Anti-Ramsey numbers for small complete bipartite graphs

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Abstract

Given two graphs G and $H \subseteq G$, we consider edge-colorings of G in which every copy of H has at least two edges of the same color. Let $f(G, H)$ be the maximum number of colors used in such a coloring of $E(G)$. Erdős, Simonovits and Sós determined the asymptotic behavior of f when $G = K_n$ and H contains no edge e with $\chi(H - e) \leq 2$. We study the function $f(G, H)$ when $G = K_n$ or $K_{m,n}$, and H is $K_{2,t}$.

1 Introduction

An edge-colored graph is *rainbow* if no two edges have the same color. Given a host graph G and subgraph H of G, anti-Ramsey theory studies edge-colorings of G that avoid rainbow copies of H. Formally, we define the anti-Ramsey function $f(G, H)$ to be the maximum number of colors used in a coloring of $E(G)$ that contains no rainbow copy of H. For the purpose of this note, we call a coloring which does not contain a rainbow copy of H an *H-free coloring.*

Given a graph G and a family $\mathcal{F}$ of graphs, the *Turán function* $ex(G, \mathcal{F})$ is defined as the maximum number of edges of a subgraph of G containing no member of $\mathcal{F}$ as a subgraph. Erdős, Simonovits and Sós [5] showed that $f(K_n, H) - ex(K_n, \mathcal{H}) = o(n^2)$ as $n \to \infty$, where $\mathcal{H} = \{H - e : e \in E(H)\}$. Hence by an earlier result of Erdős and Simonovits [6] on the asymptotics of the Turán function, we have $f(K_n, H)/\binom{n}{2} \to 1 - (1/d)$ as $n \to \infty$, where $d + 1 = \min\{\chi(H - e) : e \in E(H)\}$. This determines $f(K_n, H)$ asymptotically when $\min\{\chi(H - e) : e \in E(H)\} \geq 3$.

When $\min\{\chi(H - e) \colon e \in E(H)\} \leq 2$, the situation is more complex. Already the cases when H is a tree or a cycle are nontrivial. Simonovits and Sós [18] proved for large n that $f(K_n, P_{2t+3+\epsilon}) = tn - \binom{t+1}{2} + 1 + \epsilon$, where $\epsilon = 0, 1$ and P_k is a path on k vertices. Jiang and West [14] considered $f(K_n, T)$ when T is a general tree of a given size. For cycles, Erdős, Simonovits, and Sós [5] conjectured that for every fixed $k \geq 3$ $f(K_n, C_k) = n(\frac{k-2}{2} + \frac{1}{k-1}) + O(1)$ and proved it for $k = 3$. Alon [2] proved this conjecture for $k = 4$ and gave some upper bounds for $k \geq 5$.

In this note, we initiate the study of $f(K_n, H)$ and $f(K_{m,n}, H)$ for complete bipartite graphs H. We focus on the case when one of the bipartite sets of H has size 2. For all positive integers t, we determine $f(K_n, K_{2,t})$ and $f(K_{n,n}, K_{2,t})$ asymptotically by proving that $f(G, K_{2,t}) - ex(G, K_{2,t-1}) = O(n)$ holds when $G = K_n$ or $K_{n,n}$. We start with a useful notion and a proposition providing general bounds on $f(G, H)$ in terms of Turán numbers.

Definition 1.1 Let G be a graph and $c : E(G) \to \mathbb{Z}$ be a coloring of $E(G)$. A *representing graph* of c is a spanning subgraph L of G containing exactly one edge of each color of c (L may contain isolated vertices).

The following proposition was proved in [5]. We include the proof for self-containment.

Proposition 1.2 *Given graphs G, H, we have $ex(G, \mathcal{H}) + 1 \leq f(G, H) \leq ex(G, H)$, where $\mathcal{H} = \{H - e : e \in E(H)\}$.*

Proof. The upper bound follows from the fact that a representing graph of any H-free coloring of $E(G)$ is a subgraph of G containing no H as subgraph. For the lower bound, let G' be a subgraph of G with $ex(G, \mathcal{H})$ edges which does not contain any member of $\mathcal{H}$ as a subgraph. Color the edges in G' with distinct colors, and then assign a new color to all the remaining edges in G. It is easy to see that this coloring contains no rainbow copy of H, and uses $ex(G, \mathcal{H}) + 1$ colors. Thus, $f(G, H) \geq ex(G, \mathcal{H}) + 1$. ■

The Turán function for complete bipartite graphs has been extensively studied. In particular, $ex(K_n, K_{2,t})$ has been determined asymptotically for all $t \geq 2$.

Theorem 1.3 ([11]) $ex(K_n, K_{2,t}) = (\sqrt{t-1}/2)n^{3/2} + O(n^{4/3})$.

The bipartite Turán function $ex(K_{m,n}, K_{s,t})$ is closely related to the Zarankiewicz function $z(m,n,s,t)$, which is defined as the the maximum number of 1's in an m by n 0-1 matrix containing no s by t submatrix of 1's. Naturally, we always have $ex(K_{m,n}, K_{s,t}) \leq z(m,n,s,t)$. In [10], Füredi obtained the following upper bound on $z(m,n,s,t)$ which is asymptotically optimal for $s = 2$ and for $s = t = 3$ ([10]).

Theorem 1.4 ([10]) *$z(m,n,s,t) \leq (t-s+1)^{\frac{1}{s}} mn^{1-\frac{1}{s}} + sm + sn^{2-\frac{2}{s}}$ holds for all $m \geq s, n \geq t$, and $1 \leq s \leq t$.*

This upper bound on $z(m,n,s,t)$ together with Theorem 1.3 and the observation that $2ex(K_n, K_{s,t}) \leq ex(K_{n,n}, K_{s,t}) \leq z(n,n,s,t)$ yields

Theorem 1.5 ([10]) $ex(K_{n,n}, K_{2,t}) = \sqrt{t-1}\; n^{3/2} + O(n^{4/3})$.

Asymptotically optimal bounds on $ex(K_{n,n}, K_{3,3})$ and an upper bound on $ex(K_{n,n}, K_{s,s} \setminus e)$ can be found in [3] and [9], respectively.

2 Anti-Ramsey numbers for small complete bipartite graphs

In this section, we consider $f(K_n, K_{s,t})$. Jiang [13] proved for $n > t$ that $f(K_n, K_{1,t}) = \lfloor n(t-2)/2 \rfloor + \lfloor n/(n-t+2) \rfloor$, except when n, t, and $\lfloor 2n/(n-t+2) \rfloor$ are all odd, in which case the value of f may be larger than above by 1. Alon [2] proved that $f(K_n, K_{2,2}) = n + \lceil \frac{n}{3} \rceil - 1$. In this section, we asymptotically determine $f(K_n, K_{2,t})$ for $t \geq 3$ by proving that $f(K_n, K_{2,t}) - ex(K_n, K_{2,t-1}) = O(n)$. Our strategy is to consider a representing graph H of a $K_{2,t}$-free coloring of $E(K_n)$ and to show that one can delete the edges of $O(n)$ copies of $K_{2,t-1}$ to make H $K_{2,t-1}$-free.

We introduce some notions for convenience. Let $p \geq 2$. Given a copy B of a $K_{2,p}$, we use $X(B)$ and $Y(B)$ to denote the bipartite sets of B of size 2 and p, respectively. A graph R is a $K_{2,p}$*-string* of length k if the edges of R can be partitioned into k copies $B_1, \cdots, B_k$ of $K_{2,p}$, such that $X(B_i) = \{u_i, u_{i+1}\}$, for $i \in [k]$, where $u_1, u_2, \cdots, u_{k+1}$ are distinct vertices. The sets $X(R) = \bigcup_{i=1}^k X(B_i) = \{u_1, u_2, \cdots, u_{k+1}\}$ and $Y(R) = \bigcup_{i=1}^k Y(B_i)$ are the *interior* and *exterior* of R, respectively. Note that in general $X(R)$ and $Y(R)$ are not necessarily disjoint. Vertices u_1 and u_{k+1} are called the two *ends* of R. If in the above definition $u_1, \cdots, u_k$ are distinct and $u_{k+1} = u_1$, where $k \geq 2$, then R is a $K_{2,p}$*-ring* of length k.

Lemma 2.1 *Let G be a graph on n vertices and G' be a subgraph of G with more than $ex(G, K_{2,p}) + 2p(n-1)$ edges. Then G' contains a $K_{2,p}$-ring, where $p \geq 2$.*

Proof. Let $\mathcal{K}$ be a maximal collection of pairwise edge-disjoint copies of $K_{2,p}$ in G'. Suppose $\mathcal{K}$ contains q copies of $K_{2,p}$. By the maximality of $\mathcal{K}$, $G' - E(\mathcal{K})$ contains no copies of $K_{2,p}$. Hence $e(G' - E(\mathcal{K})) \leq ex(G, K_{2,p})$. Thus, $e(G') \leq ex(G, K_{2,p}) + q(2p)$. Since $e(G') > ex(G, K_{2,p}) + 2p(n-1)$, it follows that $q > n-1$. Now, construct a graph F with $V(F) = V(G')$ as follows. For each member B (which is a copy of $K_{2,p}$) of $\mathcal{K}$, where $X(B) = \{u, v\}$, we include uv as an edge in F. By our discussion above, F is a loopless multigraph on at most n vertices with $q > n-1$ edges. Hence F contains a cycle C. The members of $\mathcal{K}$ which correspond to the edges on C form a $K_{2,p}$-ring in G'. ■

A graph T obtained from a $K_{2,p}$-string R of length k by adding a new vertex x not in R and making it adjacent to the two ends of R is a $K_{2,p}$-*string-tie* of length k. The *interior* $X(T)$ of T is defined as the interior $X(R)$ of R, and *exterior* $Y(T)$ of T is defined as $Y(R) \cup x$.

Lemma 2.2 *Let R be $K_{2,p}$-ring, where $p \geq 2$. Then R contains a $K_{2,p}$-string-tie.*

Proof. Suppose R has length k, $X(R) = \{u_1, \cdots, u_k\}$, and the copies of $K_{2,p}$ forming R are $B_1, \cdots, B_k$ with $B_i = \{u_i, u_{i+1}\}$ (indices taken modulo k). Since B_i and B_{i+1} are edge-disjoint and $u_{i+1} \in X(B_i) \cap X(B_{i+1})$, we have $Y(B_i) \cap Y(B_{i+1}) = \emptyset$ for all $i \in [k]$ (indices taken modulo k). Suppose first that the $Y(B_i)$'s are pairwise disjoint. Then we have $|Y(R)| = kp > k = |X(R)|$, thus there exists $w \in Y(R) \setminus X(R)$. Without loss of generality, suppose $w \in Y(B_1)$. Then $(\bigcup_{i=2}^{k} B_i) \cup \{wu_1, wu_k\}$ is a $K_{2,p}$-string-tie.

Hence we may assume that there exist $l_1 < l_2$ such that $Y(B_{l_1}) \cap Y(B_{l_2}) \neq \emptyset$. Without loss of generality, suppose $l_1 = 1$ and l_2 is chosen to be as small as possible. Let $v \in Y(B_1) \cap Y(B_{l_2})$. Let $l_3 = \max\{i \in [k] : v \in Y(B_i)\}$, we have $l_3 \geq l_2$. By our observation above, we have $l_2 - 1 \geq 2$ and $l_3 \leq k-1$. Since $u_1 \cdots, u_k$ are distinct, one of $\{u_2, \cdots, u_{l_2}\}$ and $\{u_{l_3+1}, \cdots, u_k, u_1\}$ avoids v. Without loss of generality suppose the former does. By our choice of l_2, we have $v \notin Y(\bigcup_{i=2}^{l_2-1} B_i)$, and hence $v \notin \bigcup_{i=2}^{l_2-1} B_i$. Now, $\bigcup_{i=2}^{l_2-1} B_i \cup \{vu_2, vu_{l_2}\}$ is a $K_{2,p}$-string-tie. ■ We prove the following lemma using the argument similar to one used in [5].

Lemma 2.3 *Let c be a coloring of $E(K_n)$ that contains a rainbow $K_{2,t-1}$-string-tie. Then c contains a rainbow copy of $K_{2,t}$.*

Proof. Let T be a rainbow $K_{2,t-1}$-string-tie in c of minimum length. Suppose T is obtained from a string R of length k by adding a vertex x not in R and making it adjacent to the two ends of R. If $k = 1$ then T is a rainbow $K_{2,t}$. So we may assume $k \geq 2$. Suppose $X(R) = \{u_1, \cdots, u_{k+1}\}$, and the copies of $K_{2,t-1}$ forming R are $B_1, \cdots, B_k$ with $X(B_i) = \{u_i, u_{i+1}\}$. Let $T_1 = B_1 \cup xu_1$ and $T_2 = B_2 \cup \cdots \cup B_k \cup xu_{k+1}$. Since T is rainbow, the color $c(xu_2)$ cannot be used in both T_1 and T_2. Now xu_2 completes a rainbow $K_{2,p}$-string-tie with either T_1 or T_2 which is shorter than T, a contradiction. ■

Theorem 2.4 $f(K_n, K_{2,t}) - ex(K_n, K_{2,t-1}) = O(n)$.

Proof. By Proposition 1.2, we have $f(K_n, K_{2,t}) \geq ex(K_n, K_{2,t} - e) \geq ex(K_n, K_{2,t-1})$. Now, consider a $K_{2,t}$-free coloring c of $E(K_n)$ using $f(K_n, K_{2,t})$ colors. Let H be a representing graph of c, we have $e(H) = f(K_n, K_{2,t})$. By Lemma 2.3, H contains no $K_{2,t-1}$-string-tie. By Lemma 2.1 and 2.2, we have $e(H) \leq ex(K_n, K_{2,t-1}) + (2t-2)(n-1)$. ■

Theorem 1.3 and Theorem 2.4 yield

Corollary 2.5 $f(K_n, K_{2,t}) = (\sqrt{t-2}/2)n^{3/2} + O(n^{4/3})$.

■

3 Bipartite anti-Ramsey numbers for small complete bipartite graphs

In this section, we study $f(K_{m,n}, K_{s,t})$. In [13], Jiang proved for $n \geq t$ that $f(K_{n,n}, K_{1,t}) = n(t-2) + \lfloor \frac{n}{n-t+2} \rfloor$. By considering a shortest rainbow cycle in a coloring of $E(K_{m,n})$ using at least $m+n$ colors, one can easily prove that $f(K_{m,n}, C_4) = f(K_{m,n}, K_{2,2}) \leq m+n-1$. On the other hand, a coloring that assigns distinct colors to a spanning subgraph of $K_{m,n}$ isomorphic to $K_{1,m-1} + K_{1,n-1}$ and a new color to the remaining edges in $K_{m,n}$ is a $K_{2,2}$-free coloring of $E(K_{m,n})$. Therefore, we have $f(K_{m,n}, K_{2,2}) = m+n-1$. (A proof of this fact was also given in [17].)

For $t \geq 3$, we show that $f(K_{m,n}, K_{2,t}) - ex(K_{m,n}, K_{2,t-1}) = O(n+m)$. Thus, we determine $f(K_{n,n}, K_{2,t})$ asymptotically for all $t \geq 3$. We use the same method as in the previous section. The next lemma is a bipartite graph version of Lemma 2.3. We omit the proof, since it is just a slight modification of the proof of Lemma 2.3.

Lemma 3.1 *Let c be a coloring of $E(K_{m,n})$ that contains a rainbow $K_{2,t-1}$-string-tie. Then c contains a rainbow copy of $K_{2,t}$.*

■

Theorem 3.2 $f(K_{m,n}, K_{2,t}) - ex(K_{m,n}, K_{2,t-1}) = O(m+n)$.

Proof. By Proposition 1.2, we have $f(K_{m,n}, K_{2,t}) > ex(K_{m,n}, K_{2,t-1})$. Let c be a coloring of $E(K_{m,n})$ using $f(K_{m,n}, K_{2,t})$ colors that does not contain a rainbow copy B of $K_{2,t}$. Let H be a representing graph of c. It suffices to show that $e(H) \leq ex(K_{m,n}, K_{2,t-1}) + O(m+n)$. By Lemma 3.1, H does not contain a $K_{2,t-1}$-string-tie. By Lemmas 2.1 and 2.2, we have $e(H) \leq ex(K_{m,n}, K_{2,t-1}) + O(m+n)$. ∎

Theorem 1.5, and Theorem 3.2 yield

Theorem 3.3 $f(K_{n,n}, K_{2,t}) = \sqrt{t-2}\; n^{3/2} + O(n^{4/3})$.

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Finally, we consider colorings of $E(K_{m,n})$ avoiding rainbow copies of $K_{2,t}$ whose two bipartite sets are contained in specified bipartite sets of $K_{m,n}$. Let $G = K_{m,n}$ with M, N being the two bipartite sets of sizes m and n, respectively. Let $g(m,n,2,t)$ denote the maximum number of colors in a coloring of $E(G)$ that does not contain a rainbow copy B of $K_{2,t}$ with $X(B) \subseteq M$ and $Y(B) \subseteq N$. Clearly, we have $g(m,n,2,t) \geq f(K_{m,n}, K_{2,t})$ always. The proofs of Lemmas 2.1 and 2.3 can be easily modified to give

Lemma 3.4 *Let G' be a subgraph of G with more than $z(m,n,2,p) + 2p(m-1)$ edges. Then G' contains a $K_{2,p}$-ring R with $X(R) \subseteq M$ and $Y(R) \subseteq N$.*

∎

Lemma 3.5 *Let c be a coloring of $E(G)$ that contains a rainbow $K_{2,t-1}$-string- tie T with $X(T) \subseteq M$ and $Y(T) \subseteq N$. Then c contains a rainbow copy B of $K_{2,t}$ with $X(B) \subseteq M$ and $Y(B) \subseteq N$.*

∎

Now, by giving distinct colors to the edges of a subgraph of G with $z(m,n,2,t-1)$ edges which does not contain a copy B' of $K_{2,t-1}$ with $X(B') \subseteq M$ and $Y(B') \subseteq N$ and a new color to the remaining edges of G, we obtain a coloring of G using $z(m,n,2,t-1)$ colors with no rainbow copy B of $K_{2,t}$ with $X(B) \subseteq M$ and $Y(B) \subseteq N$. On the other hand, Lemmas 3.4, 2.2 and 3.5 imply that $g(m,n,2,t) \leq z(m,n,2,t-1) + 2p(m-1)$. Hence we have

Theorem 3.6 $g(m,n,2,t) - z(m,n,2,t-1) = O(m)$.

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