ON INTERVAL COLORINGS OF PLANAR GRAPHS

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ABSTRACT. An edge-coloring of a graph is called interval if the colors used on edges incident to any vertex form an interval of non-repeated integers. There are only few nontrivial infinite classes of graphs for which existence of interval coloring has been studied. We consider planar graphs. First we find an infinite class of planar graphs with no interval coloring. Second, we give an algorithmic proof for interval colorability of a large class of outerplanar graphs. Third, we improve the general upper bound on the maximal number of colors used in an interval coloring of a planar graph with a fixed number of vertices.

1. INTRODUCTION

For a graph $G = (V, E)$, an interval coloring $c\colon E(G) \to [k]$ is a proper coloring such that $\{c(xy) : y \in N(x)\}$ is an interval for each x. Interval coloring of graphs corresponds to the following scheduling problem. In a group of n people, each person tries to arrange personal conferences with several other people, such that each conference is 10 minutes long and there are no time gaps between conferences. If we represent people as vertices of a graph G and pairs of people who agreed to participate in the conferences as edges of this graph then the existence of the desired schedule is equivalent to the existence of an interval coloring of G.

This problem was introduced by Asratian and Kamalian [1] and has been studied extensively for bipartite graphs [4], [2], [5]. The main classes of bipartite graphs for which interval coloring exists include regular bipartite graphs and bi-regular bipartite graphs with minimal degree 2 (here, bi-regular means that all vertices in the same partite set have the same degree). It is still an open question whether any bi-regular bipartite graph is interval colorable. Among other classes of graphs not much is known. Trees are easy to show to have interval colorings. If edge-chromatic number of a graph is $\Delta + 1$, i.e., the graph is of Class II, then such graph is not interval colorable. Indeed, by considering any interval coloring and taking the label of each edge modulo Δ one produces a proper edge-coloring of a graph with Δ colors. The only nontrivial example of Class I graphs with no interval coloring and infinitely many members includes all wheels on n vertices $n \neq 4, 7, 10$ as was shown using computer analysis by Giaro et. al. [3]. In sections 1 and 2 we investigate interval colorable and interval noncolorable planar graphs. We prove the result for wheels analytically and find interval colorings for a large class of outerplanar triangulations.

In section 3 we investigate the extremal problem of finding the maximal number of colors used in an interval coloring of a planar graph on n vertices. Asratian and Kamalian [1] proved that for general graphs this number is at most $2n$. We improve this bound for planar graph to $(11/6)n$. We conclude the paper with several conjectures and open questions.

2. Interval colorings of wheels

A wheel W_n is formed by joining a vertex v to all vertices of an n-cycle C_n. We shall call the edges incident to v *spokes.* In the following calculations we always assume without loss of generality that any coloring of W_n uses labels $\{1, \cdots, n\}$ on its spokes.

Lemma 1. *Let c be an interval coloring of a graph G and $P = e_1, \dots, e_q$ be a path with $c(e_1) < c(e_q)$. Then for every $i \in [c(e_1), c(e_q)]$ there is an edge e incident to an internal vertex of P such $c(e) = i$.*

Proof. We use induction on the length of P. If $q = 2$ then the result is obvious from the definition. Now, fix i and consider a color of e_2. If $c(e_2) < c(e_q)$ and $i \geq c(e_2)$ then we apply induction to $P \setminus e_1$. If $c(e_2) > c(e_q)$ or $i < c(e_2)$, then we apply induction hypothesis to $P' = e_1 e_2$. □

Lemma 2. *In any interval coloring of W_n the set of colors used on the cycle includes $\{2, \dots, n-1\}$.*

Proof. For a fixed $i \in [2, n-1]$ consider the spokes e_i, e_1, e_n colored i, 1 and n respectively. Consider the segment P of the cycle between the endpoints of e_1 and e_n which does not contain an endpoint of e_i. By lemma 1 some of the vertices of P must have an incident edge colored i. It can not be used on a spoke, so it must be used on the cycle.

□

Lemma 3. *In any interval coloring of W_n the set of colors used on the cycle is a proper subset of* $\{1,\dots,n\}$.

Proof. By lemma 2, colors $2,\dots,n-1$ are used on the cycle. First assume that $1,\cdots,n$ are the labels used on the cycle and v is the center of the wheel. Consider an edge on the cycle of color 1. The adjacent spokes and cycle edges must have colors 2 and 3. It is easy to see that there are consecutive vertices x_1,x_2,x_3,x_4 on the cycle such that $c(x_1x_2)=3$, $c(x_2v)=2$, $c(x_2x_3)=1$, $c(x_3v)=3$, $c(x_3x_4)=2$. Now, there are two choices of sets of colors for other neighbors of x_4. Namely, $1,3$ and $3,4$. Color 3 can not be used on a spoke, so in any case we shall have a cycle edge colored 3 which is impossible since we already have a x_1x_2 colored 3. This shows that $\{1,\dots,n\}$ can not be a set of colors used on the cycle.

Now, assume that there is a color q less than 1 used on a cycle. Then edges adjacent to this must have colors from the set $\{q-2,q-1,q+1,q+2\}$. Since we have set of colors $[n]$ on the spokes, $q+1$ and $q+2$ must be used on the spokes (this implies that $q=0$). Easy analysis shows that in this case we are forced to have 1 on the cycle and also two edges colored 2 on the cycle, thus the multiset of colors used on the cycle is $0,1,2,2,3,\cdots,n-1$ which is impossible since the number of edges on the cycle is n.

Similarly, we show that there are no edges on the cycle colored with numbers greater than n which concludes the proof.

□

Corollary 4. *In any interval coloring of W_n, $n\geq 5$, there is a color $i\in\{3,\dots,n-2\}$ used at least twice on the edges of the cycle. Colors $1,2,n-2,n-1$ can not be repeated.*

Proof. Since there are n edges on the cycle, Lemma 3 implies that some color i is repeated on the cycle. There are at least four spokes adjacent to these cycle edges of color i. They must have distinct colors chosen from the set $\{i-2,i-1,i+1,i+2\}$. Since the colors used on the spokes are $1,2,\cdots,n$; i can not be equal to 1, 2, $n-1$ or n. □

Lemma 5. *Let color i be repeated on the cycle in the interval-colored W_n, $n\geq 9$. If $i<n-3$ then color $i+2$ is repeated. If $i>4$ then color $i-2$ is repeated.*

Proof. Assume that i is a repeated color on edges e_1 and e_2. The spokes adjacent to cycle edges of color i have colors $i-2, i-1, i+1, i+2$. Consider paths P_1' and P_2' formed by the disjoint segments on the cycle between e_1 and e_2 and the edge e of color $n-1$ respectively. Let $P_1 = P_1' \cup e$ and $P_2 = P_2' \cup e$. We might assume that the spoke adjacent to e_1 which is incident to P_1 is not colored $i+2$, then by lemma 1, $i+2$ is used on some edge of P_1. If $i+2$ is not used on the corresponding spoke of e_2 then we have $i+2$ used on some edge of P_2. Since P_1 and P_2 share only edge e of color $n-1$, we have repeated $i+2$ for $i+2 < n-1$. If the corresponding spoke of e_2 is colored $i+2$, then, applying Lemma 1 to $P_2 \cup e_2$, we have an edge e' colored $i+1$ on P_2. Now, apply lemma 1 again to the segment of P_2 from e' to e including both e and e', we get an edge of color $i+2$ on that path. This again gives a second edge of color $i+2$. In case $i > 4$, we can find two edges of color $i-2$ on the cycle. □

Lemma 6. *If W_n is interval-colored and $n > 9$ then there are three distinct colors repeated on the cycle.*

Proof. By Corollary 4 there is a repeated color $i \in \{3, \ldots, n-2\}$. If $4 < i < n-3$, then by Lemma 5 colors $i+2$ and $i-2$ are repeated at least twice on the cycle. If $i \geq n-3$, then $i-2 \geq n-5 > 4$ for $n > 9$, thus using Lemma 5 again, we have colors $i-2$ and $i-4$ repeated at least twice on the cycle. Similarly, when $i \leq 4$, $i+2 \leq 6 < n-3$ for $n > 9$, thus we have colors $i+2$ and $i+4$ repeated on the cycle. □

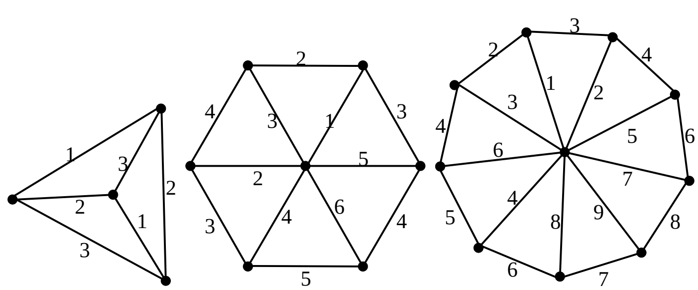

FIGURE 1. The only interval colorable wheels.

Theorem 7. *Interval coloring of W_n exists if an only if $n = 3, 6, 9$.*

Proof. Consider an interval coloring of W_n, $n > 9$. Assume without loss of generality that the colors used on the spokes are $\{1, \ldots, n\}$. Then, using lemmas 2 and 3, there are at least $n - 2$ distinct colors used on the cycle. Using lemma 6 there are three distinct colors which are repeated at least twice on the cycle, a contradiction. This shows that there is no interval coloring of W_n for $n > 9$. For $n = 3, 6, 9$ Figure 1 gives an interval coloring. For $n = 4, 5, 7, 8$ case analysis shows that there is no interval coloring. □

3. On interval colorings of outerplanar graphs

An *outerplanar triangulation* is a graph which has a planar embedding with all vertices belonging to one (unbounded) face and all bounded faces being triangles. Here we say that an edge which does not belong to unbounded face is *internal* edge. Not all outerplanar triangulations have interval colorings. The simplest example is a triangle. Another example is

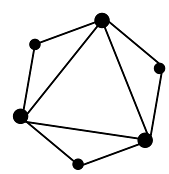

FIGURE 2. Outer planar triangulation with no interval coloring.

shown in Figure 2, that graph has no interval coloring.
On the other hand, we show that outerplanar triangulations with more than 3 vertices without separating triangles are interval colorable. Here, separating triangle is a triangular face none of whose edges belongs to the unbounded face.

We show that any outerplanar triangulation of size more that three and with no separating triangle can be nicely "decomposed" into small graphs for which we construct interval colorings as shown in Figure 3. An interval coloring of the original graph is then constructed using these building blocks.

Definition 8. We say that G is obtained from G_1 by *attaching* G_2 via edge e, $G = G_1 e G_2$ if $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2)$, $V(G_1) \cap V(G_2) = \{x, y\}$ where $e = xy$ and $e \in E(G_1) \cap E(G_2)$. If G is obtained from several graphs by attaching them as follows $G = G_1 e_1 G_2 e_2 \cdots G_m e_m G_{m+1}$, where $e_1, \cdots, e_m$ are distinct edges, then we say that G is *decomposed* into graphs $G_1, \cdots, G_{m+1}$ with *connecting edges* $e_1, \cdots, e_m$. When it is clear from the context, we shall simply write $G = G_1 \cdots G_{m+1}$.

Definition 9. Let c be an interval coloring of a graph G. We say that an edge e has an *extremal coloring* under c if either it has a maximal label among all labels on the edges adjacent to e or it has a minimal such label.

For example, the bold edges of the graphs in Figure 3 have extremal colorings in corresponding graphs. Note that if e has a maximal label among all edges incident to it, then it is easy to construct an interval coloring where e gets a minimal such label by defining $c_1(e) = t - c(e)$ where t is the maximal label used on G.

Theorem 10. *Suppose $G = G_1 e G_2$, where G_1 and G_2 have interval colorings in which e has an extremal label. Then G has an interval coloring.*

Proof. Take an interval coloring c_1 of G_1 such that $e = xy$ gets the smallest label among edges incident to x and y. Assume that $c_1(e) = 0$. Consider an interval coloring c_2 of G_2 where e gets the largest label among edges incident to x and y. Say, $c_2(e) = i$. Then we construct an interval coloring c of G as follows:

$$c(e') = \begin{cases} c_1(e') & \text{if} \quad e' \in E(G_1) \\ c_2(e') - i & \text{if} \quad e' \in E(G_2) \end{cases}$$

□

We need some more definitions. A graph obtained by joining a vertex x to all vertices on a path $P = x_1 x_2 \ldots x_k$ is called a *k − block*. Next we shall show that we can decompose any outerplanar triangulation with no separating triangles into 4-blocks and 3-blocks and at most two other small graphs for which we construct interval colorings separately. In the figures illustrating the following results we arrange the building blocks vertically so that we can always easily identify lower and upper connecting (bold) edges.

Theorem 11. *Let G be an outerplanar triangulation of order at least four and without separating triangles. Then one of the following holds*
a) G is a 5-block
b) $G = G_1 e_1 \cdots e_m G_{m+1}$ where G_i is isomorphic to one of H_4, H_4', H_3,

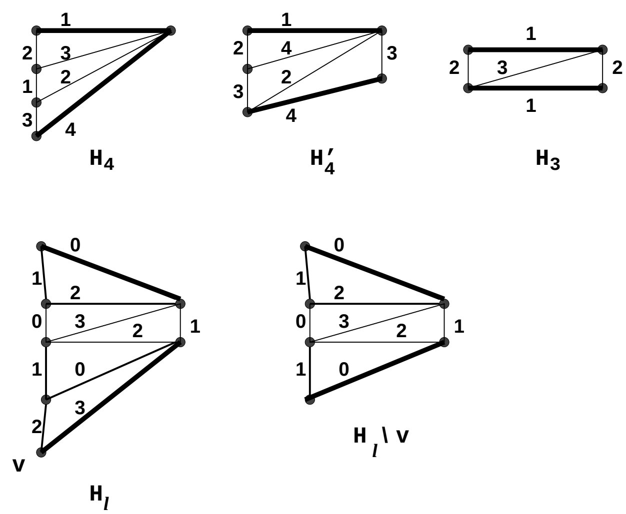

FIGURE 3. Building block of decomposition with bold edges having extremal colors.

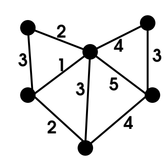

FIGURE 4. Interval coloring of a 5-block

H_l, $H_l - v$ for each $i = 1, \cdots, m+1$ with $e_1, \cdots, e_m$ being corresponding bold edges. Moreover, there is at most one graph in the decomposition isomorphic to H_l or $H_l - v$.

Proof. We give an explicit algorithm for such a decomposition. First we describe a greedy decomposition using only H_4, H_4' and H_3 except perhaps for the last graph in the decomposition. The procedure is illustrated in Figure 5. We start with a vertex v of degree two in G, "peel off" a desired small subgraph containing this vertex, and repeat the same for the leftover graph.

Formally, we define G_1 to be a subgraph of G containing v, isomorphic to one of H_4, H_4' or H_3 such that an edge e_1 corresponding to one (lower)

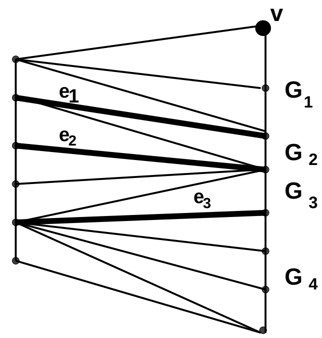

FIGURE 5. Greedy decomposition. Bold edges are connecting edges.

bold edge is internal in G. It is easy to see that such G_1 always exists and such e_1 exists unless $G = G_1$.

Suppose that $G_1, \cdots, G_i$ and corresponding edges $e_1, \cdots, e_i$ are constructed. Note that $G_1 e_1 G_2 e_2 \cdots e_{i-1} G_i$ is a decomposition of a subgraph of G and e_i is an edge from G_i not used in other G_ks , $k < i$ such that it corresponds to a bold edge and it is an internal edge of G. As before, we define a subgraph G_{i+1} in undecomposed part G' of G, i.e., in an induced subgraph of G not containing any vertices of G_j, $j = 1, \cdots, i$ except the endpoints of e_i.
Case 1. If $|V(G')| \geq 4$ we let G_{i+1} be the graph containing e_i, being isomorphic to one of H_4, H_4' or H_3 such that edges e_i and e_{i+1} correspond to the bold edges and e_{i+1} is an internal edge in G. Such G_{i+1} always exists and e_{i+1} exists unless $G = G_1 e_1 \cdots e_i G_{i+1}$.
Case 2. If $|V(G')| = 3$ then we are creating the last graph in decomposition by modifying G_i. We let G_{m+1} be a subgraph of G induced by vertices of G_i and G'. Now, $G = G_1 e_1 \cdots G_{i-1} e_{i-1} G_{m+1}$. Note that in this case G_{m+1} has either three or four triangular faces.
If we never applied Case 2 in constructing the decomposition, we are done using only H_4, H_4' and H_3. But Case 2 might give us the last graph G_{m+1} not isomorphic to any of H_4, H_4' or H_3 (and this is the case depicted in the Figure 5). Note that this might happen only if G_{m+1} has 4 triangular faces. If G_{m+1} is not a 5-block then it must be isomorphic to $H_l \setminus v$ from Figure

3 and we are done. The only situation left is when G_{m+1} is a 5-block. We need to change our decomposition in this case. Observe that any greedy decomposition may start either with a 3-block or a 4-block. If both of such decompositions lead to the last graph being a 5-block then we show that there is a very specific subgraph of G which we can use to create a new decomposition without 5-blocks. We leave the details for Appendix.

□

Theorem 12. *If G is an outerplanar graph of order at least 4 without separating triangles then it has an interval coloring.*

Proof. If G is a 5-block then Figure 4 gives an interval coloring. So, we might assume from now on that G is not a 5-block. We consider the decomposition of a given outerplanar triangulation G without separating triangles as given in Theorem 11. Next we use our constructions shown in Figure 3 as building blocks using Theorem 10. The conditions of Theorem 10 are satisfied since the connecting edges have extremal colors in our colorings. □

Next we observe that deleting certain edges from outerplanar triangulations with no separating triangles does not prevent the resulting graph from being interval-colorable. We say that an edge is *dividing* in interval coloring c of a graph H if $H = H_1 e H_2$ and e has an extremal label in H_1 under c as well as in H_2 under c.

We need the following lemma.

Lemma 13. *Let graph H have interval coloring c such that e is a dividing edge under coloring c. Then $H \setminus e$ has interval coloring c' such that all dividing edges in $H \setminus e$ under coloring c are dividing edges in $H \setminus e$ under coloring c'.*

Proof. Let e be dividing edge in H under interval coloring c and $H = H_1 e H_2$. Then, without loss of generality, $c(e') > c(e)$ for all $e' \in E(H_1)$ adjacent to e and $c(e') < c(e)$ for all $e' \in E(H_2)$ adjacent to e. Moreover, the coloring c restricted to H_1 as well as to H_2 is an interval coloring. Now, we define c' on $H \setminus e$ as follows.

$$c'(e) = \begin{cases} c(e) & \text{if} \quad e \in E(H_1) \\ c(e)+1 & \text{if} \quad e \in E(H_2) \end{cases}$$

It is easy to see that c' is an interval coloring of $H \setminus e$ satisfying the conditions of the lemma.

□

Lemma 14. *Let G be a graph obtained by deleting some dividing edges under some interval coloring of a graph H. Then G is interval colorable.*

Proof. Consider an interval coloring c of H and dividing edges $e'_1, \cdots, e'_k$ to be deleted. We delete edges e'_i, $i = 1, \cdots, k$ one by one and apply Lemma 13 at each step. After each step we obtain an interval coloring c of a new graph with all remaining edges e'_is being dividing edges under c. □

Corollary 15. *Let H be an outerplanar triangulation with no separating triangles and let $H = H_1 \cdots H_m$ be decomposition given by Theorem 11 with connecting edges $e_1, \cdots, e_{m-1}$. If G is obtained from H by deleting some connecting edges, then G has an interval coloring.*

Proof. If c is the coloring of H given by Theorem 12 then all connecting edges are dividing edges under this coloring. □

Observation The above corollary remains valid if some of the edges besides connecting edges are deleted, i.e., ones corresponding to certain non bold edges from the graphs in Figure 3. For example, the internal edge of color 3 in H_3. It is also easy to see that if all bounded faces of an outerplanar graph have even length then this graph is interval colorable.

4. General Upper Bound for planar graphs

Let G be an interval colorable graph then $t(G)$ is the maximal number of colors used in any interval coloring of G. In [1] Asratian and Kamalian showed that if a graph G on n vertices has no triangles then the maximal number of colors used in an interval coloring is $n-1$. For arbitrary graphs on n vertices, they proved that the maximal number of colors used in an interval coloring is at most $2n$. Here we improve the upper bound of $2n$ to $(11/6)n$ for planar graphs. The major idea of our proof is based on the proof of Asratian and Kamalian for triangle-free graphs.

First, we give a construction (and a planar embedding of the corresponding graph) which is illustrated in Figure 6. This construction can be described as follows. If $n = 2k+2$, then we let G be the graph obtained from k copies $T_1, T_2, \ldots, T_k$ of K_4 which are arranged as a path as follows. Label vertices of each T_i by u_i, v_i, u'_i, v'_i and identify u'_{i+1} with u_i and v'_{i+1}

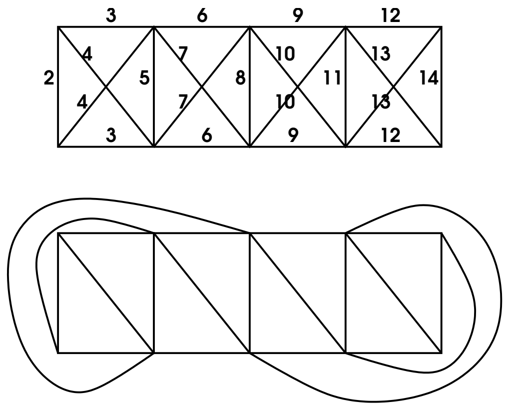

FIGURE 6. A planar graph on n vertices whose interval coloring uses $(3/2)n$ colors.

with v_i for $i = 1, \dots, k-1$. Now, we color the edges of each T_i with colors $3i-1, 3i, 3i+1, 3i+2$ so that the resulting coloring is interval. If $n = 2k+3$ we add a pendant edge to u_1 and color it with 1. In the following proof we denote by $[a, b]$ the set of integers $\{a, a+1, \cdots, b\}$.

Theorem 16. *$t(G) \leq (11/6)n$ for any planar interval colorable graph G on n vertices.*

Proof. Let $t = t(n)$ be the maximal number of colors used in an interval coloring of a planar graph on n vertices. Among all planar graphs on n vertices with interval coloring using t colors choose one with minimal number of edges. Let this graph be G. Assume without loss of generality that t and 1 are the maximal and minimal labels used. By a proof of Asratian and Kamalian [1] there is a path between an edge colored t and an edge colored 1 with labels decreasing along the path. As in the proof in [1] we choose a specific path among all paths S described above. Namely, first we choose a subset S' of S of shortest paths, then, among paths from S' with the same labels on the first i edges we choose one with larger label on $(i+1)$st edge.

Call the resulting path $P = e_1 v_1 e_2 v_2 \dots e_{q+1}$. If $q = n$ then we are done since the number of colors used on P is at most t. Otherwise, there are some missed colors on P. For each vertex v_i let $C_i = [c(e_{i+1}) + 1, c(e_i) - 1]$

be the set of colors between the colors of the two edges of P incident to v_i. Note that colors from C_i are not present on P and they must occur on the edges incident to v_i. Let the endpoints of these edges different from v_i be T_i. We call a graph formed by P and T_is H, i.e., $H = (V, E)$ where $V(H) = V(P) \cup \bigcup_{i=1}^{q} T_i$ and $E(H) = E(P) \cup \bigcup_{i=1}^{q} e(v_i, T_i)$, where $e(v_i, T_i)$ is the set of edges between v_i and T_i. Now, the set of colors used on H is $\{1, \cdots, t\}$ with no two edges having the same color. Thus the number of edges of H is equal to the total number of colors used in G. If the T_i's are pairwise disjoint and do not intersect the set of vertices of P then H has $t+1$ vertices, thus $t \leq n-1$. In general, we shall show that even if T_is share vertices, the total number of vertices in G is large in comparison with number of edges in H. For this we analyse the structure of H.

First, we give several observations describing the intersecting properties of the T_i's. We are going to show that sets T_i do not intersect P and do not share many vertices between themselves. For every i, T_i can only share vertices with T_{i-1} or T_{i+1} but no other T_js. So, the graph H looks like the one in Figure 7.

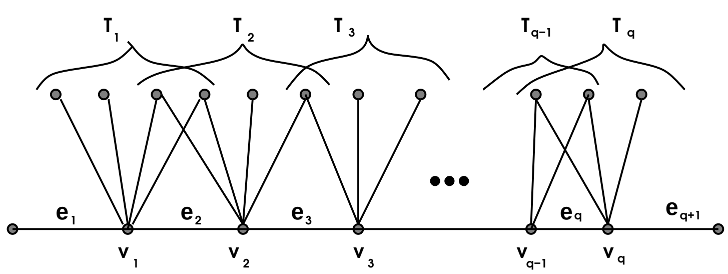

FIGURE 7. Path P and edges of intermediate colors not used on P but incident to vertices on P.

Claim 1. $T_i \cap T_j = \emptyset$ if $j - i > 1$.
Indeed, if $v \in T_i \cap T_j$ for some $i < j-1$ then the path

$$P' = e_1 v_1 \cdots v_i v v_j e_{j+1} \ldots e_{q+1}$$

is either shorter than P or has the same length but has larger label than P on the $(i+1)$st edge.
Claim 2. $T_i \cap V(P) = \emptyset$ for all $i = 1, \cdots, q$.
If $v_a \in T_i \cap V(P)$ and then either $P' = e_1, \ldots v_a v_i e_{i+1}, \ldots e_{q+1}$ (if $a < i$) or

$P'' = e_1, \dots v_i v_a e_{a+1}, \dots e_{q+1}$ (if $a > i$) is shorter than P.

For each $i = 1, \cdots, q-1$ consider $Q_i = T_i \cap T_{i+1}$, let $Q_q = \emptyset$. Define an **intermediate point** of $u_j \in Q_i$ to be a vertex v such that $c(vu_j)$ is between the colors of $u_j v_{i+1}$ and $u_j v_i$. Note that there are $c(v_i u_j) - c(v_{i+1} u_j) - 1$ such intermediate points and, correspondingly, edges incident to u_j besides the edges $u_j v_i$ and $u_j v_{i+1}$. For each Q_i and $u \in Q_i$, let $ex(u)$, **excessive degree** of u be the number of intermediate points of u. Note that $ex(u) > 0$ for any $u \in Q_i$, $i = 1, \cdots, q-1$.
Claim 3 For $u_1, \dots, u_m \in Q_i$ we have $\sum_{j=1}^{m} ex(u_j) \geq m^2$.
If $c(v_i v_{i+1}) = \alpha$ then the largest possible colors on edges $u_j v_i, \quad j = 1, \dots, m$ are $A^- = \{\alpha - 1, \alpha - 2, \cdots, \alpha - m\}$. Correspondingly, the smallest possible labels used on $u_j v_{i+1}, \quad j = 1, \cdots, m$ are $A^+ = \{\alpha + 1, \alpha + 2, \cdots, \alpha + m\}$. Now

$$\sum_{j=1}^{m} ex(u_j) \geq \sum_{j=1}^{m} (a + j - (a - \pi(j)) - 1),$$

where $\pi(j)$ is some permutation of $(1, \dots, m)$. Then

$$\sum_{j=1}^{m} ex(u_j) \geq \sum_{j=1}^{m} (j + \pi(j) - 1) =$$

$$(1 + \cdots + m) + (1 + \cdots + m) - m = m(1+m) - m = m^2.$$

We say that a T**-region** of Q_i is a region in the planar embedding of H which is bounded by v_i, v_{i+1}, u, u'; $\quad u, u' \in Q_i$ and which does not contain any vertices of Q_i in its interior. Note that u might coincide with u'. The total number of T regions is exactly $|Q_i|$.

We choose a planar embedding of H such that $T_i \setminus Q_i$ belongs to the same face as edge e_i for each $i = 1, \cdots, q$. Thus there are at most two T-regions containing some vertices of H in their interior. We say that a T-region is *good* if it is a face in the planar embedding of H, i.e., it does not contain any vertices of H in its interior. As we just observed, the number of good T-regions of Q_i is at least $|Q_i| - 2$. Next we estimate the number of intermediate vertices of Q_i which are located inside the good T-regions of Q_i, call such vertices R_i. Note that no vertices from R_i belong to T-regions of other Q_j, $j \neq i$ which will allow us to count all R_is separately. The following claim gives the lower bound on the size of R_i.

Claim 4 If $q = |Q_i| \geq 7$ then $|R_i| \geq (q-4)^2/2 - (q-3)$. If $q = |Q_i| = 6$ then $|R_i| \geq 1$.
Since there are at most two T-regions which are not good, there are at most 4 vertices from Q_i on the boundaries of these regions. Thus at least $q-4$ vertices from Q_i belong to the boundaries only of good T-regions, call the set of that vertices Q_i'. For each vertex x in Q_i' consider edges incident to x different from xv_i and xv_{i+1}, i.e., in particular those corresponding to excessive degree. The total excessive degree of these vertices is at least $(q-4)^2$ according to Claim 3. Some of these edges might have both end-points in Q_i'. There are at most $q-3$ such edges due to planarity of G. Since each T-region contains at most two elements of Q_i' on its boundary, any vertex not in Q_i' might be an end-point of at most two edges considered above. Thus we have at least $(q-4)^2/2 - (q-3)$ vertices different from Q_i' in the T-regions not containing P in their interior. If $q = |Q_i| = 6$ a simple analysis of excessive degrees similar to the general case shows that there is at least one vertex in some T-region of Q_i not containing p in its interior.

Now, we are ready to estimate the number of vertices in the graph G. Note that we have the following sets of vertices under consideration. $V(P)$, Q_i, $S_i := T_i \setminus \{Q_i \cup Q_{i-1}\}$, R_i. Vertices in $V(P)$, Q_i, S_i, $i = 1, ..., q$ are all distinct, but vertices of R_i might coincide with some vertices in S_i or S_{i+1}. If $x \in R_i \cap S_{i+1}$ we shift x from S_i to S_{i+1}. This way the $R_i \cap S_j = \emptyset$ for $i \neq j$. We have

$$V(G) \supseteq V(P) \cup \bigcup_{i=1,\ldots,q-1} (Q_i \cup S_i \cup R_i) \cup S_q.$$

Thus

$$n \geq \sum_{i=1}^{q-1} (1 + |Q_i| + max\{|S_i|, |R_i|\}) + 1 + |S_q|.$$

The total number of edges in H is

$$e = t = \sum_{i=1}^{q-1} (1 + 2|Q_i| + |S_i|) + 1 + |S_q|.$$

Therefore

$$\frac{t}{n} \leq \max_{i=1,\cdots,q-1} \frac{1+2|Q_i|+|S_i|}{1+|Q_i|+max\{|S_i|,|R_i|\}} = \mu.$$

If $|S_i| > |R_i|$ then

$$\mu \leq \frac{1+2|Q_i|+|S_i|}{1+|Q_i|+|S_i|} \leq 1 + \frac{|Q_i|}{1+|Q_i|+|S_i|} \leq 1 + \frac{|Q_i|}{1+|Q_i|+|R_i|}.$$

If $|S_i| \leq |R_i|$ then

$$\mu \leq \frac{1+2|Q_i|+|S_i|}{1+|Q_i|+|R_i|} \leq \frac{1+2|Q_i|+|R_i|}{1+|Q_i|+|R_i|} = 1 + \frac{|Q_i|}{1+|Q_i|+|R_i|}.$$

So, in both cases we have

$$\frac{t}{n} \leq \mu \leq 1 + \frac{|Q_i|}{1+|Q_i|+|R_i|}.$$

Now, using Claim 4 and the fact that $\frac{|Q_i|}{1+|Q_i|+|R_i|} \leq \frac{|Q_i|}{1+|Q_i|} \leq 5/6$ for $|Q_i| \leq 5$, we have

$$\frac{t}{n} \leq \begin{cases} 1 & \text{if } |Q_i| = 0, \\ 1+5/6 & \text{if } |Q_i| \leq 5, \\ 1+6/8 & \text{if } |Q_i| = 6, \\ 1+q/(1+q+(q-4)^2/2-(q-3)) & \text{if } |Q_i| \geq 7. \end{cases}$$

Since $1+q/(1+q+(q-4)^2/2-(q-3)) \leq 1+5/6$ for $q \geq 7$, we have $t \leq (11/6)n$ and this concludes the proof.

□

5. Appendix

Consider two decompositions of G as follows. $G = G_1 \cdots G_q$ and $G = G'_1 \cdots G'_p$ such that G_1 corresponds to H_3, G'_1 corresponds to H_4 or H'_4 and all other graphs except perhaps the last graph in each decomposition are isomorphic to either H_4, H'_4 or maximal even iterated triangulation. Here even iterated triangulation is an outerplanar triangulation with even number of vertices and internal edges forming a path. Such outerplanar trianglulation can be easily composed from copies of H_3. Also, if there is a choice between the graphs to be used in the decomposition, we choose the largest one. Assume that $G_q = G'_p$ are 5-blocks. Let G_a and G'_b be the last graphs different in the decomposition, i.e., $G_a \neq G'_b$, but $G_{a+i} = G'_{b+i}$ for all $i > 0$. Since G_a and G'_b share the connecting edge, it is easy to see that the only possible situation is as shown in the Figure 8. Now, consider

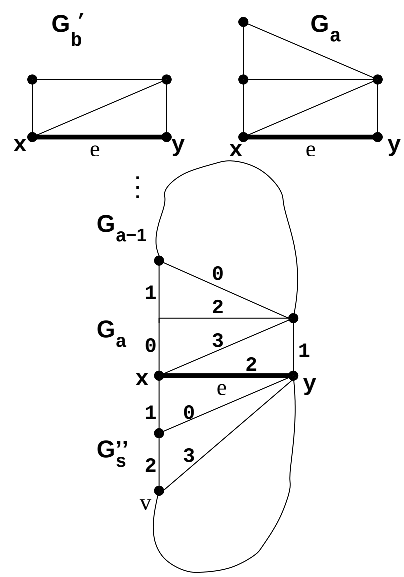

FIGURE 8. Last different graphs in decomposition

the triangles following the bold edge e in the next graph of decomposition. There is only one way to attach them, see rightmost on Figure 8, otherwise we can either take larger G_a or larger G'_b. Note that the resulting graph is isomorphic to H_l.

Next we shall keep the graphs $G_1, \cdots, G_a$ of our decomposition and decompose the rest G' of the graph by greedy algorithm starting at the "other end", i.e., with other vertex (not equal to v) of degree 2 in G.

Formally we have G' a graph induced by $V(G) \setminus (V(G_1) \cup \cdots \cup V(G_a)) \cup \{x, y\}$. Let $G' = G''_1 \cdots G''_s$ where all G''_i except maybe G''_s are isomorphic to H_4, H'_4 or H_3 with corresponding connecting edges and x, y belonging to the last graph G''_s. We treat the cases for G''_s, it has α triangular faces, $1 \leq \alpha \leq 3$. If $\alpha = 3$ then G''_s is isomorphic to H_4 or H'_4 and we are done by taking the following decomposition

$$G = G_1 ... G_a G''_s G''_{s-1} \cdots G''_1.$$

If $\alpha = 1$ or 2, let $Q = G_a G''_s$. Note that it is isomorphic to $H_l \setminus v$ or H_l respectively and

$$G = G_1 ... G_{a-1}(Q) G''_{s-1} G''_{s-2} \cdots G''_1.$$

Now we are done using only graphs isomorphic to H_4, H_4', H_3, H_l, or $H_l \setminus v$ with connecting edges being corresponding bold edges.

6. Open Problems and Conjectures

The method of proof used to determine the maximal number of colors used in an interval coloring of a planar graph suggests improvements. For instance, more careful analysis of the case when $|Q_i| = 3$ is likely to improve the upper bound. We conjecture that the given construction gives correct answer of $(3/2)n$ as a maximal number of colors used in an interval coloring of a planar graph on n vertices.

Conjecture 1 If G is a planar graph on n vertices then the maximal number of colors used in an interval coloring of G is at most $(3/2)n$.

Several examples suggest that the presence of separating triangles in outerplanar triangulations prevents a graph from being interval-colorable.
Question 2 Is it true that an outerplanar triangulation has interval coloring if and only if it does not have a separating triangle?

Conjecture 3 An outerpanar graph obtained from an outerplanar triangulation with no separating triangles by deleting internal edges is interval-colorable.

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