3 - e) > 2^{ck}$. For the upper bound, one can use a variation of the classical Erdős-Rado pigeonhole argument to obtain $r_k(K_4^3 - e) < 2^{(k+1)\log k}$. We will, however, use a different approach to prove this fact, which also shows some connection between the multicolor Ramsey number of $K_4^3 - e$ and the multicolor Ramsey number of a triangle.

Proof of Theorem 7. For the lower bound, let $n = r_k(K_3) - 1$ and $\phi : \binom{[n]}{2} \to k$ be a k-coloring of $\binom{[n]}{2}$ with no monochromatic triangles. We will construct a coloring ψ of $\binom{[n]}{3}$ with $4k$ colors with no monochromatic $K_4^3 - e$. This then would imply that $r_{4k}(K_4^3 - e) \ge n + 1 = r_k(K_3)$ as desired. Let ψ be the following coloring of the triples $i < j < k$. If P is a path with vertices i, j, k, denote by $\phi'(P)$ the color under ϕ of the edge in $\{i, j, k\}$ that is not in P. For such a path P, let the type of P, $t(P) = 1, 2$, or 3 if i, j or k is its center, respectively. If $\{i, j, k\}$ is a rainbow triangle, let $\psi(ijk) = (0, \phi(jk))$. If $\{i, j, k\}$ induces a monochromatic path P, let $\psi(ijk) = (t(P), \phi'(P))$.

Suppose there is a monochromatic copy $K = \{abc, abd, acd\}$ of $K_4^3 - e$, we will show a contradiction when the first coordinate is 0, namely all three triples $\{abc, abd, acd\}$ span rainbow triangles under ϕ. The cases when the first coordinate is $1, 2$ or 3, can be proved using a similar argument. Notice that when the first coordinate is 0, by the definition of ψ, the color of a triple depends on the color, under ϕ, of the edge spanned by the two largest elements in that triple. Since b, c, d play a symmetric role, we can assume that $b < c < d$.

If a is the smallest, then $\psi(abc) = \psi(abd) = \psi(acd)$ implies $\phi(bc) = \phi(bd) = \phi(cd)$, i.e. bcd is monochromatic under ϕ. Thus b is the smallest. But then $\psi(abc) = \psi(abd)$ implies $\phi(ac) = \phi(ad)$, which means acd is not a rainbow triangle under ϕ, a contradiction.

For the upper bound, simply notice that $K_4^3 - e = \{abc, abd, acd\}$ has a single vertex transversal $\{a\}$, and the trace of a is a triangle on $\{b, c, d\}$. Thus the upper bound follows from Lemma 16. The case with 2 colors is treated in Section 7. □

Proof of Theorem 8. The cases $k = 2, 3$ are treated in Section 7. The general lower bound follows from Proposition 4(ii), since F_5 is not 3-partite.

The upper bound follows by induction with basis $k = 4$. Suppose that the edges of K_{24}^3 with vertex set V can be 4-colored so that there is no monochromatic F_5. There are 22 triples uvx containing a fixed pair uv. Assume that uvx_1, uvx_2 are red triangles. Then x_1x_2y cannot be red for $y \in Y = V - \{u, v, x_1, x_2\}$. Thus we have a set S, $S \subseteq Y$, $|S| \geq \lceil (|V| - 4)/3 \rceil = 7$ and x_1x_2y are blue triples for all $y \in S$. Therefore, no triple in S is colored blue, and thus $\binom{S}{3}$ uses $k - 1 = 3$ colors. Applying Theorem 25 to the 3-colored subhypergraph spanned by S, we get a contradiction.

The inductive step is simply repeating the argument above in general. Suppose we already know $r_k(F_5) \leq k!$ for some $k \geq 4$ and we have a K_n^3 with a $(k+1)$-coloring such that there is no monochromatic F_5. Selecting u, v, x_1, x_2 as above and applying the same argument, we get $n - 4 \leq k(k! - 1) < (k+1)! - k$, thus $n \leq (k+1)! - k + 4 \leq (k+1)!$. This implies $r_{k+1}(F_5) \leq (k+1)!$. □

Remark. The above results slightly suggests that $r_k(F_5) \leq r_k(K_3)$ might hold. Although the bound $r_k(F_5) \leq k!$ in Theorem 8 can be improved slightly, this improvement still does not show that $r_k(F_5) \leq r_k(K_3)$.

5 Bow, Kite, $F(a, b)$

The next lemma (without the statements on the extremal configurations) is referred in [27] as an unpublished remark of Erdős and Sós.

Lemma 17.

$$\mathrm{ex}(n, bow) = \begin{cases} n & \text{if } n \equiv 0 \pmod 4 \\ n-1 & \text{if } n \equiv 1 \pmod 4 \\ n-2 & \text{if } n \equiv 2, 3 \pmod 4. \end{cases}$$

When $n \equiv 0, 1 \pmod 4$, the extremal configurations are unique, all components are K_4^3-s, (apart from a possible one vertex component). When $n \equiv 2 \pmod 4$, the extremal configuration is either $\frac{n-2}{4}$ copies K_4^3-s and two isolated vertices or any number of K_4^3-s and one star component. Similarly, when $n \equiv 3 \pmod 4$, the extremal configuration is either $\frac{n-3}{4}$ copies K_4^3-s and component with a single edge or any number of K_4^3-s and one star component.

Proof. Suppose C is the vertex set of a nontrivial connected component of a 3-uniform hypergraph without a *bow*. Then either C spans only one edge or there are two edges e_1, e_2 in C, intersecting in two vertices, u, v. Suppose that $|C| > 4$. Then every edge f that is not covered by $e_1 \cup e_2$ and intersecting $e_1 \cup e_2$ must contain u, v and a vertex w not covered by $e_1 \cup e_2$. It is easy to see that these vertices w cover C and C has no other edges, thus C has $|C| - 2$ edges, all containing u, v. Such a component is called a star component.

On the other hand, if $|C| = 4$ then we have two, three or four edges in C. From this analysis the lemma follows. $\square$

Lower bounds of $r_k(bow)$ follow from the existence of resolvable designs. A $3-(n, 4, 1)$ design is a set of 4-element subsets (blocks) of an n-element set V such that each 3-element subset of V is in precisely one block. Hanani [15] showed that $3-(n, 4, 1)$ designs exist if and only if $n \equiv 2, 4 \pmod 6$. A $3-(n, 4, 1)$ design is called *resolvable* if its blocks can be grouped so that each group (parallel class) gives a partition of V. Resolvable $3-(n, 4, 1)$ designs exist if and only if $n \equiv 4, 8 \pmod{12}$, see [18, 19], and [21].

Proof of Theorem 9. When $n \equiv 4, 8 \pmod{12}$, $k = \frac{\binom{n}{3}}{n}$, $\mathrm{ex}(n, bow) = n$, thus Lemma 15 (i) gives $r_k \le n + 1$. This is sharp, since K_n^3 can be partitioned into k matchings. The statement $r_k(bow) \approx \sqrt{6k}$ follows from considering the construction for largest n, $n \equiv 4, 8 \pmod{12}$, $k \ge \frac{\binom{n}{3}}{n}$ for the lower bound and applying the Lemma 15(i) for the upper bound. The statements about the small values are proved in Section 7. $\square$

Proof of Theorem 10. Let $H = F(2, 2)$ be the kite. Then $\mathrm{ex}(n, H)$ corresponds to the maximum number of triples on n elements such that any two triples intersect in at most one element, i.e. the maximum number of edges in a linear 3-uniform hypergraph. A well-known result of Schönheim [36] and others (the cases $n \equiv 0, 1, 2, 3 \pmod 6$ go back even to Kirkman [22]) is $\mathrm{ex}(n, H) = \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor - \epsilon$, where $\epsilon = 1$ for $n \equiv 5 \pmod 6$, otherwise $\epsilon = 0$. Lemma 15(i) gives, after some calculations, the upper bounds.

The lower bound for the cases $k \equiv 3, 4 \pmod 6$ is easy. Given $K_n^3 = (V, E)$, consider $V = Z_n$ and color triple ijk with color $i + j + k \pmod n$. Clearly this coloring yields no monochromatic H, hence $r_k(H) > k$.

For the cases $k \equiv 0, 1, 2, 5 \pmod 6$ the (difficult) constructions of J. X. Lu [29, 30] finished by Teirlinck [38] are needed: for $n > 7, n \equiv 1, 3 \pmod 6$, K_n^3 can be partitioned into $n - 2$ Steiner triple systems (called a *large set* of STS).

Indeed, for $k \equiv 0, 2 \pmod 6$ we need a kite-free k-coloring of K_{k+1}^3 i.e. $(n-1)$-coloring of K_n^3 when $n \equiv 1, 3 \pmod 6$. This can be done even with $n - 2$ colors according to the cited result of Lu. However, the case $k = 6$ is exceptional because Lu's theorem does not hold for $n = 7$. Nevertheless, there is a 6-coloring of K_7^3 without a monochromatic kite as shown in Proposition 22. Similarly, for $k \equiv 1, 5 \pmod 6$ we need a kite-free k-coloring of K_{k+2}^3 i.e. $(n-2)$-coloring of K_n^3 when $n \equiv 3, 1 \pmod 6$. This is provided by Lu's theorem, apart from the case $k \equiv 5$ $(n = 7)$ which is indeed exceptional, in Proposition 22 we prove that

$r_5(kite) = 6$ (together with the case $k = 4$). $\square$

Proof of Proposition 11. In an $F(a,2)$-free coloring of K_n^3 any pair of vertices is in at most $a-1$ edges of the same color. Thus $n \leq 2+k(a-1)$, proving the upper bound. (One can also use Lemma 16 and the multicolor Ramsey number for stars (see [1]): $r_k(K_{1,a}) \leq k(a-1)+2$.)

For the lower bound, set $n = k(a-1)$ and consider $K_n^3 = (V,E)$ with $V = Z_n$. Color a each edge with the sum of its vertices mod k. Then a monochromatic copy of $F(a,2)$ would require that for some $y, z \in V$, $y+z+x_1, ..., y+z+x_a$ are all equal (mod k) i.e. we have a different positive x_s, all equal (mod k), which is impossible. Hence $r_k(F(a,2)) > k(a-1)$. $\square$

Proof of Theorem 12. For the upper bound, let $N = r_t(K_b) + m$. Consider a k-coloring ϕ of the triples of K_N. Fix a set S of m vertices and define a t-coloring c on the pairs of the remaining $N-m$ vertices as follows. Let $c(xy) = (\phi(xys_i), s_1, s_2, ..., s_a)$, where $\phi(xys_i)$ is the majority color on triples containing x and y, and $s_1, s_2, ..., s_a \in S$ is the lexicographically first a-tuple in S such that $\phi(xys_i) = \phi(xys_j)$ for every $1 \leq j \leq a$ (by the choice of m there is such an a-tuple). Since c is a t-coloring of a complete graph on $N-m = r_t(K_b)$ vertices, there is monochromatic K_b in c, which gives a monochromatic $F(a,b)$ in ϕ.

A lower bound for $r_k(F(a,b))$ is obtained from Proposition 4 (i) since $F(a,b)$ is not 3-partite, for $b \geq 3$. $\square$

6 Windmill and tight path

The following result (conjectured by Kalai) is a special case of a theorem of Füredi and Frankl ([12], Theorem 3.8). We give their proof also, since it is extremely short in this special case.

Theorem 18. $\mathrm{ex}(n,W) \leq \binom{n}{2}$ *with equality for every* $n \equiv 1, 5$ *(mod 20).*

Proof. The lower bound comes from the following construction. Let $n \equiv 1, 5$ (mod 20) and consider a Steiner system S, a $2-(n,5,1)$ design, i.e., a set of 5-element blocks on n elements such that every pair lies in precisely one block. Its existence is proved by Hanani [16, 17]. Then the number of blocks is $\binom{n}{2}/10$. Now place 10 triples inside each block of S. The resulting triple system, H, has $\binom{n}{2}$ triples and is W-free. Indeed, a copy of W would have to be contained in one of the blocks, but each block has less vertices than the number of vertices in W.

To prove the upper bound, suppose that H is a 3-uniform hypergraph with no W. For $x, y \in V(H)$, the *codegree* $d(x,y)$ is the number of edges of H containing both x, y. Let a, b, c be codegrees of three pairs of vertices from a edge of H, $1 \leq a \leq b \leq c$. If $a = 2$, $b \geq 3$ and $c \geq 4$, then H contains a copy of W. Thus either $a = 1$ or $a = b = 2$ or $a = 2, b = 3, c = 3$.

In each of these cases we have that $1/a + 1/b + 1/c \geq 1$. For each edge $e = uvw$ of H, let

$$w(e) = \frac{1}{d(u,v)} + \frac{1}{d(v,w)} + \frac{1}{d(u,w)}.$$

We see that $w(e) \geq 1$. Let $s = \sum_{e \in H} w(e)$. Notice that $s \leq \binom{n}{2}$, since a term $\frac{1}{d(u,v)}$ appears exactly $d(u,v)$ times for each pair uv that belongs to at least one edge of H. Now, $|H| \leq |H| \min_{e \in H} w(e) \leq s \leq \binom{n}{2}$. □

For the next proof we need the following decomposition result:

Theorem 19 (Pippenger and Spencer [33]). *Let r be fixed and D be sufficiently large. Let H be an r-uniform hypergraph with $d(v) = (1+o(1))D$ for every $v \in V(H)$ and codegree of $o(D)$ for every pair $\{u,v\} \subseteq V(H)$. Then $E(H)$ can be partitioned into $(1+o(1))D$ matchings.*

Proof of Theorem 13. To prove the lower bound, let S be a $3-(n,5,1)$ design, i.e. a set of 5-element blocks of an n-element set such that each 3-element set is in precisely one block. The existence of such designs are known for infinitely many n, for example for $n = 4^s + 1, s \geq 2$ [20], see also [32]. Construct an auxiliary 10-uniform hypergraph H where $V(H)$ is the set of $\binom{n}{2}$ pairs in $V(S)$, and ten of these pairs form an edge of H if and only if they are the ten pairs in a block of S. Since every pair in $V(S)$ is in exactly $(n-2)/3$ blocks of S, H is an $(n-2)/3$-regular hypergraph. On the other hand, the codegree of any two vertices in H is at most one. Indeed, any two vertices in H (two pairs in $V(S)$) contain at least three vertices in $V(S)$, and they can be in at most one block of S. With large enough n, and with $r = 10, D = n/3$, the conditions of Theorem 19 hold so we can decompose $E(H)$ into $m = (1+o(1))n/3$ matchings $M_i, i = 1,2,\ldots,m$. Each M_i corresponds to a subset of blocks S_i of S and any two blocks in S_i share at most one element in $V(S)$. The set of triples covered by the blocks of any S_i form a W-free triple system (the center edge of a windmill W in a block $B \in S_i$ would force the other three edges of W to B, similarly as in Theorem 18). Thus K_n^3 is decomposed into $m = (1+o(1))n/3$ W-free triple systems, showing $r_k(W) \geq (1-o(1))3k$.

The upper bound follows from Theorem 18: in a k-coloring of K_n^3 with no monochromatic W, each color class has at most $\mathrm{ex}(n,W) = \binom{n}{2}$ edges. Thus $\binom{n}{3}/k \leq \binom{n}{2}$, implying $n \leq 3k+2$. So by Lemma 15(i), $r_k(W) \leq 3k+3$. □

We need the following result for tight path.

Proposition 20. *$ex(n, P_3^3) \leq \frac{n(n-1)}{3}$ with equality for $n \equiv 1, 4$ (mod 12).*

Proof of Proposition. For a P_3^3-free hypergraph T on n vertices and a vertex v, the degree $d(v) \leq ex(n-1, P_4) \leq n-1$. Thus $3|E(T)| = \sum_v d(v) \leq n(n-1)$. The statement for equality comes from a $2-(n,4,1)$ design by replacing all blocks by K_4^3-s. □

Proof of Theorem 14. Observe that the trace of P_3^3 at its transversal vertex is P_4, the path on four vertices. The upper bound can be obtained in two ways.

Applying and Lemma 15 (i) with proposition 20, we have $r_k(P_3^3) \le 2k+3$. On the other hand, we may apply Lemma 16 as well: $r_k(P_3^3) \le r_k(P_4)+1 \le 2k+3$ ([34]).

For the lower bound we start with a $3-(n,4,1)$ design F (already used in the proof of Theorem 9) and follow the construction in the proof of Theorem 13. Consider the 6-uniform hypergraph H with vertex set being the set of pairs of vertices of F and edges formed by the sets of pairs within the blocks of F. The degree of any vertex in H is $d=(n-2)/2$, the codegree of any pair of vertices is at most one, so the conditions for Pippenger-Spencer Theorem are satisfied, giving a decomposition of H into $(1+o(1))d=(1+o(1))n/2$ matchings, M_i. Each M_i corresponds to a set F_i of blocks of F, intersecting each other in at most one element. Let T_i be the set of triples covered by the blocks of F_i. The T_i-s provide the required P_3^3-free coloring of K_n^3 with $(1+o(1))n/2$ colors. □

7 Small Ramsey numbers

The only known non-trivial classical Ramsey number for triples is $r_2(K_4^3)=13$, due to McKay and Radziszkowski [31].

It is proven in ([34] that $13 \le r_3(K_4^{(3)}-e) \le 16$ and stated as an easy fact without a proof that $r_2(K_4^3-e)=7$. Here we prove this for completeness.

Proposition 21. *$r_2(K_4^3-e)=7$.*

Proof. Consider the following coloring C of K_6^3. Fix the set of five vertices, V, and let c be the 2-coloring of K_5 on vertex set V with two monochromatic C_5's. Let v be the remaining vertex of K_6^3. For any triple containing v, let $C(\{v,u,w\})=c(uw)$.
For each triple xyz, not containing v, let $C(\{x,y,z\})$ be the color different from $c(V-\{x,y,z\})$. Under coloring C, there are two triples of each color in every 4-set, hence there is no monochromatic K_4^3-e. □

The following proposition determines the small undecided cases from Theorem 10. A hypergraph is linear if every two edges share at most one vertex.

Proposition 22. *$r_4(kite)=5, r_5(kite)=6, r_6(kite)=8$.*

Proof. It is obvious that $r_4(kite)>4$. The fact that $r_4(kite)\le 5$ follows by observing that any 4-coloring of the edges of K_5^3 contains three edges of the same color.

Coloring the triple ijk, $1\le i<j<k\le 5$ by color $i+j+k$ (mod 5) gives $r_5(kite)>5$. To show that $r_5(kite)\le 6$, we need the result of Cayley [2], stating that the maximum number of pairwise disjoint Fano planes in K_7^3 is 2. Suppose K_6^3 on vertex set V is 5-colored so that each color class i is a linear hypergraph P_i. Since the average number of edges in a color class is four and no linear hypergraphs on 6 vertices can have more than four edges, it follows that each P_i must be a Pasch configuration. Therefore the pairs uncovered by the triples of P_i form a matching M_i in the complete graph on V. The M_i-s must form a factorization on V otherwise some pair in V would be covered by at most three P_i-s instead of the required four. These P_i-s can be extended by a new vertex to a decomposition of K_7^3 into five Fano planes, contradicting Cayley's theorem stated above.

The upper bound $r_6(kite) \leq 8$ is already proved (see the proof of Theorem 10). For the lower bound we need a partition of K_7^3 into six linear hypergraphs, see Figure 1. Set $V = [7]$ and let F_1, F_2 be the two Fano planes generated by shifts of $124, 134 \pmod 7$. The next two sets are isomorphic to a Fano plane from which one line is deleted:

$$F_3 = \{135, 167, 236, 257, 347, 456\}, F_4 = \{123, 146, 247, 256, 345, 367\}$$

and $F_6 = \{127, 136, 145, 246, 567\}$ (Fano plane from which two lines are deleted), $F_7 = \{125, 147, 234, 357\}$ (a Pasch configuration).

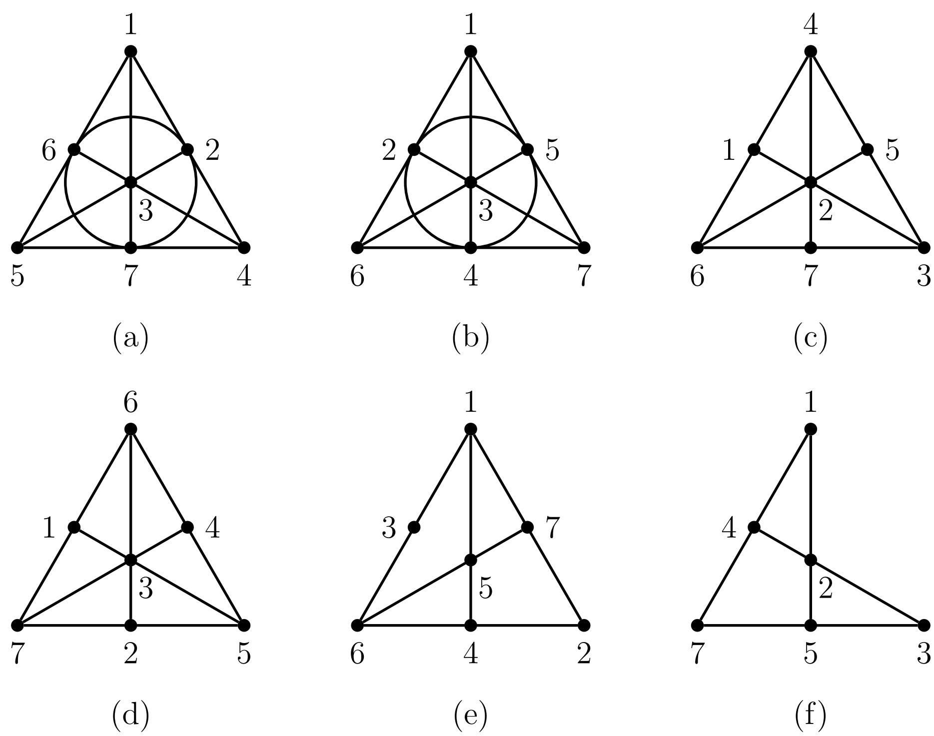

Figure 1: Partition of K_7^3 into two Fano, two Fano$-e$, Fano$-2e$, Pasch

□

Proposition 23. *Set* $r_k = r_k(bow)$*, then* $r_1 = r_2 = 5, r_3 = r_4 = r_5 = 6,\ r_6 = 7, r_7 = r_8 = r_9 = r_{10} = 9,\ 9 \le r_{11} \le r_{12} \le r_{13} \le r_{14} \le 10, r_{15} = 11.$

Proof. All but one upper bounds are obtained from Lemma 15(i). The exceptional case is when Lemma 15(i) gives $r_5(bow) \le 7$. Here we improve it as follows. Suppose K_6^3 is 5-colored without monochromatic bow. From Lemma 17 each color class is either a K_4^3 (type A) or four triples pairwise intersecting in the same base pair (type B). There are at most three type A colors. The base pairs for different type B colors must be vertex disjoint. Thus there are at least two type A color classes, w.l.o.g. $abcd, cdef$. But then only the base pairs ae, af, be, bf are available for type B colors. Therefore we have two type B and three type A colors, the third is the K_4^3 spanned by $abef$. Now there is no base pair available for type B color classes since every pair of vertices is covered by a type A K_4^3.

Lower bounds should be exhibited for $r_1, r_3, r_6, r_7, r_{15}$ only. Coloring all triples of K_4^3 with the same color, $r_1 > 4$ follows. Coloring the triples of $\{1,2,3,4\}$ with color 1, the triples $125, 135, 235$ with color 2, the triples $145, 245, 345$ with color 3, $r_3 > 5$ follows. Then $r_6 > 6$ comes from the following 6-coloring with color classes $\binom{\{1,2,3,4\}}{3}$, $\binom{\{3,4,5,6\}}{3}$, $\binom{\{1,4,5,6\}}{3} - \{4,5,6\}$, $\binom{\{2,4,5,6\}}{3} - \{4,5,6\}$, $\binom{\{1,2,3,5\}}{3} - \{1,2,3\}$, $\binom{\{1,2,3,6\}}{3} - \{1,2,3\}$. The 7-coloring of K_8^3 is the 7 parallel classes of the unique $3-(8,4,1)$ design. Finally, the 15-coloring of K_{10}^3 comes from the unique $3-(10,4,1)$ design whose 30 blocks can be partitioned into 15 disjoint pairs. □

Proposition 24. $r_2(F_5) = 6$.

Proof. The lower bound is obvious, color triples of K_5^3 containing a fixed vertex with color 1 and other triples by color 2. For the upper bound, consider a 2-colored K_6^3 on vertex set $\{1,2,3,4,5,6\}$ and its 2-colored trace $K = K_5^2$ with respect to vertex 6. There is a monochromatic, say red odd cycle C in $V(K) - \{6\}$. If $C = 1,2,3,1$ then either there is a red triple in K with two vertices on C and one vertex not in C or all such triples are blue. The former gives a red, the latter a blue F_5. If $C = 1,2,3,4,5,1$ then either there is a red triple with vertices non-consecutive on C or all the five such triples are blue. Again, the former gives a red, the latter a blue F_5. □

Theorem 25. $r_3(F_5) = 7$.

Proof. For the lower bound, color the triples of K_6^3 containing v with color 1, color uncolored triples containing vertex $w \ne v$ with color 2 and color all other edges with color 3.

To prove the upper bound, call a graph G *nice* if for every triple $T = \{v_1, v_2, v_3\}$ of vertices at least one of the following holds:
1. There are two vertex disjoint edges of G, such that one of them is in T and the other meets T. 2. There is a path of length two in G connecting two vertices of T with midpoint not in T.

Observation 26. *If* H *is an* F_5*-free* 3*-uniform hypergraph, such that the trace of* v *for a vertex* v *is a nice graph, then all edges of* H *within* $V(G) \cup \{v\}$ *contain* v*.*

Indeed, otherwise from the definition of a nice graph we find F_5 in H. Thus finding a large nice subgraph in a trace one can reduce the number of colors. More generally, a graph is i-nice if the property holds for all but at most i triples of vertices.

We need a lemma on 6-vertex graphs. Since its proof is routine but lengthy, we state it without proof.

Lemma 27. *Suppose G has six vertices. If $|E(G)| \geq 9$ then G is nice. If $|E(G)| = 8$ then G is 1-nice, if $|E(G)| = 7$ then G is 2-nice. If $|E(G)| = 6$ then G is 5-nice, except in one case, when G is $K_{2,3}$ plus an isolated vertex (in this case it is 6-nice).*

With these preparations we are ready to prove the upper bound. The majority color, say red in a 3-colored K_7^3, has at least 12 edges. Some vertex v has red degree at least 6. Let G be the trace of a red hypergraph at v. We get a contradiction from Lemma 27 (and from the fact that we have 12 edges) except when G has exactly six edges and the trace is $K_{2,3} + w$. This case implies that the red color class has 12 edges forming $K_{2,2,3}$, a complete 3-partite hypergraph with parts of sizes 2, 2, and 3. However, among the $35 - 12 = 23$ edges of other colors, one color, say blue, has at least 12 edges. Repeating the argument for the blue hypergraph, we conclude that the blue hypergraph is also a $K_{2,2,3}$. However, as one can easily check, there is no way to place two edge disjoint $K_{2,2,3}$-s on 7 vertices. □

8 Concluding remarks

We determined, for 3-uniform hypergraphs, r_k ranges from $\sqrt{k}$ to double exponential in k, and showed a jump in r_k when H changes from r-partite to non-r-partite. This leads to the following question.

Problem 28. *For which 3-uniform hypergraphs F, is $r_k(F)$ double exponential? Are there other jumps that the Ramsey function r_k exhibits?*

The ramsey-numbers $r_k(bow), r_k(kite)$ are closely connected to block designs. In case of the kite the only uncertainty is whether $r_k(kite)$ is $k + 1$ or $k + 2$ when $k \equiv 4 \pmod 6$. This leads to the following problem.

Problem 29. *Suppose $n \equiv 5 \pmod 6$. Is it possible to partition the triples of an n-element set into $n - 1$ partial triple systems, i.e. into parts so that distinct triples in each part intersect in at most one vertex? By Theorem 10, this is not possible for $n = 5$ but perhaps for large enough n (possibly for $n \geq 11$) such partitions exist.*

In case of the bow, the problems related to sharper bounds of $r_k(bow)$ are not purely design theoretic, since color classes can be star components as well. We state just one of those problems.

Problem 30. *Suppose* $n \equiv 6, 10$ *(mod 12). Is it possible to partition the triples of an* n*-element set into* $\frac{n(n-1)}{2}$ *classes so that each class is the union of some disjoint* K_4^3*-s and at most one star component? (Any color class has* $n-2$ *triples.) For* $n = 6$ *there is no solution.*

Concerning $r_k(K_3 - e)$ the most challenging (perhaps difficult) problem is to decrease the upper bound of Theorem 7 by one.

Problem 31. $r_k(K_4^3 - e) < r_k(K_3) + 1$ *for every* $k \geq 3$*?*

A challenging open problem is to improve the estimates of $r_k(P)$ (and/or $ex(n, P)$) where P is the *Pasch configuration* with edges $\{abc, bde, cef, adf\}$. (It can be obtained from the Fano plane by deleting a vertex.) Presently only the following is known.

Proposition 32. *For positive constants* c, c', $c\left(\frac{k}{\log k}\right)^2 < r_k(P) < c'k^4$.

Proof. The lower bound is based on the following P-free hypergraph, showing that $\mathrm{ex}(n, P) = \Omega(n^{5/2})$, [26]. Take an incidence graph G of a projective plane with n points and n lines. It has $\Omega(n^{3/2})$ edges. Add n new vertices $x_1, ..., x_n$ and add all triples of the form $x_i \cup e$, where e is an edge of G. The resulting 3-uniform hypergraph, call it H, has $3n$ vertices and $\Omega(n^{5/2})$ edges.

Notice that the edge-density of H is $d(H) = cn^{-1/2}$ for some constant $c > 0$. From Lemma 15(ii) we see that there is a coloring of K_n^3 with $(c'n^{1/2} \log n)$ colors and no monochromatic P. Thus $r_k(P) > n$ with $k = c'n^{1/2} \log n$. Expressing n in terms of k gives the desired lower bound.

The upper bound follows from Lemma 15(i) and the fact that $\mathrm{ex}(n, P) = O(n^{11/4})$ [26]. This is based on the claim that $\mathrm{ex}(n, K(2, 2, 2)) = O(n^{11/4})$ proved by Erdős [8], where $K(2, 2, 2)$ is the complete 3-partite 3-uniform hypergraph with two vertices in each part. □

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