Bipartite anti-Ramsey numbers of cycles

Maria Axenovich
DEPARTMENT OF MATHEMATICS
IOWA STATE UNIVERSITY
AMES, IA 50011
axenovic@math.iastate.edu

Tao Jiang*
DEPARTMENT OF MATHEMATICS AND STATISTICS
MIAMI UNIVERSITY
OXFORD, OH 45056
jiangt@muohio.edu

André Kündgen
DEPARTMENT OF MATHEMATICS
CALIFORNIA STATE UNIVERSITY SAN MARCOS
SAN MARCOS, CA 92096
akundgen@csusm.edu

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ABSTRACT

We determine the maximum number of colors in a coloring of the edges of $K_{m,n}$ such that every cycle of length $2k$ contains at least two edges of the same color. One of our main tools is a result on generalized path covers in balanced bipartite graphs. For positive integers $q \leq a$ let $g(a,q)$ be the maximum number of edges in a spanning subgraph G of $K_{a,a}$ such that the minimum number of vertex-disjoint even paths and pairs of vertices from distinct partite sets needed to cover $V(G)$ is q. We prove that $g(a,q) = a^2 - aq + \max\{a, 2q-2\}$.

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1. INTRODUCTION

A coloring of the edges of a graph is a *rainbow coloring* if the colors used are distinct. A *rainbow copy* of a graph H in an edge-colored graph G is a subgraph of G isomorphic to H such that the coloring restricted to this subgraph is a rainbow coloring. Given two graphs G and H, let $f(G,H)$ denote the maximum number of colors in a coloring of the edges of G that has no rainbow copy of H. When $G = K_n$, $f(G,H)$ is called the *anti-Ramsey number* of H. Anti-Ramsey numbers were introduced by Erdős, Simonovits and Sós in the 1970s. They showed that these are closely related to Turán numbers. Anti-Ramsey numbers are studied in [1, 3, 2, 5, 7, 8, 9, 11, 13] and elsewhere. The only graphs whose anti-Ramsey numbers have not been asymptotically determined are bipartite graphs and graphs that become bipartite upon deletion of an edge. On the other hand, there are very few graphs whose anti-Ramsey numbers have been determined exactly. To the best of our knowledge, $f(K_n,H)$ is known exactly for large n only when H is a complete graph, a path, a star,a broom whose maximum degree exceeds its diameter (a broom is obtained by identifying an end of a path with a vertex of a star), or a cycle of length at most seven (see [1, 7, 10, 9, 11, 13]).

An interesting open problem concerning anti-Ramsey numbers is the determination of the anti-Ramsey number of a cycle. Erdős, Simonovits, and Sós [5] conjectured that $f(K_n, C_k) = (\frac{k-2}{2} + \frac{1}{k-1})n + O(1)$ and provided a construction achieving this bound. They also observed that the conjecture holds for $k = 3$. Alon [1] proved this conjecture for $k = 4$ and provided a general upper bound of $(k-2)n - \binom{k-1}{2}$. Jiang, Schiermeyer, and West [10] proved the conjecture for $k \leq 7$, while Jiang and West [8] improved the general upper bound to $f(K_n, C_k) \leq (\frac{k+1}{2} - \frac{2}{k-1})n - (k-2)$.

The main focus of this paper is to consider the analogous problem for cycles when the host graph G is a complete bipartite graph. For all positive integers m, n, k, where $k \geq 2$, we determine $f(K_{m,n}, C_{2k})$ exactly.

Definition. For positive integers m, n, k, let

$$\theta(m,n,k) = \begin{cases} (k-1)(m+n) - 2(k-1)^2 + 1 & \text{for } m \geq 2k-1, \\ (k-1)n + m - (k-1) & \text{for } k-1 \leq m \leq 2k-1, \\ mn & \text{for } m \leq k-1. \end{cases}$$

Note that θ is well-defined for $m = k-1$ or $m = 2k-1$. We can now formulate our main result.

Theorem 1. For all positive integers m, n, k with $m \leq n$ and $k \geq 2$,

$$f(K_{m,n}, C_{2k}) = \theta(m,n,k).$$

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* Part of the research of this author was done at Michigan Technological University, Houghton, MI 49931

In order to prove Theorem 1, we introduce the notion of an even quasi-path cover of a balanced bipartite graph.

Definition. A bipartite graph is *balanced* if its partite sets are of equal size. A *doubleton* in a bipartite graph G with a fixed bipartition (X, Y) is a pair of non-adjacent vertices (x, y) with $x \in X$ and $y \in Y$. An *even path* is a path with an even number of vertices. An *even quasi-path* is either an even path or a doubleton. An *even quasi-path cover* (or eqp-cover) of a balanced bipartite graph G is a cover of $V(G)$ by vertex-disjoint even quasi-paths. Given positive integers a, q with $a \geq q$, let $g(a, q)$ be the maximum number of edges in a spanning subgraph G of $K_{a,a}$ whose smallest even quasi-path cover consists of exactly q even quasi-paths.

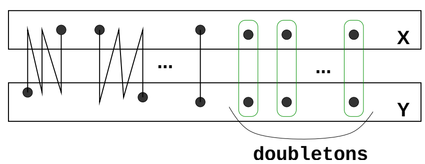

FIGURE 1. An even quasi-path cover

Theorem 2. If q and a are positive integers with $q \leq a$, then $g(a, q) = a^2 - aq + \max\{a, 2q - 2\}$.

The rest of the paper is organized as follows. In Section 2, we show that $f(K_{m,n}, C_{2k}) \geq \theta(m, n, k)$ for all positive integers m, n, k with $m \leq n$ and $k \geq 2$. We also give a simple proof that $f(K_{m,n}, C_{2k}) \leq \theta(m, n, k) + (k-1)^2 - 1$, thus determining $f(K_{m,n}, C_{2k})$ up to a term depending only on k and determining $f(K_{m,n}, C_4)$ precisely (the latter has appeared in [3] and [12]). In Section 3, the largest part of the paper, we finish the proof of Theorem 1 by establishing $f(K_{m,n}, C_{2k}) \leq \theta(m, n, k)$ using various tools, including a slight refinement of Theorem 2. Finally, in Section 4, we prove the refinement and therefore Theorem 2.

Before we proceed, we introduce some definitions and notation. For other undefined terms we refer the reader to West [14].

Definition. Given a graph G and a coloring c of $E(G)$, a *representing graph* is a spanning subgraph L of G having exactly one edge of each color under c (L may have isolated vertices). The number of edges of L, denoted by $e(L)$, equals the number of colors used by c. Given a graph H, a coloring of $E(G)$ is *H-good* if it contains no rainbow copy of H.

2. GENERAL BOUNDS ON $F(K_{M,N}, C_{2K})$

We start with a lower bound on $f(K_{m,n}, C_{2k})$ by exhibiting C_{2k}-good colorings of the edges of $K_{m,n}$ using many colors. Frequently we refer to a rainbow copy of a cycle simply as a *rainbow cycle.*

Constructions. Consider $K_{m,n}$ with $m, n \geq k$. Let X and Y denote the two partite sets of $K_{m,n}$ with sizes m and n, respectively. We define two C_{2k}-good colorings c_1 and c_2 of $E(K_{m,n})$ as follows. First, partition X into two subsets X_1 and X_2 of sizes $k-1$ and $m-(k-1)$, respectively, and partition Y into Y_1 and Y_2 of sizes $n-(k-1)$ and $k-1$, respectively. Let G_1 and G_2 be the subgraphs of $K_{m,n}$ induced by $X_1 \cup Y_1$ and $X_2 \cup Y_2$, respectively.

We obtain c_1 by giving distinct colors to the edges of $G_1 \cup G_2$ and assigning a new color to the remaining edges. We obtain c_2 by first assigning distinct colors to each of the edges induced by $X_1 \cup Y$, and then for each n-edge star centered at a vertex in X_2, we assign a new color to all the edges in the star.

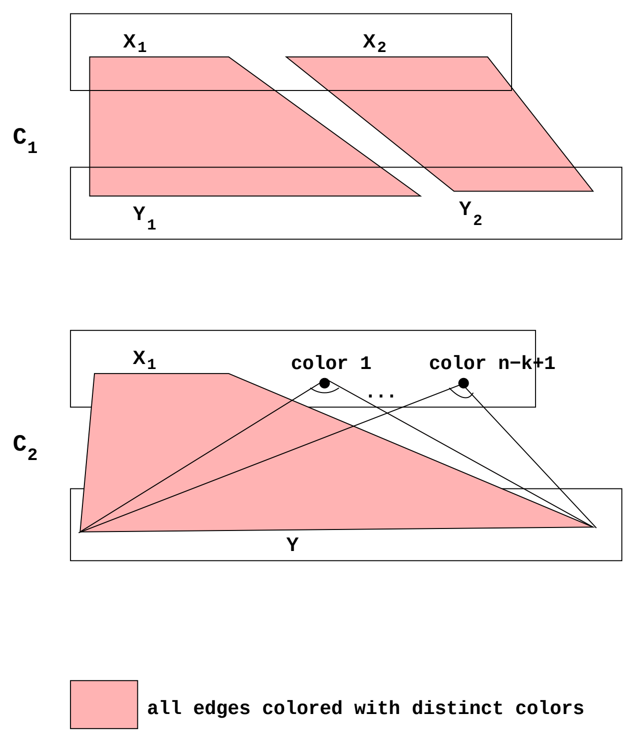

FIGURE 2. Extremal construction

Proposition 1. Given positive integers m, n, k with $m \leq n$ and $k \geq 2$, we have $f(K_{m,n}, C_{2k}) \geq \theta(m, n, k)$.

Proof. The colorings c_1 and c_2 defined above are C_{2k}-good colorings of $E(K_{m,n})$ using $(k-1)(m+n) - 2(k-1)^2 + 1$ and $(k-1)n + m - (k-1)$ colors, respectively. For $m \leq k-1$, we give distinct colors to $E(K_{m,n})$; for $k \leq m \leq 2k-2$, we use coloring c_2, and for $m \geq 2k-1$ we use coloring c_1. In each case, we get the stated number as a lower bound on $f(K_{m,n}, C_{2k})$. ∎

In the next section we show that the opposite inequality also holds, which establishes that $f(K_{m,n}, C_{2k}) = \theta(m, n, k)$. The equality holds trivially when $m \leq k-1$. Next, we develop some standard tools that we will use in the proof of Theorem 1. By themselves these tools yield an upper bound on $f(K_{m,n}, C_{2k})$ that differs from the lower bound $\theta(m, n, k)$ by at most a quadratic term in k.

The following lemma is a slight variant of a lemma by Alon [1]; their proofs are virtually identical.

Lemma 1. If an edge-coloring of $K_{m,n}$ has a rainbow copy C of a cycle of length congruent to 2 modulo $2k-2$, then it has a rainbow copy of C_{2k}.

Proof. If C has length $2k$, then there is nothing to prove. Otherwise, let x and y be two vertices on C at distance $2k-1$ along C. Note that x and y are adjacent in $K_{m,n}$ and that C consists of two paths from x to y of length congruent to 1 modulo $2k-2$. Since C is a rainbow cycle, one of these paths avoids the color on xy. Adding xy to that path completes a rainbow cycle shorter than C whose length is also congruent to 2 modulo $2k-2$. The result follows by induction. ∎

Lemma 2. Let d be an integer greater than 1, and let G be a bipartite graph. If u is an endpoint of a maximal path in G and u has degree at least d, then G contains a cycle of length 2 mod $(2d-2)$.

Proof. Let P be a maximal path in G, and let u be an endpoint of P. Since P cannot be extended to a longer path, all neighbors of u in G lie on P. Let $v_0, v_1, \ldots, v_{d-1}$ be d distinct neighbors of u, where v_0 is the vertex next to u along P. For $0 \leq i \leq d-1$, let l_i denote the distance along P between v_0 and v_i. Since G is bipartite, each l_i is even. Thus the pigeonhole principle yields $l_i \equiv l_j \bmod (2d-2)$ for some $i, j \in [d-1]$, since there are only $d-1$ even congruence classes. Now, the portion of P between v_i and v_j has length 0 mod $(2d-2)$, and together with edges uv_i, uv_j, it completes a cycle of length 2 mod $(2d-2)$. ∎

Lemma 3. Let G be a bipartite graph with partite sets X and Y. If $|X| \leq |Y|$, then there is a maximal path with at least one endpoint in Y.

Proof. Let M be a maximum matching in G, and let Y' denote the set of vertices in Y that are endpoints of edges in M. If $Y' = Y$, then M is a perfect matching and every maximal M-alternating path has the desired properties.

So suppose that $y \in Y \setminus Y'$. If y is an isolated vertex, then the desired path is trivial. Otherwise consider a maximal M-alternating path P starting at y. Since M is maximum this path cannot be M-augmenting and must thus end in a vertex $y' \in Y'$ all of whose neighbors are in P. If we now grow P to a maximal path P' by extending it at y, then P' and y' have the desired properties. ∎

Combining Lemmas 1, 2 and 3, we obtain the following recursive upper bound on $f(K_{m,n}, C_{2k})$.

Proposition 2. If $n \geq m \geq 1$ and $k \geq 2$, then $f(K_{m,n}, C_{2k}) \leq (k-1) + f(K_{m,n-1}, C_{2k})$.

Proof. Let c be a C_{2k}-good coloring of $E(K_{m,n})$ using the maximum number of colors. Furthermore let L be a representing graph of c with bipartition X, Y, where $|X| = m$ and $|Y| = n$. By Lemma 3 we can find a vertex $u \in Y$ that is an endpoint of a maximal path in L. If $d(u) \geq k$, then by Lemma 2 L contains a cycle of length 2 mod $(2k-2)$, which is a rainbow cycle in c. This contradicts Lemma 1. Hence $e(L) \leq (k-1) + e(L-u)$. Since $e(L-u)$ is at most the number of colors used by c on $G-u$, and c restricted to $G-u$ is C_{2k}-good, we have $e(L) \leq (k-1) + f(K_{m,n-1}, C_{2k})$, and the claim follows. ∎

Proposition 2 and the trivial fact that $f(K_{k-1,n}, C_{2k}) = (k-1)n$ yield the following simple upper bound.

Theorem 3. If $m, n \geq k-1$, then $f(K_{m,n}, C_{2k}) \leq (k-1)(m+n) - (k-1)^2 \leq \theta(m,n,k) + (k-1)^2 - 1$.

This simple bound, together with Proposition 1, gives the following result, which was also obtained in [3] and [12] and possibly elsewhere.

Corollary. For all positive integers m and n, $f(K_{m,n}, C_4) = \theta(m,n,2) = m+n-1$.

In general, however, the bound from Theorem 3 is not tight. In the next section we improve it to meet the lower bound $\theta(m,n,k)$.

3. DETERMINING $F(K_{M,N}, C_{2K})$

In this section we use a strengthening of Theorem 2 whose proof we postpone to the last section of the paper. We need some further definitions. Let a be a positive integer and G be a bipartite graph with partite sets X, Y, where $|X| = |Y| = a$. Recall that an even quasi-path cover (eqp-cover) $\mathcal{C}$ of G is a collection of vertex disjoint even paths and doubletons that cover $V(G)$, where a doubleton consists of one vertex of X and one vertex of Y. For the rest of the paper, we will be particularly interested in optimal covers:

Definition. An eqp-cover $\mathcal{C}$ of G is called an *optimal cover* if

(1) $\mathcal{C}$ has the minimum size among all eqp-covers of G, and
(2) among all eqp-covers of G with minimum size $\mathcal{C}$ has the minimum number of doubletons.

Suppose $\mathcal{C}$ consists of s even paths $P_1, \dots, P_s$ and t doubletons $(u_1, v_1), \dots, (u_t, v_t)$ (so the size of $\mathcal{C}$ is $s+t$). For each $i \in [s]$, let x_i and y_i denote the endpoints of P_i in X and Y, respectively. We classify doubletons into two types:

Definition. A doubleton (u_i, v_i) in an eqp-cover $\mathcal{C}$ is *attachable* if there are two even paths $P_j, P_k \in \mathcal{C}$ such that u_i is adjacent to y_j and v_i is adjacent to x_k (note that if $\mathcal{C}$ is an optimal cover then j, k must be different). Otherwise (u_i, v_i) is *unattachable*.

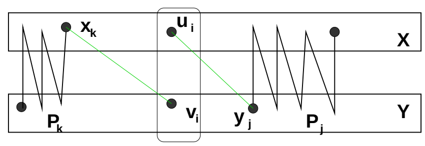

FIGURE 3. An attachable doubleton

Definition. For $q \le a$, let $H_{a,q}$ denote the subgraph of $K_{a,a}$ that consists of $K_{a,a-q+1}$ and $q-1$ isolated vertices.

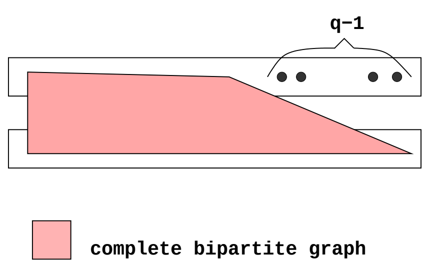

FIGURE 4. Densest graph $H_{a,q}$ with an eqp-cover of size q

The graph $H_{a,q}$ has $a^2 - aq + a$ edges, and an optimal cover consists of a path with $2(a-q+1)$ vertices and $q-1$ unattachable doubletons. We will prove the following strengthening of Theorem 2.

Theorem 4. Let G be a bipartite graph with partite sets of size a. Let q be the size of an optimal cover $\mathcal{C}$ of G. Suppose that $\mathcal{C}$ consists of s even paths, t_1 attachable doubletons, and t_2 unattachable doubletons. If $s \ge 2$, then $e(G) \le a^2 - aq + q + t_1$. If $s = 1$ (and thus $t_1 = 0$) and P is the unique path in $\mathcal{C}$, then $e(G) \le e(G[V(P)]) + (a - q + 1)(q-1) \le a^2 - aq + a$, with equality achieved only by $H_{a,q}$. Furthermore, if $s = 1$ and $e(G) = a^2 - aq + a - 1$, then G is formed by deleting an edge from $H_{a,q}$ or by joining the centers of two stars $K_{1,a-1}$ (i.e. G is a double-star).

This result easily implies Theorem 2. Given graphs G, H, let $G+H$ denote the vertex disjoint union of G and H.

Proof of Theorem 2. For the upper bound on the function g note that if $s \geq 2$, then $t_1 \leq q-2$, whereas if $s=0$, then $e(G)=0$. The bound is achieved by $H_{a,q}$ and by $K_{a-1,a-q+1}+K_{1,q-1}$. ∎

Next, we prove a variation of a lemma of Jiang and West [8]. For two disjoint subsets A and B of vertices in a graph G we use $e(A,B)$ to denote the number of edges of G with one endpoint in A and the other endpoint in B. We denote the number of edges induced by a subset of vertices A in a graph G by $e(A)$. Given $H \subseteq G$ and $v \in V(G)$ we let $N_H(v) = N(v) \cap V(H)$, and let $d_H(v) = |N_H(v)|$.

Lemma 4. Let G be a bipartite graph with partite sets X, Y. Suppose that G contains a cycle C of length $2k-2$ but contains no cycle of length 2 mod $(2k-2)$.

(1) If P is a path disjoint from C, then $e(V(P), V(C)) \leq k-1$.
(2) If x and y are vertices not on C, with $x \in X$ and $y \in Y$, then $e(\{x,y\}, V(C)) \leq k$.
(3) If T is a nontrivial tree with l leaves that is disjoint from C, then $e(V(T), V(C)) \leq l(k-1)/2$.

Proof. Fix a direction along C. Given $W \subseteq V(C)$, and a nonnegative integer s, let W^s denote the shift of W by s positions in this direction. Let v_i be the ith vertex of P, and let $N_i = N_C(v_i)$ for $1 \leq i \leq q$.

To prove the first inequality we first show that $N_1^{q-1}, N_2^{q-2}, \dots, N_q^0$ are pairwise disjoint. If N_i^{q-i} and N_j^{q-j} have a common vertex, then choose $r \in \{0, \dots, 2k-3\}$ such that $r \equiv j-i \bmod (2k-2)$. Now v_i and v_j have neighbors on C that are connected by a path of length r along C. If $r>0$, then replacing this portion of C with the edges from its endpoints to v_i and v_j and the v_i, v_j-path along P yields a cycle in G with length 2 mod $(2k-2)$, which is forbidden in G. If $r=0$, then v_i and v_j have a common neighbor that together with the v_i, v_j-path along P yields a contradiction as well. Furthermore we see that $N_1^{q-1}, \dots, N_q^0$ lie in the same partite set of G. Since they are pairwise disjoint, their sizes sum to $\frac{1}{2}|V(C)|$, which equals $k-1$. Thus $e(V(P), V(C)) \leq k-1$, as desired.

To prove the second inequality, we first prove that $|N_C^1(x) \cap N_C^0(y)| \leq 1$. If there exist $w, w' \in N_C^1(x) \cap N_C^0(y)$, then deleting the edges of C from w and w' to their predecessors and adding the four edges from x, y to these vertices completes a cycle of length $2k$ in G, which is forbidden. Since $N_C^1(x) \cup N_C^0(y) \subseteq X \cap V(C)$, we have $e(\{x,y\}, V(C)) = |N_C(x)| + |N_C(y)| = |N_C^1(x)| + |N_C^0(y)| \leq |N_C^1(x) \cup N_C^0(y)| + 1 \leq k$.

Finally, we prove the third inequality. If l is even, then $V(T)$ can be covered by $l/2$ paths in T, each joined by at most $k-1$ edges to C by (1). Hence $e(V(T), V(C)) \leq (l/2)(k-1)$. So suppose l is odd and the leaves of T are $u_1, \dots, u_l$. For each $i \in [l]$, let B_i denote the branch in T (excluding the branching vertex) that connects u_i to the rest of T. If $e(V(B_i), V(C)) > (k-1)/2$ for all $i \in [l]$, then one can find a path in T that is joined by more than $k-1$ edges to C, a contradiction. Hence we may assume without loss of generality that $e(V(B_1), V(C)) \leq (k-1)/2$. Now, $T - V(B_1)$ is a tree with $l-1$

leaves. Since $l-1$ is even, we have proved that $e(V(T-B_1), V(C)) \leq (l-1)(k-1)/2$. Hence $e(V(T), V(C)) \leq (l-1)(k-1)/2 + (k-1)/2 = l(k-1)/2$. ∎

Lemma 5. Let G be a spanning subgraph of $K_{a,a}$. If an optimal cover $\mathcal{C}$ of G consists of s even paths, t_1 attachable doubletons, and t_2 unattachable doubletons, then $V(G)$ can be covered by s non-trivial trees $T_1, \dots, T_s$ and t_2 doubletons such that $T_1, \dots, T_s$ altogether have at most $2s + 2t_1 - 2\varepsilon$ leaves, where $\varepsilon = 1$ if $t_1 \geq 1$ and $\varepsilon = 0$ if $t_1 = 0$.

Proof. Let the s even paths in $\mathcal{C}$ be $P_1, \dots, P_s$. For each $i \in [s]$, let x_i and y_i be the endpoints of P_i, with $x_i \in X$ and $y_i \in Y$. We apply the following procedure to obtain $T_1, \dots, T_s$. Initially, let $T_i = P_i$ for all $i \in [s]$. For each attachable doubleton (u_i, v_i), by definition, there exist $P_j, P_k \in \mathcal{C}$ such that $u_i y_j \in E(G)$ and $v_i x_k \in E(G)$. Hence we can extend T_j and T_k through these two edges to include u_i, v_i. As the procedure terminates, the T_i's remain as non-trivial trees. If $t_1 \geq 1$ then at most $2t_1 - 2$ leaves are added in the procedure (adding the first attachable doubleton does not increase the leaf count). Hence, the total number of leaves in $T_1, \dots, T_s$ is at most $2s + 2t_1 - 2$. The vertices that remain not covered by $T_1, \dots, T_s$ are the t_2 unattachable doubletons. ∎

Definition. Let G be a bipartite graph with partite sets X, Y of equal size. We say that G is *bi-Hamiltonian-connected* if whenever $x \in X$ and $y \in Y$ there is a Hamiltonian path of G with endpoints x and y.

Lemma 6. Let G be a bipartite graph with partite sets X, Y of size p. If G has at least $p^2 - (p-2)$ edges, then G is bi-Hamiltonian-connected and Hamiltonian.

Proof. We use induction on p. When $p \leq 2$, the only such graph is $K_{2,2}$. For the induction step, assume $p \geq 3$. First we show that G is bi-Hamiltonian-connected. Consider an arbitrary vertex $x \in X$. It suffices to show that for any $y \in Y$ there is a Hamiltonian path from x to y in G. Note that each vertex in G has degree at least two. We consider two cases.

Case 1. $d_G(x) < p$ or $d_G(y) < p$.

In this case, let $G' = G - \{x, y\}$. Since there are at most $2p - 2$ edges in G incident to $\{x, y\}$, we have $e(G') \geq p^2 - (p-2) - (2p-2) = (p-1)^2 - (p-3)$. By the induction hypothesis, G' is bi-Hamiltonian-connected. Since each vertex of G has degree at least two, x has a neighbor v in $V(G') \cap Y$ and y has a neighbor u in $V(G') \cap X$. Let P be a Hamiltonian u, v-path in G'. Now, $yu \cup P \cup vx$ is a Hamiltonian x, y-path in G.

Case 2. $d_G(x) = p$ and $d_G(y) = p$.

In this case, there exists $z \in Y$ with $d_G(z) < p$; otherwise $G = K_{p,p}$ and the claim holds trivially. Let $G' = G - \{x, z\}$. By the same argument as in Case 1, G' is bi-Hamiltonian-connected. Since z has degree at least two in G, it has a neighbor u in $V(G') \cap X$. Let P be an Hamiltonian u, y-path in G'. Since $d_G(x) = p$, $xz \in E(G)$. Now $xz \cup zu \cup P$ is a Hamiltonian x, y-path in G. This completes the proof that G is bi-Hamiltonian-connected.

To show that G is Hamiltonian, observe that G contains at least one edge xy. By the above discussion, there is a Hamiltonian x,y-path P. Since $p > 1$, we conclude that $P \cup xy$ is a Hamiltonian cycle of G. ∎

Equipped with all these tools, we now prove the main result we need to establish Theorem 1.

Theorem 5. For $3 \le k \le n \le 2k-1$, every C_{2k}-good coloring of $K_{n,n}$ that is not C_{2k-2}-good uses at most $\theta(n,n,k) = kn-(k-1)$ colors.

Proof. Let $G = K_{n,n}$ with X, Y being the two partite sets. Consider a C_{2k}-good coloring c of $E(G)$ which contains a rainbow copy C^* of C_{2k-2}. Let $U = V(C^*)$ and $W = V(G) - U$. Choose a representing graph L of c satisfying

(1) $L[U]$ is Hamiltonian.
(2) subject to (1), $L[W]$ has maximum possible number of edges, and
(3) subject to (1) and (2), $L[W]$ has an optimal cover $\mathcal{C}$ with the minimum possible size, and $\mathcal{C}$ is such a cover with the smallest number of doubletons.

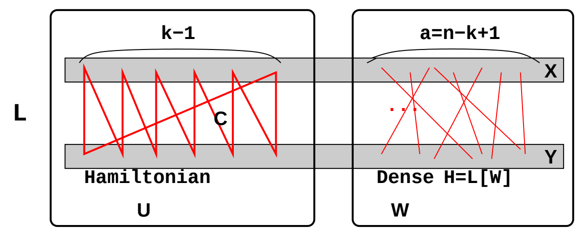

FIGURE 5. A representing graph

Let C be a Hamiltonian cycle of $L[U]$, which must therefore be of length $2k-2$. Note that $L[W]$ is a balanced bipartite graph with $n-(k-1)$ vertices in each partite set. Let $a = n-(k-1)$; since $k \le n \le 2k-1$, we have $1 \le a \le k$. The number of colors used by c is the number of edges in L, which we express as $e(U) + e(U,W) + e(W)$. Hence we need to prove that $e(U) + e(U,W) + e(W) \le kn-(k-1) = (k-1)^2 + ak$.

Our strategy in deriving this upper bound is roughly as follows: we show that $e(U,W)$ or $e(W)$ is "small". More specifically, if $L[W]$ has many edges then it has a small eqp-cover. By applying Lemma 4 to the cycle C paired with each path and each doubleton in the cover, we find that there are few edges joining U and W, and hence $e(U,W)$ is small. Otherwise, $L[W]$ has few edges and thus $e(W)$ is small.

Let $H = L[W]$. Suppose that the optimal cover $\mathcal{C}$ of H has size q and consists of s even paths, t_1 attachable doubletons, and t_2 unattachable doubletons. Observe that $q = s + t_1 + t_2 \le a$, since each quasi-path in the cover contains at least 2 vertices from a copy of $K_{a,a}$. We will repeatedly use the inequalities $q \le a \le k$.

If (x,y) is a doubleton of H, then $e(U, \{x,y\}) \le k$, by Lemma 4. If $s = 0$, then our choice of $\mathcal{C}$ implies that $e(W) = 0$ and that W is covered by a doubletons. Thus

$e(U) + e(U, W) + e(W) \le (k-1)^2 + ak + 0$, as desired. We may therefore assume henceforth that $s \ge 1$.

Note that if T is a tree in H then $e(U, V(T)) \le l(k-1)/2$ H with l leaves and By Lemma 5, W can be covered by s nontrivial trees $T_1, \ldots, T_s$ and t_2 doubletons such that the total number of leaves in $T_1, \ldots, T_s$ is at most $2s + 2t_1 - 2\varepsilon$, where $\varepsilon = 1$ if $t_1 \ge 1$ and $\varepsilon = 0$ if $t_1 = 0$. Let $F = T_1 \cup \cdots \cup T_s$. By Lemma 4.(3), $e(U, V(F)) \le (2s + 2t_1 - 2\varepsilon)(k-1)/2 = (s + t_1 - \varepsilon)(k-1)$. We consider two main cases depending on whether $e(U)$ is "large" or"small".

Case 1. $(k-1)^2 - (k-3) + 1 \le e(U) \le (k-1)^2$.

It suffices to show that $e(U, W) + e(W) \le ak$. First, we note that given any edge $uv \in E(G)$ with $u, v \in W$, the color $c(uv)$ is used on some edge in H. Suppose otherwise, that $c(uv)$ is used on an edge $e \in E(L)$ with at least one endpoint in U. Now $L' = L \cup uv - e$ is a representing graph of c. Furthermore, $L'[U]$ has still at least $(k-1)^2 - (k-3)$ edges and hence is Hamiltonian, while $H' = L'[W]$ has more edges than H, contradicting condition (2) satisfied by our choice of L.

Next, we show that either $e(U, W \cap X) = 0$ or $e(U, W \cap Y) = 0$. Suppose to the contrary that there exist $u \in W \cap X$ and $v \in W \cap Y$ each having a neighbor in U. Let $x \in U \cap X$ be a neighbor of v and $y \in U \cap Y$ a neighbor of u. Since $e(U) \ge (k-1)^2 - (k-3) + 1$, Lemma 6 implies that $L[U]$ contains a Hamiltonian x, y-path P. By the preceding paragraph the color $c(uv)$ is used in H and hence not on $uy \cup P \cup xv$. Thus adding uv completes a rainbow cycle of length $2k$, a contradiction.

Our discussion above in particular shows that $e(U, \{u, v\}) \le k - 1$ for all $u \in W \cap X$ and $v \in W \cap Y$. Hence the t_2 doubletons not covered by F are joined by at most $t_2(k-1)$ edges to U. This yields

$$
\begin{aligned}
e(U, W) &\le e(U, V(F)) + t_2(k-1) \le (s + t_1 - \varepsilon)(k-1) + t_2(k-1) \\
&= (s + t_1 + t_2)(k-1) - \varepsilon(k-1) = q(k-1) - \varepsilon(k-1),
\end{aligned}
$$

where $\varepsilon = 1$ if $t_1 \ge 1$ and $\varepsilon = 0$ otherwise.

Subcase 1a. $s \ge 2$.

By Theorem 4, $e(W) \le a^2 - aq + q + t_1$. Using that $q \le a \le k, t_1 = q - s - t_2 \le a - 2 \le k - 2$, and that $\varepsilon = 1$ if $t_1 > 0$, we obtain

$$
\begin{aligned}
e(U, W) + e(W) &\le q(k-1) - \varepsilon(k-1) + a^2 - aq + q + t_1 \\
&= q(k-a) + a^2 + [t_1 - \varepsilon(k-1)] \le a(k-a) + a^2 = ak.
\end{aligned}
$$

Subcase 1b. $s = 1$ (and thus $t_1 = 0$ and $\varepsilon = 0$).

If $a \le k - 1$ then

$$\begin{aligned} e(U,W)+e(W) &\le q(k-1)+a^2-aq+a = q(k-1-a)+a^2+a \\ &\le a(k-1-a)+a^2+a = ak. \end{aligned}$$

Hence we may assume that $a=k$. Let P denote the only path in $\mathcal{C}$. Thus P has $2p$ vertices, where $p=a-q+1$. Suppose first that $H[V(P)]$ is Hamiltonian (thus $p \ge 2$). If $p=a=k$, then we obtain a cycle of length $2k$ in L, a contradiction. So $p<a$ and $\mathcal{C}$ contains a doubleton (u,v). By our earlier discussion, $c(uv)$ appears on some edge e within H. Now the representing graph $L' = L \cup uv - e$ satisfies conditions (1) and (2). Furthermore $H' = L'[W]$ has an optimal cover with the same size as an optimal cover of H but with fewer doubletons, since $L'[V(P)]$ has a Hamiltonian path and uv now is an edge in L'. This contradicts our choice of L. Hence, we may assume that $H[V(P)]$ is non-Hamiltonian. Lemma 6 now implies that $e(V(P)) \le p^2-(p-1)$. By Theorem 4 $e(W)-e(V(P)) \le (a-q+1)(q-1) = p(a-p)$. Thus, since $a=k$, we obtain

$$\begin{aligned} e(W)+e(U,W) &\le p^2-(p-1)+(a-p)p+q(k-1) \\ &= ap-p+1+(a-p+1)(a-1) = a^2 = ak. \end{aligned}$$

Case 2. $e(U) \le (k-1)^2-(k-3)$.

In this case, it suffices to prove that $e(U,W)+e(W) \le ak+k-3$. By Lemma 4.(2), the t_2 doubletons not covered by F altogether are joined by at most t_2k edges to U. Using Lemma 5, Lemma 4 and $q=s+t_1+t_2$, we also get

$$e(U,W) \le (s+t_1-\varepsilon)(k-1)+t_2k = q(k-1)-\varepsilon(k-1)+t_2.$$

We will use the fact that equality can only be achieved in $(*)$ if every doubleton not covered by F is joined by exactly k edges to U and F is joined by exactly $(s+t_1-\varepsilon)(k-1)$ edges to U. Consequently, if $t_1=0$ and we have equality, then F consists of s paths each of which is joined by $k-1$ edges to U. In this case, we claim that there can be no edge joining a vertex $v \in W-V(F)$ to an endpoint u of a path in F (call such an edge *bad*): such a vertex v is in a doubleton and is not covered by F. Since equality in $(*)$ requires that doubletons send k edges to U, which has $k-1$ edges in each partite set, v has a neighbor in U. This implies that the bad edge uv could be used to extend the path in F to a path P such that $e(U,V(P)) \ge k$, contradicting Lemma 4.(1).

Subcase 2a. $s \ge 2$.

Theorem 4 states that $e(W) \le a^2-aq+q+t_1$. As in Subcase 1a, $q \le a \le k$ and $t_2 = q-s-t_1 \le k-2$. Hence

$$\begin{aligned} e(U,W)+e(W) &\leq q(k-1)-\varepsilon(k-1)+t_2+a^2-aq+q+t_1 \\ &= q(k-a)+a^2+t_2+[t_1-\varepsilon(k-1)] \\ &\leq a(k-a)+a^2+t_2 \leq ak+k-2. \end{aligned}$$

It remains to show that equality cannot hold throughout. Observe that we can only have equality if $t_2 = k-2$, which implies $q = k = a$, $s = 2$ and $t_1 = 0$. Hence $\mathcal{C}$ consists of 2 independent edges and $a-2$ unattachable doubletons. Furthermore, we also must have $e(W) = a^2 - aq + q + t_1 = k$. By the hypothesis of the theorem we have $k \geq 3$ so that $H[W]$ has an edge not used in $\mathcal{C}$. By the optimality of $\mathcal{C}$ this must be a bad edge joining a vertex in a doubleton to the end of a path in $\mathcal{C}$. But this is a contradiction, since we have shown that there are no bad edges.

Subcase 2b. $s = 1$.

If $s = 1$, then $t_1 = \varepsilon = 0$ and by Theorem 4, $e(W) \leq a^2 - aq + a$. Since $q = s + t_1 + t_2$, we have $t_2 + 1 = q \leq a \leq k$. Thus

$$\begin{aligned} e(U,W)+e(W) &\leq q(k-1)+t_2+a^2-aq+a = q(k-a)+a^2+a-1 \\ &\leq a(k-a)+a^2+a-1 = ak+a-1 \leq ak+k-1. \end{aligned}$$

We need to improve this inequality by 2. If $q < a < k$, then the last two inequalities yield the desired improvement. If $q = a < k$, then $a = k-1 \geq 2$ (or we are done) and we need to save one more. Now we may assume that $e(W) = a$, and thus Theorem 4 implies that $H = H_{a,a} = K_{1,a} + (a-1)K_1$. Since $a \geq 2$, we conclude that $\mathcal{C}$ has at least one doubleton. We have shown (following $(*)$) that when $t_1 = 0$ and $e(U,W) = q(k-1)+t_2$, there is no bad edge joining $W - V(F)$ and an endpoint in a path of F. Here F consists of a one path of length 1, and thus there is a bad edge, a contradiction.

Hence we may assume for the rest of the proof that $a = k$, so the last two inequalities in the display above hold with equality.

Let us now consider the case when $e(W) < a^2 - aq + a$. It suffices to consider the case when $e(W) = a^2 - aq + a - 1$, since otherwise we have again improved our inequality by 2. Hence by Theorem 4, either H is a double-star with $2a-2$ leaves or H is $H_{a,q} - e$, where $H_{a,q} = K_{a,a-q+1} + (q-1)K_1$. If H is a double-star, then by Lemma 4.(3) we get $e(U,W) + e(W) \leq (2a-2)(k-1)/2 + (2a-1) = ak + a - k \leq ak + k - 3$. If however $H = (K_{k,k-q+1} - e) + (q-1)K_1$, then certainly $q > 1$, since otherwise H is Hamiltonian and c was not C_{2k}-good. Thus H again we have a bad edge joining an endpoint of a path in F with a vertex of a doubleton, which prevents $e(U,W) = q(k-1)+t_2$.

Thus, we may henceforth assume that $e(W) = a^2 - aq + a$. This requires that $H = K_{k,k-q+1} + (q-1)K_1$ and that $\mathcal{C}$ consists of one path and $t_2 = q-1$ doubletons. Since $t_2 = q-1$, again $t_2 > 0$, because otherwise H is Hamiltonian and c is not C_{2k}-good. The set Y_0 of isolated vertices in H lies in one partite set of G; by symmetry, we may assume that $Y_0 \subseteq Y$. Let $X_1 = W \cap X$ and $Y_1 = W \cap Y - Y_0$. Thus X_1 and Y_1 are the large and small partite sets in the copy of $K_{k,k-q+1}$ in H. By changing $\mathcal{C}$, we can make any $q-1$ vertices in X_1 belong to doubletons; the rest lie in a single path in $\mathcal{C}$.

If some vertex in X_1 is joined by d edges to U, then we can choose $\mathcal{C}$ such that the path is joined by at most $k-1-d$ edges to U, thus allowing us to save d in the inequality (else the path can be extended again to get a contradiction). We may thus assume that no vertex in X_1 is joined by more than one edge to $\mathcal{C}$. From this it also immediately follows that every vertex $y \in Y_0$ is joined by exactly $k-1$ edges to U: if some vertex $x \in X_1$ is joined by 0 edges to U then (x, y) forms an unattached doubleton which is joined by at most $k-2$ edges to U (and we are done); otherwise the path is joined by at most $k-2$ edges to U and the doubleton containing y is joined by at most $k-1$ edges to U (again we are done).

Next we observe that the color of every edge e between Y_0 and X_1 must appear somewhere in C, since otherwise we could include e in L instead of the edge originally chosen and thus violate conditions 2. or 3. in the definition of L. Let C consist of the vertices $u_1, u_2, \dots u_{2k-2}$ in order. Now consider the case that $|Y_0| = t_2 > 1$ and let $y, y' \in Y_0$. If every vertex in X_1 is joined by 0 edges to U, then the doubletons involving y, y' are joined by only $k-1$ edges each to U and we are done. If on the other hand $x \in X_1$ is adjacent to, say, u_1, then we may assume without loss of generality that $c(xy) \neq c(u_1 u_{2k-2})$.

Now the cycle $u_1, x, y, u_2, \dots, u_{2k-2}$ has only one repeated color: the color $c(xy)$ appears on some $c(u_i u_{i+1})$. But if we now replace either u_i or u_{i+1} with y' then we obtain a rainbow copy of C_{2k}, a contradiction.

Finally, consider the case when $t_2 = 1$, so that $Y_0 = \{y\}$. Suppose there is a vertex $y' \in Y_1$ that is joined by an edge to some u_i. Let P be a x, y'-path of length $2k-1$ in H, where $x \in X_1$.. By our earlier discussion $c(xy)$ is used on C and thus not used on $P' = P \cup y' u_i y$. Now, $P' \cup xy$ is a rainbow copy of C_{2k}, a contradiction.

Suppose next that there is a vertex $x \in X_1$ that is joined by an edge to some u_i, and pick $x' \in X_1 - x$. Without loss of generality $c(yx') \neq c(u_i u_{i+1})$, and in this case the path $x' y u_{i+1} u_i x$ can be concatenated with a path in H to obtain a rainbow copy of C_{2k} in a similar fashion. If neither X_1 nor Y_1 is joined by edges to U, then we have $e(W) + e(U, W) = k(k-1) + (k-1) = ak - 1$. ∎

Now, using Proposition 2 and Theorem 5 we can finish the proof of Theorem 1.

Theorem 6. Let m, n, k be positive integers such that $m \leq n$ and $k \geq 2$. We have

$$f(K_{m,n}, C_{2k}) \leq \theta(m,n,k) = \begin{cases} (k-1)(m+n) - 2(k-1)^2 + 1 & m \geq 2k-1 \\ (k-1)n + m - (k-1) & k-1 \leq m \leq 2k-1 \\ mn & m \leq k-1. \end{cases}$$

Proof. We proceed by induction on $k+m+n$. The base case when $k=2$ is given by Corollary and we already observed that the case $m \leq k-1$ is trivial. For the induction step, by Proposition 2 we observe that for $m < n$

$$f(K_{m,n}, C_{2k}) \leq f(K_{m,n-1}, C_{2k}) + (k-1) \leq \theta(m, n-1, k) + (k-1) = \theta(m, n, k).$$

The same argument applies in the case when $m = n > 2k-1$. So we may now assume that $k \leq m = n \leq 2k-1$.

Consider a C_{2k}-good coloring c of $K_{n,n}$ that uses $f(K_{n,n}, C_{2k})$ colors. If c is not C_{2k-2}-good, then the desired inequality follows from Theorem 5. It can be easily checked that if $3 \leq k \leq 2k-1$, then $\theta(n, n, k-1) \leq \theta(n, n, k)$. So when c is C_{2k-2}-good it follows that

$$f(K_{n,n}, C_{2k}) \leq f(K_{n,n}, C_{2k-2}) \leq \theta(n, n, k-1) \leq \theta(n, n, k).$$

∎

4. EVEN QUASI-PATH COVERS IN BALANCED BIPARTITE GRAPHS

In this section we prove Theorem 4 via a series of claims. Let a be a positive integer, and let $G \subseteq K_{a,a}$. Let $\mathcal{C}$ be an optimal cover of G consisting of s even paths, t_1 attachable doubletons, and t_2 unattachable doubletons. Let $|\mathcal{C}| = q = s + t_1 + t_2$ and $t = t_1 + t_2$. Let $P_1, \ldots, P_s$ denote the s even paths in $\mathcal{C}$. For each $i \in [s]$, let $x_i \in X, y_i \in Y$ denote the two endpoints of P_i, and suppose P_i has $2p_i$ vertices. Let $(u_1, v_1), \ldots, (u_t, v_t)$ denote the t doubletons, where $u_j \in X$ and $v_j \in Y$ for all $j \in [t]$.

Orient each P_i from x_i to y_i. For a vertex w on P_i denote its predecessor and successor on P_i (if they exist) by w^- and w^+, respectively. For a subset $W \subseteq V(P_i) - x_i$ we define $W^- = \{w^- : w \in W\}$; note that $|W^-| = |W|$. Similarly, for $W \subseteq V(P_i) - y_i$, define $W^+ = \{w^+ : w \in W\}$; thus $|W^+| = |W|$.

Claim. Let $i \in [s]$. Then $N_{P_i}(x) \cap N^+_{P_i}(y) = \emptyset$ holds in the following cases: (1) $x = x_i, y = y_i$ (unless if $G_i = G[V(P_i)]$ is Hamiltonian or $G_i = K_2$), (2) $x = u_j, y = v_j$, where $j \in [t]$, and (3) $x = x_j, y = y_j$, where $j \in [s] - i$. In particular, if G_i is neither Hamiltonian nor K_2, then $d_{P_i}(x_i) + d_{P_i}(y_i) \leq p_i$.

Proof. Recall that $\mathcal{C}$ is an eqp-cover of G of minimum size and with a minimum number of doubletons. Suppose there exists $w \in N_{P_i}(x) \cap N^+_{P_i}(y)$, then $xw, yw^- \in E(G)$. If $x = x_i, y = y_i$ then $P_i \cup \{wx, w^-y\} - w^-w$ is a Hamiltonian cycle of G_i (except if $G_i = K_2$), a contradiction. If $x, y \notin V(P_i)$ then $P_i \cup \{wx, w^-y\} - w^-w$ is the union of two vertex disjoint even paths Q_1, Q_2 covering $V(P_i) \cup \{x, y\}$. If $x = u_j, y = v_j$ for $j \in [t]$, then replacing $P_i, (u_j, v_j)$ with Q_1, Q_2 in $\mathcal{C}$ yields an eqp-cover with fewer doubletons than $\mathcal{C}$, a contradiction. If $x = x_j, y = y_j$ for $j \in [s] - i$, then Q_1, Q_2 can be concatenated with P_j to form a single even path covering $V(P_i) \cup V(P_j)$, again contradicting our choice of $\mathcal{C}$.

Now, suppose G_i is not Hamiltonian. By our discussion above, we have $N_{P_i}(x_i) \cap N^+_{P_i}(y_i) = \emptyset$. Since $N_{P_i}(x_i) \cup N^+_{P_i}(y_i) \subseteq V(P_i) \cap Y$, it follows that $d_{P_i}(x_i) + d_{P_i}(y_i) = |N_{P_i}(x_i)| + |N_{P_i}(y_i)| = |N_{P_i}(x_i)| + |N^+_{P_i}(y_i)| = |N_{P_i}(x_i) \cup N^+_{P_i}(y_i)| \leq p_i$. ∎

Claim. Let $i, j \in [s]$, $i \neq j$. Then $x_i y_j, x_j y_i \notin E(G)$ and $e(\{x_j, y_j\}, V(P_i)) \leq p_i - 1$. Furthermore, if $G_i = G[V(P_i)]$ is Hamiltonian, then $e(\{x_j, y_j\}, V(P_i)) = 0$.

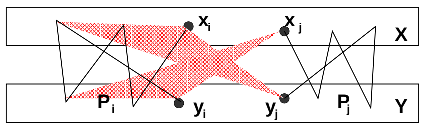

FIGURE 6. Edges between the end-points of an even path and another even path

Proof. If $x_i y_j \in E(G)$ or $x_j y_i \in E(G)$ then we can concatenate P_i and P_j into a single even path, contradicting our choice of $\mathcal{C}$. By arguments similar to those in Claim 4., one can show that $N^+_{P_i}(x_j) \cap N_{P_i}(y_j) = \emptyset$ (recall that $y_i \notin N_{P_i}(x_j)$). Note also that $N^+_{P_i}(x_j) \cup N_{P_i}(y_j) \subseteq V(P_i) \cap X - x_i$. Hence, $e(\{x_j, y_j\}, V(P_i)) = |N_{P_i}(x_j)| + |N_{P_i}(y_j)| = |N^+_{P_i}(x_j)| + |N_{P_i}(y_j)| = |N^+_{P_i}(x_j) \cup N_{P_i}(y_j)| \leq p_i - 1$.

If G_i is Hamiltonian, then there can be no edge between $\{x_j, y_j\}$ and $V(P_i)$; otherwise we can obtain a single even path that covers $V(P_i) \cup V(P_j)$, contradicting our choice of $\mathcal{C}$. ∎

Claim. Let $i \in [t], j \in [s]$. We have $e(\{u_i, v_i\}, V(P_j)) \leq p_j$. Furthermore, equality holds for at most two choices of j if (u_i, v_i) is an attachable doubleton, and for at most one choice of j if $\{u_i, v_i\}$ is an unattachable doubleton.

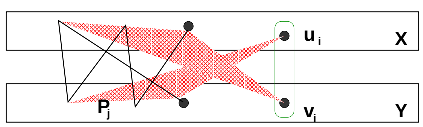

FIGURE 7. Edges between a doubleton and an even path

Proof. By Claim 4. $N_{P_j}(u_i) \cap N^+_{P_j}(v_i) = \emptyset$. Since $N_{P_j}(u_i) \cup N^+_{P_j}(v_i) \subseteq Y$, we have $e(\{u_i, v_i\}, P_j) = |N_{P_j}(u_i)| + |N_{P_j}(v_i)| = |N_{P_j}(u_i)| + |N^+_{P_j}(v_i)| = |N_{P_j}(u_i) \cup N^+_{P_j}(v_i)| \leq p_j$. Equality requires that $N_{P_j}(u_i) \cup N^+_{P_j}(v_i) = Y \cap V(P_j)$. For convenience, we say that P_j is *extremal* with respect to $\{u_i, v_i\}$ if equality holds in the above inequality.

We classify extremal paths P_j with respect to $\{u_i, v_i\}$ into three types: P_j is of type 1 if $u_i y_j \in E(G)$), P_j is of type 2 if $v_i x_j \in E(G)$, and P_j is of type 3 if $u_i y_j, v_i x_j \notin E(G)$. Note that if P_j is of type 3, then the fact that $x_j^+, y_j \in V(P_j) \cap Y = N_{P_j}(u_i) \cup N^+_{P_j}(v_i)$

implies that $u_i x_j^+, v_i y_j^- \in E(G)$. As we traverse P_j from x_j to y_j, let z_j be the last vertex belongs to $N_{P_j}(u_i)$; clearly z_j exists, $z_j \neq y_j$, and $z_j^+ \in N_{P_j}(v_i)$. In other words, we have $u_i z_j, v_i z_j^+ \in E(G)$.

Now, we show that if P_j and P_j' are two distinct extremal paths then one of them has to be of type 1 and the other has to be of type 2. It then follows that there are at most two extremal paths with respect to $\{u_i, v_i\}$. Furthermore, note that having both an extremal path of type 1 and and extremal path of type 2 with respect to $\{u_i, v_i\}$ implies that (u_i, v_i) is an attachable doubleton. Hence if (u_i, v_i) is an unattachable doubleton, then there is at most one extremal path. We consider two cases

Case 1. neither of P_j and $P_{j'}$ is of type 3

It suffices to show that they cannot be both of type 1 or 2. Without loss of generality, suppose P_j and $P_{j'}$ are both of type 1. Then $V(P_j) \cup V(P_{j'}) \cup \{u_i, v_i\}$ can be covered by a single even path and a doubleton, contradicting our choice of $\mathcal{C}$.

Case 2. one of $P_j, P_{j'}$ is of type 3

We may assume that P_j is of type 3. If one of $P_{j'}$ is of type 1 or 2, then again, one can find a single even path and a doubleton to cover $V(P_j) \cup V(P_{j'}) \cup \{u_i, v_i\}$, a contradiction. So we may assume that P_j and $P_{j'}$ are both of type 3. Let z_j be defined as at the end of last paragraph. Let P denote the even path obtained from $P_j \cup P_{j'} \cup \{u_i, v_i\}$ by adding edges $u_i z_j, v_i z_j^+, u_i x_{j'}^+, v_i y_{j'}^-$ and deleting edges $z_j z_j^+, x_{j'} x_{j'}^+, y_{j'}^- y_{j'}$. Now $V(P_j) \cup V(P_{j'}) \cup \{u_i, v_i\}$ can be covered by P and $\{x_{j'}, y_{j'}\}$, contradicting our choice of $\mathcal{C}$. ∎

Let $S = \bigcup_{i=1}^{s} V(P_i), T = \bigcup_{i=1}^{t} \{u_i, v_i\}$. Note that our choice of $\mathcal{C}$ implies that T induces an empty subgraph of G.

Claim. If $s = 1$, then $e(G) \leq e(G[V(P_1)]) + (a-q+1)(q-1) \leq a^2 - aq + a$. Equality is only achieved by $H_{a,q}$. Furthermore, if $e(G) = a^2 - aq + a - 1$, then G is either obtained by deleting an edge from $H_{a,q}$ or by joining the centers of two stars $K_{1,a-1}$ (i.e. G is a double-star).

Proof. We have $S = V(P_1)$ and $q = t+1$. By Claim , there are at most $tp_1 = t(a-t) = (q-1)(a-q+1)$ edges of G between S and T. Hence $e(G) \leq e(G[V(P)]) + (a-q+1)(q-1)$. Since $e(G[V(P)]) \leq p_1^2 = (a-t)^2 = (a-q+1)^2$, we have

$$e(G) \leq (a-q+1)^2 + (a-q+1)(q-1) = a^2 - aq + a. \tag{1}$$

To achieve equality in (1), S must induce $K_{a-t,a-t}$ and every doubleton is joined by $a-t$ edges to S. It is not possible that vertices from both $T \cap X$ and $T \cap Y$ are joined by edges to S, since otherwise we could lengthen the path and thus obtain a smaller cover. Thus $G = K_{a,a-t} + tK_1 = H_{a,q}$.

If $e(G) = a^2 - aq + a - 1$, but S still forms a $K_{a-t,a-t}$, then by a similar argument we conclude that $G = H_{a,q} - e$. So we may assume that S forms a $K_{a-t,a-t} - e$ and that every doubleton is joined by $a - t$ edges to S. If $G \neq H_{a,q} - e$, then there must be an edge from some $x \in T \cap X$ and $y \in T \cap Y$ to S. The only way for S to contain a spanning path, but not being extendable to a longer path using these two edges is that S induces $K_{2,2} - e$, i.e. a path v_1, v_2, v_3, v_4. Since each doubleton is joined by exactly 2 edges to S it follows that x and y each are joined by exactly 1 edge to S, as they can form a doubleton. But now combining x, y with the remaining vertices in T we see that every such vertex is joined by 1 edge to S. If all edges from T to S go to v_2 or v_3, then we have a double-star. So suppose that x is adjacent to v_1. Now y must be connected by an edge to v_2 or v_4. In the latter case we obtain a longer even path (and thus a smaller eqp-cover), whereas in the former case x, v_1, v_2, y and v_3, v_4 combine to yield an eqp-cover with fewer doubletons. ∎

Claim. If $s \geq 2$, then $e(G) \leq a^2 - aq + q + t_1$.

Proof. Regarding G as a spanning subgraph of $K_{a,a}$, let $G^- = K_{a,a} - G$. It suffices to show that $e(G^-) \geq aq - q - t_1$. Given disjoint subsets U and V in G^-, let $l(U, V)$ denote the numbers of edges of G^- with one endpoint in U and the other in V. Let $l(U)$ denote the number of edges of G^- with both endpoints in U. We first give a lower bound on $l(S)$.

For $i \in [s]$, let $Q_i = \{x_i, y_i\}$, $Q = \bigcup_{1<i<s} Q_i$ and $R_i = V(P_i) - Q_i$. Note that $|R_i| = 2p_i - 2$. We show that $l(Q_i) + l(Q, R_i) \geq s(p_i - 1)$. We consider three cases.

Case 0. If $p_i = 1$, then there is nothing to show.

Case 1. Suppose that $G[V(P_i)]$ is Hamiltonian.

In this case, by Claim 4., for all $j \in [s] - i$, $\{x_j, y_j\}$ is joined by no edges of G to $V(P_i) \supseteq R_i$. This implies that $l(Q, R_i) \geq (s-1)(2p_i - 2)$. In particular, we have $l(Q_i) + l(Q, R_i) \geq (s-1)(2p_i - 2) \geq s(p_i - 1)$, recalling that $s \geq 2$.

Case 2. $G_i = G[V(P_i)]$ is not Hamiltonian or K_2.

In this case, by Claim 4., for all $j \in [s]-i$, $\{x_j, y_j\}$ is joined by at most $p_i - 1$ edges of G to $V(P_i) \supseteq R_i$. Hence, $l(Q_j, R_i) \geq (2p_i - 2) - (p_i - 1) = p_i - 1$. By Claim 4., Q_i is joined by at most $d_{P_i}(x_i) + d_{P_i}(y_i) \leq p_i$ edges of G to R_i, hence $l(Q_i, R_i) \geq (2p_i - 2) - p_i = p_i - 2$. So, we have $l(Q, R_i) \geq (s-1)(p_i - 1) + (p_i - 2) = s(p_i - 1) - 1$. Since G_i is not Hamiltonian, we have $x_i y_i \notin E(G)$, hence $l(Q_i) = 1$. Thus, $l(Q_i) + l(Q, R_i) \geq s(p_i - 1)$.

For $i, j \in [s], i \neq j$, by Claim 4., there is no edge of G between Q_i and Q_j. Hence $l(Q_i, Q_j) = 2$. Now, we have $l(S) \geq \Sigma_{i=1}^{s}[l(Q_i) + l(Q, R_i)] + \Sigma_{i,j \in [s], i<j} l(Q_i, Q_j) \geq \Sigma_{i=1}^{s} s(p_i - 1) + 2[s(s-1)/2] = s\Sigma_{i=1}^{s} p_i - s = s(a-t) - s$, noting that $\Sigma_{i=1}^{s} p_i = a - t$.

Let $i \in [t], j \in [s]$. By Claim , for fixed i, $l(\{u_i, v_i\}, P_j) \geq 2p_j - p_j = p_j$, with equality holding for at most two choices of j if (u_i, v_i) is attachable and for at most one choice of j if (u_i, v_i) is unattachable. Hence $l(\{u_i, v_i\}, S) \geq \Sigma_{j=1}^{s}(p_j + 1) - 2 = (a-t) + (s-2)$ if (u_i, v_i) is attachable and $l(\{u_i, v_i\}, S) \geq \Sigma_{j=1}^{s}(p_j + 1) - 1 = (a-t) + (s-1)$ if (u_i, v_i) is unattachable. Now we have $l(S, T) \geq t_1[(a-t) + (s-2)] + t_2[(a-t) + (s-1)] = t[(a-t) + (s-1)] - t_1$ (note that $t = t_1 + t_2$).

Finally, we have $l(T) = t^2$, recalling that T induces an empty subgraph of G. Now,we have (noting that $s + t = q$)

$$\begin{aligned} e(G^-) &= l(S) + l(S, T) + l(T) \\ &\geq [s(a-t) - s] + t[(a-t) + (s-1)] - t_1 + t^2 \\ &= (s+t)a - (s+t) - t_1 \\ &= aq - q - t_1. \end{aligned}$$

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