Canonical Pattern Ramsey Numbers

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Abstract. A *color pattern* is a graph whose edges have been partitioned into color classes. A family $\mathscr{F}$ of color patterns is a *Ramsey family* provided there is some sufficiently large integer N such that in any edge coloring of the complete graph K_N there is an (isomorphic) copy of at least one of the patterns from $\mathscr{F}$. The smallest such N is the *Ramsey number* of the family $\mathscr{F}$. The classical Canonical Ramsey theorem of Erdős and Rado asserts that the family of color patterns is a Ramsey family if it consists of monochromatic, rainbow (totally multicolored) and lexically colored complete graphs. In this paper we treat the asymmetric case by studying the Ramsey number of families containing a rainbow triangle, a lexically colored complete graph and a fixed arbitrary monochromatic graph. In particular we give asymptotically tight bounds for the Ramsey number of a family consisting of rainbow and monochromatic triangle and a lexically colored K_N. Among others, we prove some canonical Ramsey results for cycles.

1. Introduction

A graph whose edge set has been partitioned into color classes will be called a *(color) pattern*. It is the partition of the edge set into equivalence classes rather than the specific colors assigned to the classes which is of interest. Classical Ramsey theory of graphs [9] is concerned with finding *monochromatic* patterns — that is, patterns in which all edges have the same color. Anti-Ramsey theory [2] is concerned with finding *rainbow* patterns — that is, patterns in which all edges have different colors. There is a third type of pattern which is of fundamental importance. An edge coloring of a graph is *lexical* provided there is *some* linear order of the vertices so that two edges get the same color iff they have the same smaller endpoint. We denote a monochromatic, rainbow and lexically colored graph H as H^{mono}, H^{rain} and H^{lex} respectively.

According to the classical *Canonical Ramsey Theorem* of Erdős and Rado [1], for any positive integer n there is a sufficiently large integer $N = ER(n)$ such that *any* edge coloring of K_N contains at least one the three patterns K_n^{mono}, K_n^{rain}, or K_n^{lex}. Lefmann and Rödl [8] proved the following bounds on the Erdős-Rado number

$$2^{c_1 n^2} \leq ER(n) \leq 2^{c_2 n^2 \log n}$$

for every positive integer $n \geq 3$, where c_1, c_2 are positive constants.

A family $\mathscr{F}$ of color patterns is a *Ramsey family* provided there is some sufficiently large integer N such that any arbitrary edge coloring of the complete graph K_N contains at least one of the patterns from $\mathscr{F}$. The smallest such N, which will be denoted by $\Re(\mathscr{F})$, is the *Ramsey number* of the family $\mathscr{F}$.

The canonical Ramsey theorem immediately implies that a family of color patterns is a Ramsey family if and only if it contains a monochromatic pattern, a rainbow pattern, and a lexical pattern. The importance of the types monochromatic, rainbow, and lexical is thus apparent. Any color pattern of one of these three types will be called a *canonical* pattern. Note that in acyclic graphs, lexical coloring might be monochromatic or rainbow. Thus there are Ramsey families consisting only of two color patterns, studied for example in [8], [7], [6], [5].

The main task of this paper is to study the behavior of the pattern Ramsey function

$$f(n, H) = \Re(K_n^{lex}, H^{mono}, K_3^{rain}),$$

where H is an arbitrary fixed graph.

In section 3 we derive several bounds on $f(n, H)$ for an arbitrary graph H. One of the conclusions of that section is that $f(n, K_n)$ does not differ much from $ER(n)$, i.e., replacing rainbow K_n with rainbow K_3 does not make a big difference in the behavior of a pattern Ramsey number. In section 4 we find tight bounds for $f(n, H)$ when $H = K_3$. In section 5 we treat the case when H is bipartite. Finally, in the last section we prove some results for canonical Ramsey numbers of cycles.

2. Notations and constructions

Let K_n and C_n be the complete graph and the cycle on n vertices. Let $C[a_1, a_2, a_3, \ldots, a_n]$ denote C_n with the colors a_i in cyclic order as given. To indicate that symbol b is taken m times, we write b^m. Thus $C[1^n]$ is a monochromatic C_n. The *monochromatic degree*, or simply mono-degree of a vertex in an edge-colored graph is the maximal number of edges of the same color incident to that vertex. We denote by $N_i(x)$ and $N_i[x]$ an open and a closed neighborhoods of x in color i, respectively, i.e., the sets of vertices adjacent to x via an edge of color i and (in case of a closed neighborhood) a vertex x itself. Let $d_i(x) = |N_i(x)|$ and $d(x) = \max_i d_i(x)$. For all other standard graph-theoretic definitions we refer the reader to [11].

Next we shall describe several constructions which provide us with useful lower bounds for Pattern Ramsey numbers. If not stated otherwise, all patterns are edge-colored complete graphs here. For a pattern P, we denote $V(P)$ a set of vertices of the colored graph corresponding to P.

Let P be a pattern with $V(P)=\{p_1,\ldots,p_n\}$ and Q be another color pattern. Let c and c' be colorings of P and Q respectively such that c and c' use disjoint sets of colors.

The *pattern product* of P with Q, denoted $P(Q)$, is the pattern obtained by taking $n = |V(P)|$ disjoint copies of a pattern Q with coloring c' and assigning $c(p_ip_j)$ to all edges between ith and jth copies of Q. $P(Q)$ may alternatively be defined as follows.

$$c((a,x),(b,y)) = \begin{cases} c(a,b) & \text{in } P \text{ if } a \neq b \\ c'(x,y) & \text{in } Q \text{ if } a = b \text{ and } x \neq y. \end{cases}$$

Thus $P(Q)$ consists of $|V(P)|$ disjoint copies of Q joined by complete bipartite graphs whose colorings come from P.

For a pattern P, let $\mathfrak{lex}(P)$ be the largest n such that P contains a K_n^{lex}. This is the *lexical index* of the pattern. The *chromatic number* $\chi(P)$ of a pattern P is the maximum chromatic number of a color class of P. The definition of a pattern product easily implies the following lemma.

Lemma 1. *Let F be a color-pattern product $F = P(Q)$. Then*

1. $\chi(F) = \max\{\chi(P), \chi(Q)\}$.
2. *If P and Q are K_3^{rain}-free, then so is F.*
3. *If P and Q do not have monochromatic non-bipartite H, then neither does F.*
4. $\mathfrak{lex}(F) \leq \mathfrak{lex}(P) + \mathfrak{lex}(Q) - 1$

We call a pattern *Ramsey-H* (and denote it $Ram(H)$) if it corresponds to an edge-bicolored K_n in which each color class does not contain H and n is maximal. One of the constructions we are going to use is the power of Ramsey patterns. Let $P = Ram(H)$ then $Ram(H)^l = P(P^{l-1}) = P(P(P...(P)...))$.

For example $Ram(K_3)$ is K_5 whose edges are partitioned into two cycles of length 5. It is easy to visualize $Ram^l(K_3)$ by taking five disjoint sets of vertices of size 5^{l-1} each, say $S_1,\ldots,S_5$. We give all edges between S_i and S_j color 1 if $i<j$ and $j-i=1$. We give all other edges between S_i's color 2. Now, inside each S_i, $i=1,\ldots,5$ we use the same set of colors $\{3,4,\ldots\}$ and apply the coloring recursively.

Corollary 1. *Let H be a non-bipartite graphs and $Q = Ram(H)^l$. Then Q does not have any of the following sub-patterns: monochromatic H, rainbow triangle, lexical K_{2l+2}.*

3. Forbidding K_n^{lex}, H^{mono}, and K_3^{rain}

In this section, as before, we use $f(n,H)$ for $\Re(K_n^{lex}, H^{mono}, K_3^{rain})$. We shall provide three bounds on $f(n,H)$: first in terms of order of H, second in terms of classical multicolor Ramsey numbers, and the last using recursion involving graph Ramsey numbers and a graph H with deleted vertex. Here, we denote by $R(H;n)$ the smallest N such that any coloring of $E(K_N)$ in n colors has a monochromatic subgraph H, i.e., a classical multicolor Ramsey number.

Theorem 1.

$$f(n,H) \leq 3^{n|V(H)|}.$$

Corollary 2. $\sqrt[4]{2}^{(n-2)k} \leq f(n, K_k) \leq 3^{nk}$.

Proof. The lower bound follows from Corollary 1 and the fact [4] that classical Ramsey number $R(k,k) \geq \sqrt{2}^k$. The upper bound follows from Theorem 1.

Theorem 2. *For any connected graph H and any n there is a constant $c = c(n)$ such that $f(n, H) \leq cR(H; n-1)$.*

Let $\mathcal{F}$ be a class of graphs. We say that a graph G is *point-$\mathcal{F}$* if there is a vertex of G whose deletion results in the disjoint union of graphs from $\mathcal{F}$. Call such a vertex, a *blocking* point. For example, if $\mathcal{F}$ is the set of trees, then point-$\mathcal{F}$ graphs are those whose cycles share a common point. Let H be a point-$\mathcal{F}$ graph. For each blocking point z of H, let λ_z the maximum order of a component in $H \setminus \{z\}$. Let $\mu(H)$ be the minimum of λ_z over all blocking points z of H. For an integer i denote by $R(\mathcal{F}_i, K_s)$ the minimal order of complete graph such that any edge-coloring of it in red and blue results in either red K_s or some blue member of $\mathcal{F}$ on i vertices.

Theorem 3. *Let H be a point-$\mathcal{F}$ graph on k vertices. Then*

$$f(n, H) \leq 3[R(\mathcal{F}_{\mu(H)}, K_{f(n-1,H)}) + n(H)] - 4.$$

Note that the Theorem 1 provides a good bound when H is fixed and n large. Theorems 2 and 3 can provide us with the better upper bounds when the number of vertices of H is much larger than n.

Two main lemmas we use to prove the above results evaluate the maximal number of colors incident to every vertex in such a coloring and the maximal number of edges of the same color incident to some vertex.

Lemma 2. *Let c be a coloring of $E(K_N)$ with no rainbow triangle. Then there is a vertex with at least $(N+1)/3$ edges incident to it of the same color.*

Proof. A vertex p not incident with an edge xy will be called an associate of xy provided that some of the edges xp and yp get the same color as xy. Form a bipartite graph $\mathcal{A}$ whose vertices are the edges and vertices of K_N. Join vertex p to edge xy iff p is an associate of xy. The number of associates of xy is its degree in this graph. Each edge arises from a unique triangle xyp. A monochromatic triangle contributes 3 edges. A triangle with exactly two colors on its edges contributes 2 edges. A rainbow triangle contributes no edges, but rainbow triangles are excluded by hypothesis. Hence the number of edges of $\mathcal{A}$ is at least $2\binom{N}{3}$. Therefore there is an edge xy with at least $2\binom{N}{3}/\binom{N}{2} = 2(N-2)/3$ associates. Assume that $c(xy) = 1$. Then $N_1(x) + N_1(y) \geq 2(N-2)/3 + 2$ since x and y were not counted as associates, but $x \in N_1(y)$ and $y \in N_1(x)$. Assume that $|N_1(x)| \geq |N_1(y)|$ then $N_1(x) \geq (N+1)/3$.

Lemma 3. *Let c be a coloring of $E(K_N)$ with no rainbow triangles and no lexically colored K_n. Then, each vertex is incident to at most $n-1$ distinct colors.*

Proof. Assume now that there is a vertex v with at least n edges of different colors incident to it. Let $v_1, \ldots, v_n$ be the end-points of these edges such that $c(vv_i) = i$. Then, since there are no rainbow triangles $c(v_iv_j) \in \{i, j\}$. It is easy to see using induction on n that there is a monochromatic star spanning vertices $v_1, \ldots, v_n$. Assume, without loss of generality that the center of this star is v_1. Then, the edges of this star are colored with color 1. Similarly, there is a spanning monochromatic star in $G[v_2, \ldots, v_n]$, without loss of generality assume that its center is v_2. Proceeding in this manner we have that $c(v_iv_j) = i$ if $1 \le i \le n-2$ and $i < j \le n$. Now, $c(v_{n-1}v_n) \in \{n, n-1\}$ and we have that $v_1, \ldots, v_n$ induce a lexically colored K_n, a contradiction. Thus, every vertex is incident to at most $n-1$ colors.

3.1. Proof of Theorem 1

Let $G_1 = K_N$. Consider a coloring c of $E(G_1)$ with no rainbow triangle, no lexically colored K_n and no monochromatic H. Then, by Lemma 2 there is a vertex $v_1 \in V(G_1)$ having at least $N/3$ edges of color a_1 incident to it. Let G_2 be a complete graph induced by neighbors of v_1 in color a_1. Then there is $v_2 \in V(G_2)$ with at least $(N/3)/3$ edges of color a_2 incident to it in G_2. Using this observation it is easy to see that we can continue this process and find vertices $v_1, \ldots, v_k$ such that, for some colors $a_1, \ldots, a_{k-1}$, $c(v_iv_j) = a_i$ for $i < j$. Moreover,

$$k \ge \log_3 N. \tag{1}$$

Claim 1. $|\{a_1, \ldots, a_{k-1}\}| \le n-2$. Indeed, assume without loss of generality that $a_1, a_2, \ldots, a_{n-1}$ are distinct colors. Then $v_1, \ldots, v_n$ form a lexically colored K_n, a contradiction.
Claim 2. For every a, $|\{i : a_i = a\}| \le |V(H)| - 1$. Assume without loss of generality that $a_1 = a_2 = \ldots = a_{|V(H)|-1} = a$. Then $v_1, \ldots, v_{|V(H)|}$ induce a monochromatic $K_{|V(H)|}$ with all edges of color a, a contradiction. Therefore $k \le (n-2)(|V(H)|-1)$. Combining this with (1) we have that

$$N \le 3^{(n-2)(|V(H)|-1)}.$$

□

3.2. Proof of Theorem 2

Our main tool here is the result of Truzhinski and Tuza [10] on local colorings. A coloring of the edges of a graph is called a local k-coloring if every vertex is incident to edges of at most k distinct colors. For a given graph G, the local Ramsey number, $R_{\text{loc}}(G; k)$, is the smallest integer n such that any local k-coloring of K_n (the complete graph on n vertices) contains a monochromatic copy of G. The following was conjectured by Gyárfás, Lehel, Schelp and Tuza and proved in [10].

Theorem 4. *For each positive integer k there exists a constant $c = c(k)$ such that $R_{\text{loc}}(G; k) \le cR(G; k)$ for every connected graph G.*

Consider an edge-coloring c of K_N with no rainbow triangle, no lexically colored K_n and no monochromatic H. Using lemma 3 we see that every vertex is incident to at most $n-1$ colors and thus our coloring is a local $(n-1)$-coloring. Now, Theorem 2 follows immediately from Theorem 4.

□

3.3. Proof of Theorem 3

Let c be a coloring of $E(K_N)$ containing none of the patterns K_n^{lex}, H^{mono}, or K_3^{rain} where H is point-$\mathcal{F}$. Let $s = f(n-1, H)$. Suppose there is a vertex $w \in X$ and a color i with $\deg_i(w) > R(\mathcal{F}_{\mu(H)}, K_s) + n(H) - 1$. Assume that $i = 1$, call this color "white", and let $W = N_1(w)$. Consider the edges inside W as colored by two colors, white and non-white.

Suppose first that there is a non-white K_s where $s = f(n-1, H)$ inside of W. Return to viewing the original colors in this K_s. Since there is no monochromatic H and no K_3^{rain}, it follows that there must be a K_{n-1}^{lex}. Since all edges in this K_{n-1}^{lex} are non-white, we can add w as an initial vertex with all white edges to the points in the K_{n-1}^{lex}. But this produces a K_n^{lex}, contrary to our assumptions on X. Thus there can be no non-white K_s inside of W.

Now choose a blocking point z of H which minimizes λ_z. Let the components of $H \setminus \{z\}$, in increasing order, be $C_1, C_2, C_3, \ldots, C_m$. Then $\mu(H) = |V(C_m)| \geq |V(C_i)|$ for all i. From the definition of classical Ramsey number we have that there is a white copy of C_m induced by W. Lets delete $V(C_m)$ from W to obtain W_1. Let $H_1 = H - V(C_m)$, it is clear that $H_1 + z$ is point-$\mathcal{F}$ and $\mu(H_1) \leq \mu(H)$.

$$|W_1| = |W \setminus V(C_m)| > R(\mathcal{F}_{\mu(H)}, K_s) + n(H) - 1 - |V(C_m)| \geq R(\mathcal{F}_{\mu(H)}, K_s) + n(H_1) - 1 \geq$$
$$R(\mathcal{F}_{\mu(H_1)}, K_s) + n(H_1) - 1.$$

As before, there is no non-white copy of K_s induced by W_1, thus there is a white copy of C_{m-1}. Proceeding in this manner we shall obtain disjoint white copies of $C_1, \ldots, C_m$, which, together with z form a monochromatic copy of H, a contradiction.

Therefore, for each vertex w,and each color i, $\deg_i(w) \leq R(\mathcal{F}_\mu(H), K_s) + n(H) - 1$. Combining this with Lemma 2 completes the proof of the theorem.

□

4. Forbidding K_n^{lex}, K_3^{mono}, and K_3^{rain}

Let $f(n, K_3) = \Re(K_n^{lex}, K_3^{mono}, K_3^{rain})$. The following is the main result of this section.

Theorem 5. $5^{\lfloor n/2 \rfloor - 1} + 1 \leq f(n, K_3) \leq 5^{n/2}$.

Let c be a coloring of $E(K_n)$ with no monochromatic or rainbow triangles. First, we shall describe some properties of such a coloring. For two disjoint subsets of vertices X, X' having all edges between them of the same color l we denote l by $c[X, X']$. Moreover, for simplicity, we denote $c[\{x\}, X']$ by $c[x, X']$. By $E[X, X']$ we denote the set of edges with one endpoint in X and another in X'.

Fix a vertex x and let k be the number of colors incident to x. Assume, without loss of generality that those colors are $\{1, \ldots, k\}$. Let $S_i = N_i(x)$, the neighborhood of x via edges of color i, $i = 1, \ldots, k$. Because there is no monochromatic or rainbow triangle in a coloring c, the following two claims hold.

Claim 1 There is no edge of color i spanned by vertices of S_i for each $i = 1, \ldots, k$.

Claim 2 $c(xy) \in \{i, j\}$ for each $x \in S_i$ and $y \in S_j$.

Next we shall describe all possible ways the edges between S_i and S_j, $1 \leq i < j \leq k$, might be colored.

Definition 1. *Let $S_i = N_i(x)$ and $S_j = N_j(x)$ be a pair of sets such that there are both colors, i and j present on edges between them. Call such a pair, a mixed pair.*

We shall show that in a mixed pair the coloring of the edges between the sets falls into two possible simple classes. Moreover we shall show that there are not too many mixed pairs, namely, that each set S_i, $i = 1, \dots, k$ belongs to at most one mixed pair. All other pairs induce monochromatic complete bipartite graphs.

Lemma 4. *Let S_1, S_2 be a mixed pair. For $y \in S_1$, let $A_y = \{z \in S_2 : c(yz) = 1\}$ and $B_y = \{z \in S_2 : c(yz) = 2\}$. Then, for all $y, y' \in S_1$ either ($A_y = A_{y'}$ and $B_y = B_{y'}$) or ($A_y = B_{y'}$ and $B_y = A_{y'}$). Moreover $c[A_y, B_y] = 1$.*

Proof. Let y be a vertex in S_1 sending edges of both colors, 1 and 2 to S_2. It is clear that $c[A_y, B_y] = 1$ otherwise we either have a monochromatic triangle in color 2 or a rainbow triangle. This implies that there are no other edges of color 1 spanned by the vertices in S_2. If there is $y' \in S_1$ such that $A_y \neq A_{y'}$ and $B_y \neq A_{y'}$ then y' must send all edges of the same color to S_2 otherwise, $E[A_y, B_y] \cup E[A_{y'}, B_{y'}]$ will contain a monochromatic triangle in color 1. Lets fix $z \in A_y$ and $z' \in B_y$. Now, $c[y', S_2] \neq 1$ since then y', z, z' form a monochromatic triangle in color 1. Thus $c[y', S_2] = 2$, and, because of triangle y, y', z we have $c(y, y') = 2$. But then y', z, z' form a monochromatic triangle in color 2.

As an immediate consequence of Lemma 4 we obtain the following three possible types of mixed ordered pairs (S_i, S_j).
$(\mathbf{S_i}, \mathbf{S_j})$ is of Type 1 if each vertex of S_j sends all edges of the same color to S_i. Then we denote by $S_j(i)$ and $S_j(i)'$ two disjoint sets such that

$$S_j = S_j(i) \cup S_j(i)', \quad c[S_i, S_j(i)] = i, \quad c[S_i, S_j(i)'] = j.$$

$(\mathbf{S_i}, \mathbf{S_j})$ is of Type 1′ if (S_j, S_i) is of Type 1.
$(\mathbf{S_i}, \mathbf{S_j})$ is of Type 2 if each vertex in S_i sends edges of two distinct colors to S_j and each vertex in S_j sends edges of two distinct colors to S_i. Then we denote by $S_i(j), S_i(j)', S_j(i)$ and $S_j(i)'$ disjoint nonempty sets of vertices such that

$$S_i = S_i(j) \cup S_i(j)', \qquad S_j = S_j(i) \cup S_j(i)',$$

$$c[S_i(j), S_j(i)] = c[S_i(j)', S_j(i)'] = i,$$
$$c[S_i(j), S_j(i)'] = c[S_i(j)', S_j(i)] = j.$$

Moreover, $c[S_i(j), S_i(j)'] = j$ and $c[S_j(i), S_j(i)'] = i$. Note that (S_i, S_j) is of Type 2 if and only if (S_j, S_i) is of Type 2.

Lemma 5. *Each set S_i, $i = 1, \dots, k$ belongs to at most one mixed pair.*

Proof. Assume the opposite, without loss of generality, let us have mixed pairs (S_1, S_2) and (S_2, S_3). We shall consider all possible combinations of types these pairs might have.

Case 1
a) (S_1, S_2) is of Type 2 and (S_2, S_3) is of Type 2.

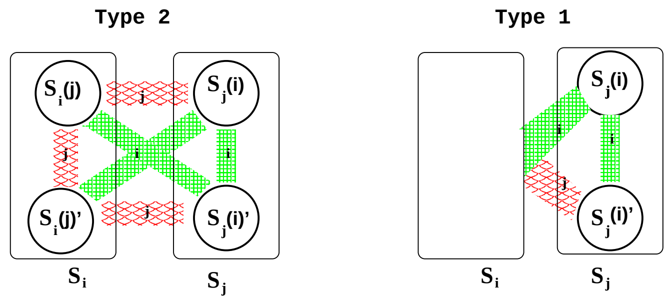
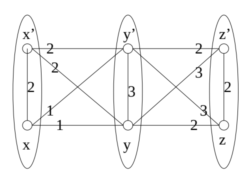


Fig. 1. Types of mixed pairs

b) (S_1, S_2) is of Type 1 and (S_2, S_3) is of Type 2.
c) (S_1, S_2) is of Type 2 and (S_3, S_2) is of Type 1.
d) (S_1, S_2) is of Type 1 and (S_3, S_2) is of Type 1.
If any of the cases a)-d) hold then $c[S_2(1), S_2(1)'] = 1$ and $c[S_2(3), S_2(3)'] = 3$. Note that $E[S_2(3), S_2(3)']$ and $E[S_2(1), S_2(1)']$ are the edges of two complete bipartite subgraphs spanning S_2. Moreover they are edge-disjoint since they have different colors. This is impossible.

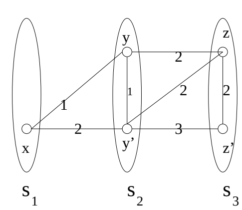
Case 2 (S_2, S_3) is of Type 2 and (S_2, S_1) is of Type 1.
Lets fix vertices x, x', y, y', z, z' such that $x \in S_1(2)$, $x' \in S_1(2)'$, $y \in S_2(3)$, $y' \in S_2(3)'$, $z \in S_3(2)$, $z' \in S_3(2)'$.
Then $c(xz) = 1$ otherwise x, y, z forms a rainbow triangle; $c(x'z) = 3$ otherwise x', y', z form



a rainbow triangle. But now x, x', z form a rainbow triangle.

Case 2' (S_1, S_2) is of Type 2 and (S_2, S_3) is of Type 1.
This case is obtained from Case 2 by interchanging indexes 1 and 3.

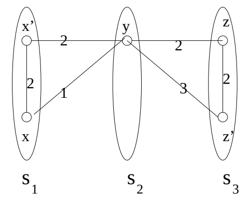
Case 3 (S_1, S_2) is of Type 1 and (S_2, S_3) is of Type 1.
Let $x \in S_1$, $y \in S_2(1)$, $y' \in S_2(1)'$, $z \in S_3(2)$, $z' \in S_3(2)'$.



Then $c(xz) = 1$ otherwise x, y, z form a rainbow triangle; $c(xz') = 3$ otherwise x, y', z' form a rainbow triangle. But now x, z, z' is a rainbow triangle.

Case 3' (S_3, S_2) is of Type 1 and (S_2, S_1) is of Type 1.
This case is obtained from Case 3 by interchanging indexes 1 and 3.

Case 3 (S_2, S_1) is of Type 1 and (S_2, S_3) is of Type 1.
Let $y \in S_2$, $x \in S_1(2)$, $x' \in S_1(2)'$, $z \in S_3(2)$, $z' \in S_3(2)'$. Then, using similar analysis, we see that $c(xz) = 1$, $c(x'z') = 3$. Using triangle x', z, z' we see that $c(x', z) = 3$. But then x, x', z is rainbow.



Lemma 6. *Let S_i, S_j, S_k be the sets described above such that (S_i, S_j) is a mixed pair and both (S_k, S_i) and (S_k, S_j) are not mixed pairs. Then either a) $c(S_k, S_i) = c(S_k, S_j) = k$ or b) $c(S_k, S_i) = i$ and $c(S_k, S_j) = j$.*

Proof. We know that $c(S_x, S_k) \in \{x, k\}$ for $x \in \{i, j\}$. Assume that $c(S_k, S_i) = i$ and $c(S_k, S_j) = k$. There is an edge uv of color j with $u \in S_i, v \in S_j$ since (S_i, S_j) is a mixed pair. Then for any $w \in S_k$, uvw is a rainbow triangle using colors i, j, k, a contradiction. Similarly, it is impossible to have $c(S_k, S_i) = k$ and $c(S_k, S_j) = j$.

Lemma 7. *Let G be a graph on vertices $v_1, \ldots, v_k$ corresponding to the sets $S_1, \ldots, S_k$, $k \geq 3$. Let v_iv_j be an edge in G if S_i, S_j is not a mixed pair. Let $c'(v_i, v_j) = c[S_i, S_j]$. Then there is a monochromatic star in c' spanning all edges of G incident to its center.*

Proof. Observe first that $c'(v_i, v_j) \in \{i, j\}$ and there are no rainbow triangles in G. Note that G is obtained from K_k by deleting the edges of a (possibly empty) matching. If $G = K_k$ then the statement of the theorem is easily provable by induction. So, we assume that there are some nonedges in G. If $k = 3$, and $G \neq K_3$ then an edge incident to a vertex of degree one is a desired star. If $k = 4$ and $G \neq K_4$, assume that $v_1v_2 \notin E(G)$. Thus, by Lemma 6 we have either
a) $c(\{v_1, v_2\}, v_3) = 3$ and $c(\{v_1, v_2\}, v_4) = 4$.
b) $c(\{v_1, v_2\}, v_4) = 4$, $c(v_1, v_3) = 1$, $c(v_2, v_3) = 2$.
c) $c(\{v_1, v_2\}, v_3) = 3$, $c(v_1, v_4) = 1$, $c(v_2, v_4) = 2$.
d) $c(v_1, \{v_3, v_4\}) = 1$ and $c(v_2, \{v_3, v_4\}) = 2$.
In cases a)-c) it is easy to see that either v_3 or v_4 will be a center of a desired star whether v_3v_4 forms an edge of G or not. In case d), v_1 is such a center.

Now, let $k > 4$. Consider $G' = G - v_k$. Assume without loss of generality that the center of a desired monochromatic star in G' is v_1. Since $k > 4$, all edges from v_1 must have color 1. If $v_1v_k \notin E(G')$ we are done. Otherwise, if $c(v_1v_k) = 1$ we are also done. So, assume that $c(v_1v_k) = k$.

Case 1. v_1 has degree $k-1$ in G. For each $i = 2, \ldots, k-1$, the edge $v_i v_k$ can not have color i otherwise $v_i v_k v_1$ is rainbow. Thus we have that v_k is a center of spanning monochromatic star of color k.

Case 2. v_1 has degree $k-2$ in G. Let $v_1 v_j \notin E(G)$ for some j. As we observed, $j \neq k$. Now, if $c(v_j v_k) = k$ we are done as in the previous case since $c(v_i v_k) = i$ for all v_i adjacent to v_k such that $i \neq j$. Thus we can assume that $c(v_j v_k) = j$. Then it is easy to see that $c(v_j v_i) = j$ for all $i \notin \{1, j\}$ since otherwise $v_i v_j v_k$ is rainbow. Thus we have a desired star in color j.

We say that two vertices x, y and a set S form a *bi-colored bi-star* if $c[x, S] = \alpha$, $c[y, S] = \beta$, and $c(xy) \in \{\alpha, \beta\}$ for some distinct colors α and β. We call S a set of leaves of that bi-star.

Theorem 6. *Let c be a coloring of $E(K_n)$ such that there is no monochromatic or rainbow triangle. Then either there is a monochromatic star with at least $(n-1)/2$ leaves or there is a bi-colored bi-star with at least $(n-1)/5$ leaves.*

Proof. Lets fix a vertex x incident to k colors $\{1, \ldots, k\}$ and let $S_i = N_i(x)$, $i = 1, \ldots, k$ as before. Now, consider a graph H on k vertices $v_1, \ldots, v_k$ such that $v_i v_j \in E(H)$ iff a pair S_i, S_j is not mixed. Then H is obtained from a complete graph by deleting some matching. For every edge $v_i v_j$ in H let $c(v_i v_j)$ be the color of edges between S_i and S_j in K_n.

Let s be the numbers of leaves in a largest monochromatic star. Then, we have that $|S_i| \leq s$ for all $i = 1, \ldots, k$. By Lemma 7 there is a monochromatic star S in H containing all the edges of H incident to the center of S, i.e., of maximum degree $k-1$ or $k-2$. Let v_1 be the center of such a star.

Assume first that S has $k-1$ edges. Then, in K_n, each vertex of S_1 sends edges of color 1 to every other S_i, $i = 2, \ldots, k$. Thus each vertex in S_1 has a monochromatic degree at least $d = n - 1 - |S_1| \geq n - 1 - s$. Since $d \leq s$, we have that $s \leq (n-1)/2$ in this case.

Next, assume that S has $k-2$ edges and $v_1 v_2 \notin E(H)$, i.e., the pair (S_1, S_2) is mixed and all edges between S_1 and S_i, $i = 3, \ldots, k$ are of color 1. Lemma 6 implies that $c[S_2, S_i] = 2$ for all $i = 2, \ldots, k$. If (S_1, S_2) is of Type 1 then there is a vertex in S_2 adjacent via an edge of color 2 to all vertices $S_1, S_3, \ldots, S_k$ thus having a monochromatic degree at least $d = n - 1 - |S_1| \geq n - 1 - s$. Again, since $d \leq s$, we have $s \leq (n-1)/2$. If (S_2, S_1) is of Type 1 then, similarly, there is a vertex in S_1 with monochromatic degree at least $d = n - 1 - |S_2| \geq n - 1 - s$ and $s \leq (n-1)/2$.

Finally, if (S_1, S_2) is a mixed pair of Type 2 then $S_1 = A \cup B$ and $S_2 = A' \cup B'$ such that $c[A, A'] = c[A, B] = c[B, B'] = 2$ and $c[A, B'] = c[A', B'] = c[A', B] = 1$. Let $X = V(K_n) \setminus (S_1 \cup S_2 \cup \{x\})$. For every $a \in A, a' \in A', b \in B, b' \in B'$ consider bi-colored bi-stars

$$(a, a', B), (a, b, B'), (b, b', A), (a, b, A'), (a, a', X).$$

Since the sets A, A', B, B', X are pairwise disjoint and their union is $V(K_n) - x$, by pigeonhole principle at least one of these bi-colored bi-stars will have at least $(n-1)/5$ leaves.

4.1. Proof of theorem 5

For the upper bound we use induction on n. Let N be the largest integer such that there a coloring c of $E(K_N)$ with no K_n^{lex}, no K_3^{mono}, and no K_3^{rain}. Then, by Theorem 6, there are two possible cases.

Case 1. There is a monochromatic star with at least $(N-1)/2$ leaves. Let S' be the set of leaves of that star. Then S' does not span any lexical K_{n-1} and it does not have K_3^{mono} and K_3^{rain}, i.e., $|S'| \leq f(n-1) - 1$. Thus

$$f(n) = N + 1 \leq 2|S'| + 1 + 1 \leq 2f(n-1).$$

Case 2. There is a bi-colored bi-star with at least $(N-1)/5$ leaves. The set of leaves S' of that star spans neither lexical K_{n-2}, nor K_3^{mono}, no K_3^{rain}, thus $|S'| \leq f(n-2) - 1$. Thus

$$f(n) = N + 1 \leq 5|S'| + 1 + 1 \leq 5f(n-2) - 3.$$

Using the easy fact that $f(3) \leq 3$ and $f(4) \leq 7$, we have that $f(n) \leq 5^{n/2}$. For the lower bound consider a pattern $Ram(K_3)^{\lfloor n/2 \rfloor - 1}$.

□

5. $f(n, H)$ for bipartite H

In this section we shall give several bounds on $f(n, H)$ for a fixed arbitrary bipartite graph H. Let $R(H, H)$ be a classical graph Ramsey number and $R_b(H, H)$ be a bipartite Ramsey number of H, i.e., the smallest integer $r = R_b(H, H)$ such that in every 2-coloring of the edges of a complete bipartite graph $K_{r,r}$, there is a monochromatic copy of H.

First we give a lower bound by providing a construction.

Theorem 7. *Let H be a bipartite graph with the smallest partite set of size m then $f(n, H) \geq (m-1)(n-4) + R(H, H)$.*

Proof. Let $N = (m-1)(n-4) + R(H, H) - 1$, consider a set V of N vertices and split it into sets $A_1, \ldots, A_{n-4}, B$ such that $|A_i| = m - 1$ for $i = 1, \ldots, n-3$ and $|B| = R(H, H) - 1$. First we color the edges between these sets as follows. For each $i = 1, \cdots, n-4$ and each $v_i \in A_i$ let

$$c(v_i v) = i \quad \text{for} \quad v \in \bigcup_{j>i} A_j \cup B.$$

Let $c(uu') = 0$ if $u, u' \in A_i$ for $i = 1, \ldots, n-4$. Finally, color the edges induced by B with two colors 0 and -1 such that neither color class has a subgraph isomorphic to H.

Clearly, this coloring does not have a rainbow triangle. Note that the total number of colors used is $n-2$ therefore there is no lexically colored K_n. Finally, in order to see that there is no monochromatic H, observe that each color class $1, \ldots, n-4$ is a complete bipartite graph with the smallest partite set of size $m-1$. By the definition of $R(H, H)$, color classes 0 and -1 contain no copy of H as a subgraph.

Theorem 8. *Let H be a bipartite graph with largest partite set of size $m_1 < n$ then $f(n, H) \leq n^2 R_b(H, H)$.*

Proof. Let us consider a coloring of $E(K_N)$ with no forbidden pattern and $N > n^2 R_b(H,H)$. Let m and m_1 be the sizes of partite sets of H, $m \leq m_1 < n$. Note first that each vertex has at most $n-1$ colors incident to it, otherwise by Lemma 3 we shall get a lexical K_n. Let $N_1 = N/n^2$. For every vertex v we say that a color i is *large* at v if the number of edges incident to v in color i is at least N_1.

Claim 1. There is at most one *large* color at each vertex.
Assume that there is a vertex v with two large colors, say 1 and 2 at it. Let A and B be the sets of neighbors of v in colors 1 and 2 respectively. The edges between A and B have colors 1 or 2 only, otherwise there will be a rainbow triangle including v, some vertex in A and some vertex in B. Since $|A|, |B| \geq N_1 \geq R_b(H,H)$ there is a monochromatic copy of H. This contradiction proves Claim 1.

Claim 2. For every vertex v there is a *large* color at v.
Let vertex v have less than N_1 edges of each color incident to it. Since the total number of colors incident to a vertex is at most $n-1$ by Lemma 3, the total number of vertices is at most $(n-1)N_1 < N$. This contradiction proves Claim 2.

Thus, for every vertex v there is a unique color c_v which is large at v. Moreover,

$$|\{uv : c(uv) = c_v\}| \geq N - N_1(n-2).$$

Let $S = \{c_v : v \in V(K_N)\}$, i.e., S is the set of *large* colors.

Claim 3. $|S| \leq n-1$.
Assume that there are distinct colors $s_1, s_2, \ldots, s_n$ which are large at vertices $v_1, v_2, \ldots, v_n$ respectively. Let U_i be the set of vertices adjacent to v_i via edges of color s_i, $i = 1, \ldots, n$. Since $|U_i| \geq N - N_1(n-2)$,

$$|\bigcap_{i=1}^{n} U_i| \geq N - N_1(n-2)n > 0.$$

In particular, there is a vertex $v \in \bigcap_{i=1}^{n} U_i$, i.e., there are at least n colors incident to v, a contradiction to Lemma 3. This proves Claim 3.

Thus, the total number of large colors is at most $n-1$ and there is a color, without loss of generality, 1, such that at least $N/(n-1)$ vertices have 1 as a large color. Consider any m of these vertices, say $v_1, \ldots, v_m$. Let U_i be the set of vertices in $V - \{v_1, \ldots, v_m\}$ adjacent to v_i via edges of color 1. We have $|U_i| \geq N - N_1(n-2) - m$, thus

$$|\bigcap_{i=1}^{m} U_i| \geq N - ((N_1)(n-2) - m)m \geq m_1,$$

therefore there is a monochromatic K_{m,m_1} colored with 1 and having partite sets $\{v_1, \ldots, v_m\}$ and $U \subseteq \bigcap_{i=1,\ldots,m} U_i$. This contradiction implies that $N \leq n^2 R_b(H,H)$.

6. Pattern Ramsey numbers of Cycles

When we talk about lexical coloring of cycles, we can adopt two definitions.

Definition 2. *An edge-coloring c of C_n is lexical-1 if there is an ordering of vertices of a cycle $v_1, \ldots, v_n$ such that $c(v_iv_j) = \min\{i, j\}$ up to renaming the colors. An edge-coloring c of C_n is lexical-2 if $c(e) = c(e')$ for two adjacent edges and all other edges have distinct colors different from $c(e)$.*

The lexical-1 coloring is more general, allowing many orderings of the vertices. The lexical-2 coloring indicates only a very specific embedding of a cycle into lexically-colored K_n. So, if an edge-coloring c of C_n is lexical-2 then it is lexical-1, but not necessarily the other way around. Let $RC_i(k)$ be the smallest number N such that in any edge-coloring of K_N there is either a monochromatic, a rainbow, or lexical-i coloring of C_k. Note that $RC_1(n) \leq RC_2(n)$.

For example, when $k = 4$, $RC_1(4) = \Re(C[1^4], C[1^2, 2, 3], C[1, 1, 2, 2], C[1, 2, 3, 4])$ and $RC_2(4) = \Re(C[1^4], C[1^2, 2, 3], C[1, 2, 3, 4])$.

Here, we start investigating the pattern Ramsey numbers for cycles by providing the following exact results.

Theorem 9. $RC_1(4) = RC_2(4) = 6$.

Proof. Assume that there is a coloring c of $E(K_6)$ with no rainbow, lexically colored and no monochromatic C_4. It is easy to see that each vertex has at most three edges of the same color incident to it. Also, the total number of colors should be at most 3. This can be verified by taking a rainbow subgraph on 4 edges spanning the smallest number of vertices.

If the total number of colors is at most 2 we get a contradiction to the classical Ramsey numbers for cycles [3]. So, next we can assume that the total number of colors is exactly three.

Case 1. There is a monochromatic triangle $T = t_1t_2t_3$, say with all edges of color 1. Then each other vertex sends at least two edges of color 2 or two edges of color 3 to T. Since there are three vertices outside of T, there will be two vertices u, v sending at least two edges of color 2 to T. It is easy to see that the only way (up to permutation of indeces) to achieve this is to have $c(ut_1) = c(vt_2) = 1$ and all other edges from u, v to T of color 2. Now, it is clear that the last vertex w does not send edges of color 3 to T and we must have $c(wt_3) = 1$ and $c(wt_1) = c(wt_2) = 2$. Now, there must be an edge of color 3, say uv, which gives a lexical 4-cycle uvt_2t_3, a contradiction.

Case 2. There is a rainbow triangle T and there are no monochromatic triangles. Simple case analysis proves this case to be impossible.

Case 3. There are no rainbow or monochromatic triangles, i.e., all triangles are lexical. If there are four vertices inducing three colors, we are done. Otherwise consider four vertices inducing only two colors. Since two-colored K_4 must have each color class being P_4, it is easy to see that this case results in a contradiction as well by analyzing the possible colors of the edges incident other two vertices.

To provide the lower bound, consider C_5 in one color and its complement in another.

Next, we consider long monochromatic even cycles versus rainbow and lexical C_4s.

Theorem 10.

$$\Re(C[1, 2, 3, 4], C[1, 1, 2, 2], C[1, 1, 2, 3], C[1^{2k}]) = \begin{cases} 2k & \text{if} \quad k \geq 3 \\ 6 & \text{if} \quad k = 2. \end{cases} \tag{2}$$

Proof. Let $f(k) = \Re(C[1,2,3,4], C[1,1,2,2], C[1,1,2,3], C[1^{2k}])$. We shall show if c is an edge-coloring of a complete graph on $2k \geq 6$ vertices having no patterns $C[1,2,3,4]$, $C[1,1,2,2]$,or $C[1,1,2,3]$ then there is a monochromatic C_{2k}.

We use induction on k. When $k=2$ the result follows from Theorem 9. Now, we can assume that there is a monochromatic C_{2k-2}, $(x_1, \dots, x_{2k-2}, x_1)$, say of color 1. Let $S = \{x_1, \dots, x_{2k-2}\}$ and let u and v be the two vertices not in S. Note first that both u and v can not have all the edges to S in color 1, otherwise we find a monochromatic C_{2k} right away. So, assume without loss of generality that $c(ux_1) = 2$. Then, it is easy to see that $c(ux_3) = 1$ (from (u, x_1, x_2, x_3, u)). Assume that $c(vx_2) = 1$ then either (u, v, x_2, x_1, u) is $C[1,1,2,3]$ or $C[1,1,2,2]$ or $(u, x_3, x_4, \dots, x_{2k-2}, x_1, x_2, v, u)$ is monochromatic C_{2k}. Thus $c(vx_2) \neq 1$ and $c(uv) = 1$ (otherwise (x_2, x_3, u, v, x_2) is one of the lexical patterns). But then $c(vx_4) = 1$ and $(u, v, x_4, x_5, \dots, x_{2k-2}, x_1, x_2, x_3, u)$ is monochromatic C_{2k}.

On the other hand, by considering a monochromatically colored K_{2k-1} we see that $f(k) \geq 2k$, for $k \geq 3$. For $k = 2$, consider C_5 in one color and its complement in another color to get K_5 with no forbidden pattern.

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