Local anti-Ramsey numbers of graphs

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Abstract

A subgraph H in an edge-coloring is *properly colored* if incident edges of H are assigned different colors, and H is *rainbow* if no two edges of H are assigned the same color. We study properly colored subgraphs and rainbow subgraphs forced in edge-colorings of complete graphs in which each vertex is incident to a large number of colors.

In an edge-colored host graph G, a subgraph H is *properly colored* if no two incident edges of H receive the same color, and *rainbow* if no two edges of H receive the same color. Given a positive integer k, a host graph G, an edge-coloring c of G (c is not necessarily proper), c is a *k-coloring* if c uses k colors overall, c is a *local k-coloring* if at most k colors are used at each vertex of G, and c is a *k-good* coloring if at least k different colors are used at each vertex of G.

The classical Ramsey and anti-Ramsey problems ask for the optimal total number of colors used on the edges of a host graph without creating a prohibited colored subgraph. The local variation of these problems is concerned only with the number of colors used on the edges incident to each vertex, instead of the total number of colors used. Local Ramsey problem, for example, asks for the minimum number k such that there exists a local k-coloring of the edges of K_n with no monochromatic subgraph isomorphic to a given graph H, see [13], [20]. In this paper we initiate the study of a similar variation of anti-Ramsey problem.

In anti-Ramsey problems, we are interested in rainbow subgraphs instead of monochromatic subgraphs in edge-colorings. Suppose we are given a graph H and a sufficiently large positive integer n. When we color the edges of K_n, we can always force a rainbow copy of H to occur by using many colors. It is natural to determine the threshold on the number of colors needed to force that. The classical anti-Ramsey number of H for fixed n, denoted by $AR(n,H)$, is defined as the maximum k such that there exists a k-coloring of $E(K_n)$ that avoids rainbow copies of H. By definition, every coloring of $E(K_n)$ using more than $AR(n,H)$ colors contains a rainbow copy of H. Anti-Ramsey numbers were introduced by Erdős, Simonovits and Sós [8] in the 1970's, and have been actively studied recently (see [2, 4, 5, 6, 15, 16]). Anti-Ramsey numbers are closely related to the extremal function $ex(n,F)$. Here $ex(n,F)$ is the maximum number of edges in a graph on n vertices with no subgraph isomorphic to F. The main result in [8] is as follows

$$AR(n,H) = \frac{n^2}{2}\left(1 - \frac{1}{\chi - 1}\right)(1 + o(1)),$$

where $\chi = \min\{\chi(H \setminus e) : e \in E(H)\}$. In particular, it shows that most Anti-Ramsey numbers are quadratic in n.

Instead of forcing rainbow copies of a given graph H, one can consider forcing properly colored copies of H by using many colors, and study the threshold on the number of colors needed. This is thoroughly studied by Manoussakis, Spyratos, Tuza, and Voigt in [18].

In this paper, we consider the local variation of the anti-Ramsey problem. Namely, we study the maximum k such that there exists a k-good edge-coloring of K_n containing no rainbow copy of a given graph H. In

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addition, we consider the related problem concerning properly colored subgraphs. We give formal definitions as follows.

Given an edge-coloring c of a host graph G and a vertex v in G, define the *color-degree* of v to be the number of different colors that are used on edges incident to v. Using this notation, a k-good edge-coloring of G is then simply an edge-coloring of G with minimum color degree at least k. Given a positive integer n and a graph H, let $f(n,H)$ denote the maximum k such that there exists a k-good coloring of $E(K_n)$ that contains no properly colored copy of H, and let $g(n,H)$ denote the maximum k such that there exists a k-good coloring of $E(K_n)$ containing no rainbow copy of H. By definition, every coloring of $E(K_n)$ with minimum color degree more than $f(n,H)$ contains a properly colored copy of H, and every coloring of $E(K_n)$ with minimum color degree more than $g(n,H)$ contains a rainbow copy of H.

So here, we impose constraints on the number of colors locally, requiring each vertex to be incident to at least a certain number of colors, and we are interested in how large this number should be in order to force certain rainbow subgraphs or properly colored subgraphs. Note, however, in a coloring of $E(K_n)$ with large minimum color degree, the total number of colors used may be as small as $n = o(n^2)$ (recalling that K_n has a proper coloring using at most n colors and such a coloring has the largest possible minimum color degree). Therefore the local anti-Ramsey problem will have a different flavor from the classical anti-Ramsey problem in the sense that one can not hope for forcing rainbow subgraphs merely by forcing a large overall number of colors used. Throughout the paper, we assume n is a sufficiently large positive integer, and H is a graph containing at least one edge. Also, we shall drop all ceiling and floor signs whenever they are not crucial.

In the first section we study properly colored subgraphs forced by large color degrees. The second section is devoted to rainbow subgraphs. As in [8] we get a close relationship between local anti-Ramsey numbers and Turán's number. Finally, in section three, we discuss rainbow and properly colored trees and cycles which is, perhaps, the most interesting topic we address. We conclude with the following question. For any n is there a constant $c = c(k)$ such that $f(n, C_k) \leq c$?

1 Properly colored subgraphs

We start with properly colored subgraphs in colorings with large minimum color degree. We first derive a general upper bound on $f(n,H)$ using the chromatic number of H.

Theorem 1.1 *Let H be a graph with $\chi(H) = k \geq 2$. Then*

$$f(n,H) \leq \left[1 - \frac{1}{2(k-1)}\right] n + o(n).$$

Proof. Let a_k be any real number with $a_k > 1 - \frac{1}{2(k-1)}$. Let c be an arbitrary $a_k(n-1)$-good coloring of the edges of $G = K_n$. It suffices to show that c contains a properly colored copy of H. Suppose the vertices of G are $v_1, \cdots, v_n$. Let $G_0 = G$, and for $i = 1, 2, \cdots, n$, define G_i iteratively as follows: G_i is obtained from G_{i-1} by deleting extra copies of each edge incident to v_i in G_{i-1} whose color appears more than once at v_i.

Clearly, by the above procedure, G_n is properly colored under c. At each step, at most $(1-a_k)(n-1)$ edges are deleted, so G_n has at least $\binom{n}{2} - n(n-1)(1-a_k) = (1 - 2 + 2a_k)\binom{n}{2} = (1 - \frac{1}{k-1} + \epsilon)\binom{n}{2}$ edges, where $\epsilon = a_k - 1 - \frac{1}{2(k-1)} > 0$. Since $ex(n,H) = (1 - \frac{1}{k-1})\binom{n}{2} + o(n^2)$, we conclude that G_n contains a copy of H, which is properly colored under c since G_n is. ■

Although the upper bound given in $f(n,H)$ appears rather high, we will soon see that there is not much room for improvement in general due to the next couple of lower bounds we establish. Suppose D is a digraph and uv is an arc in D from u to v. Then u is the *tail* of uv and v is the *head* of uv.

Theorem 1.2 *If H is a graph with $e(H) > n(H)$, then*

$$f(n,H) \geq \lfloor n/2 \rfloor.$$

Proof. We construct a coloring of $E(K_n)$ with minimum color degree at least $\lfloor n/2 \rfloor$ that contains no properly colored copy of H. Let D be an orientation of $E(K_n)$ with minimum out-degree at least $\lfloor n/2 \rfloor$. Such an orientation can be easily found using a decomposition of $E(K_n)$ or $E(K_{n+1})$ into 2-factors and cyclically orienting each cycle in these 2-factors.

Let $x_1, \cdots, x_n$ denote the vertices of K_n. Define a coloring c of $E(K_n)$ by assigning color i to all the edges with tail x_i, where $1, 2, \cdots, n$ are distinct colors. In the coloring c, each vertex has color degree at least $\lfloor n/2 \rfloor$. Consider any copy H' of H in D, let $D[H']$ denote the restriction of the orientation D on the edges of H'. Since $e(H') > n(H')$, there exists a vertex v with outdegree at least two in $D[H']$. By our definition of c, the out-edges at v in $D[H']$ have the same color, preventing H' from being properly colored. ■

Note that Theorem 1.2 applies to almost all graphs. The only graphs that it does not apply to are graphs in which each component is a tree or is unicyclic. Theorem 1.2, together with Theorem 1.1, allows us to determine the asymptotics of $f(n, H)$ for almost all bipartite graphs H.

Corollary 1.3 *Let H be a bipartite graph with $e(H) > n(H)$. Then*

$$f(n, H) = (1 + o(1))n/2.$$

Another interesting fact that follows from Theorem 1.1 and Theorem 1.2 is as follows. Recall that a graph G is a *subdivision* of another graph H if G can be obtained from H by inserting vertices of degree two into edges of H.

Corollary 1.4 *Let H be a graph. Let ϵ an arbitrary positive real and $n = n(\epsilon)$ a sufficiently large positive integer. Then every coloring of $E(K_n)$ with minimum color degree at least $\frac{1+\epsilon}{2}n$ contains a properly colored subdivision of H. Furthermore, if $e(H) > n(H)$ then there exists a coloring of $E(K_n)$ with minimum color degree at least $\lfloor n/2 \rfloor$ that contains no properly colored subdivision of H.*

Proof. For the first statement, let H' denote the graph obtained from H by subdividing each edge of H exactly once. Clearly, H' is a bipartite graph. By Theorem 1.1 $f(n, H) \leq n/2 + o(n)$. Thus for sufficiently large n, every coloring of $E(K_n)$ with minimum color degree at least $\frac{1+\epsilon}{2}n$ yields a properly colored copy of H'.

For the second statement, note that for any subdivision H'' of H, $e(H'') > n(H'')$ holds. Applying Theorem 1.2, there exists a coloring of $E(K_n)$ with minimum color degree at least $\lfloor n/2 \rfloor$ that contains no properly colored subdivision of H. ■

For non-bipartite graphs H, we have the following general lower bound on H in terms of its chromatic number.

Theorem 1.5 *Let H be a graph with $\chi(H) = k \geq 3$. Then*

$$f(n, H) \geq \left(1 - \frac{1}{\lceil k/2 \rceil - 1}\right) n + \Omega(1)$$

Proof. Let $q = \lceil k/2 \rceil - 1$. Let T_n^q denote the q-partite Turán graph on n vertices (with each part of size $\lceil n/q \rceil$ or $\lfloor n/q \rfloor$). Let $V_1, \cdots, V_q$ denote the q partite sets of T. Define a coloring c of $E(K_n)$ by assigning distinct colors to $E(T_n^q)$ and then assigning a new color 0 to all the edges in $E(K_n) - E(T_n^q)$.

It is easy to verify that each vertex has color degree at least $n(1 - \frac{1}{\lceil k/2 \rceil - 1}) + \Omega(1)$ in c. We show that c contains no properly colored copy of H. Let L be any properly colored subgraph in c, we show that $\chi(L) < k$, which would imply $L \neq H$ since $\chi(H) = k$. Suppose first that L does not use any edge of color 0, then L is a subgraph of T_n^q, and thus $\chi(L) = q = \lceil k/2 \rceil - 1 < k$. Next, suppose that L has edges of color 0. Observe that for each $i \in [q]$, $V(L) \cap V_i$ must induce a matching since L is properly colored. Thus the subgraph of L induced by $V(L) \cap V_i$ has chromatic number at most 2. It follows that $\chi(L) \leq 2q < k$, completing the proof. ■

We summarize our bounds on $f(n,H)$ for non-bipartite graphs H as follows.

Proposition 1.6 *Let H be a graph with $\chi(H) = k \geq 3$. Then*

$$\left[1 - \frac{1}{\lceil k/2 \rceil - 1}\right] n + \Omega(1) \leq f(n,H) \leq \left[1 - \frac{1}{2(k-1)}\right] n + o(n).$$

Further, if H is not a disjoint union of acyclic or unicyclic graphs , then $f(n,H) \geq \lfloor n/2 \rfloor$.

We have seen that for bipartite graphs H other than the ones in which each component is either a tree or is unicyclic, $f(n,H) = (1+o(1))n/2$, and for non-bipartite graphs H other than disjoint union of acyclic or unicyclic graphs we have lower and upper bound in the same order of magnitude. This leaves a particularly desirable task to study $f(n,H)$ when H consists entirely of components that are acyclic or unicyclic. In particular, it is interesting to study the behavior of $f(n,H)$ when H is a tree or cycle. We shall investigate local anti-Ramsey numbers of trees and cycles in Section 3.

2 Rainbow subgraphs

In this section, we study rainbow subgraphs forced by edge-colorings with large minimum color degree. We first prove a useful lemma which enables one to find a rainbow copy of a desired subgraph H in a dense properly colored graph. Let G, H be two graphs, and let c be an edge-coloring of G. A *rainbow embedding* of H in G is an embedding σ of H in G such that $\sigma(H)$ is rainbow. Given positive integers s, t, K_t^s denotes the complete t-partite graph with s vertices in each part. Let $F[A]$ be a subgraph of F induces by a vertex set A and $N_F(v)$ be a neighborhood of v in F. For convenience, we write $\sigma(A)$ for $\sigma(F[A])$ where it is clear from the context.

Lemma 2.1 *Let H be a graph with p vertices, q edges, and $\chi(H) = k \geq 2$. Let $s = pq$. Then every proper coloring of K_k^s contains a rainbow copy of H.*

Proof. Let c be a proper coloring of the edges of $G = K_k^s$. Let $A_1, \cdots, A_k$ denote the k partite sets of G, each of size s. Since $\chi(H) = k$, H is a k-partite graph; let $B_1, \cdots, B_k$ denote the partite sets of H. Let F be a maximal induced subgraph of H, such that there exists a rainbow embedding σ of F in G satisfying $V(\sigma(V(F) \cap B_i)) \subseteq A_i$ for all $i \in [k]$. Since the subgraph of H induced by B_1 can trivially be embedded in A_1, F is well defined. We show that $F = H$.

Suppose otherwise there is $v \in V(H) \backslash V(F)$. Without loss of generality, suppose $v \in B_k$. Let $N = N_F(v)$. Note that $N \subseteq \bigcup_{i=1}^{k-1} B_i$ and hence $V(\sigma(N)) \subseteq \bigcup_{i=1}^{k-1} A_i$. Let $D = A_k \backslash V(\sigma(F))$. Clearly, $|D| > s - p \geq pq - p$. If $N = \emptyset$, then we can extend σ by mapping v to any vertex in D, contradicting our choice of F. So we assume that $|N| \geq 1$.

For convenience, we call the colors used on edges in $\sigma(F)$ *bad colors*, and an edge of G with a bad color a *bad edge*. Note that there are at most $q - |N|$ bad colors. Consider any vertex $u \in D$. Since c is a proper coloring, the edges u sends to $\sigma(N)$ all have different colors. If none of these edges is a bad edge, then we can extend σ by mapping v to u, contradicting our choice of F. Hence each vertex in D sends a bad edge to $\sigma(N)$. Thus there are at least $|D| > pq - p$ bad edges between D and $\sigma(N)$. On the other hand, each vertex w in $\sigma(N)$ is incident to at most $q - |N| \leq q - 1$ bad edges (recall that c is a proper coloring). Hence, there are at most $|N|(q-1) < p(q-1) = pq - p$ bad edges between D and $\sigma(N)$, a contradiction. ■

Recall that $g(n,H)$ is the maximum k such that there exists a coloring of $E(K_n)$ with minimum color degree at least k containing no rainbow copy of G. Note that $g(n,H) \geq f(n,H)$ holds trivially.

Lemma 2.2 *Let H be a graph with p vertices, q edges, and $\chi(H) = k$. Let $s = pq$. For sufficiently large n, we have $g(n,H) \leq f(n, K_k^s)$.*

Proof. Let c be an arbitrary edge-coloring of K_n with minimum color degree at least $f(n, K_k^s) + 1$. It suffices to show that c contains a rainbow copy of H. By the definition of $f(n, K_k^s)$, the coloring c has a properly colored copy B of K_k^s. By Lemma 2.1, B contains a rainbow copy of H. ■

Theorem 1.1 and Lemma 2.2 together yield

Theorem 2.3 *Let H be graph with $\chi(H) = k \geq 2$. Then*

$$g(n, H) \leq \left[1 - \frac{1}{2(k-1)}\right] n + O(n).$$

Since $g(n, H) \geq f(n, H)$ always, Theorem 1.2 together with Theorem 2.3 yield

Corollary 2.4 *Let H be a bipartite graph with $e(H) > n(H)$. Then*

$$g(n, H) = (1 + o(1))n/2.$$

Like in the previous section, the following fact can be easily established.

Corollary 2.5 *Let H be a graph. Let ϵ an arbitrary positive real and $n = n(\epsilon)$ a sufficiently large positive integer. Then every coloring of $E(K_n)$ with minimum color degree at least $(\frac{1+\epsilon}{2})n$ contains a rainbow subdivision of H. Furthermore, if $e(H) > n(H)$ then there exists a coloring of $E(K_n)$ with minimum color degree at least $\lfloor n/2 \rfloor$ that contains no rainbow subdivision of H.*

Next, we give a lower bound on $g(n, H)$ for non-bipartite graphs H.

Theorem 2.6 *Let H be a graph with $\chi(H) = k \geq 3$. Then*

$$g(n, H) \geq \left(1 - \frac{1}{k-2}\right) n + \Omega(1).$$

Proof. Let T_n^{k-2} denote the $(k-2)$-partite Turán graph on n vertices. Consider a coloring c of $E(K_n)$ obtained by coloring the edges of T_n^{k-2} with distinct colors and assigning a new color to the remaining edges. It is easy to see that c has minimum color degree at least $(1 - \frac{1}{k-2})n + \Omega(1)$. Furthermore, if L is a rainbow subgraph under c, it is straightforward to verify that $\chi(L) \leq k - 1$. Hence, in particular, c contains no rainbow copy of H. ■

We summarize our bounds for non-bipartite graphs H as follows.

Proposition 2.7 *Let H be a graph with $\chi(H) = k \geq 3$. Then*

$$\left[1 - \frac{1}{k-2}\right] n + \Omega(1) \leq g(n, H) \leq \left[1 - \frac{1}{2(k-1)}\right] n + o(n).$$

Further, if H is not a disjoint union of acyclic or unicyclic graphs, then $g(n, H) \geq \lfloor n/2 \rfloor$.

As in the previous section, we have obtained relatively satisfactory bounds on $g(n, H)$ except when H is a disjoint union of acyclic or unicyclic graphs. It is therefore particularly interesting to study $g(n, H)$ when H is a tree or a cycle. We shall leave this for Section 3.

3 Properly colored and rainbow trees and cycles

We first consider the case when H is a tree. This case is fairly straightforward.

Proposition 3.1 *Let H be a tree with $k \geq 2$ edges. Then*

$$\begin{aligned} \Delta(H) - 2 \quad &\leq f(n, H) \quad \leq k - 1 \\ k - 1 \quad &\leq g(n, H) \quad \leq 2k - 3 \end{aligned}$$

Proof. If n is even, let c be a coloring of $E(K_n)$ in which each vertex has color degree $\Delta(H) - 1$; such a coloring can be easily obtained from a 1-factorization of $E(K_n)$. If n is odd, let c be a coloring of $E(K_n)$ in which each vertex has color degree either $\Delta(H) - 2$ or $\Delta(H) - 1$; such a coloring can be obtained from an 1-factorization of $E(K_{n+1})$. Clearly, in the coloring c defined above, no properly colored copy of H can occur. Thus, $f(n, H) \geq \Delta(H) - 1$ when n is even and $f(n, H) \geq \Delta(H) - 2$ when n is odd, respectively.

On the other hand, it can be easily proven by induction on k that every coloring of $E(K_n)$ with minimum color degree at least k contains a properly colored copy of H. Thus $f(n, H) \leq k - 1$. A similar argument establishes $g(n, H) \leq 2k - 3$.

For a lower bound on $g(n, H)$, consider a coloring using $k - 1$ colors in which each vertex is incident to all $k - 1$ colors; such a coloring can be obtained from an 1-factorization of K_n or 2-factorization of K_n (where n is sufficiently large). Since only $k - 1$ colors are used, no rainbow copy of H can arise. Thus $g(n, H) \geq k - 1$. ■

It is likely that the upper bound on $g(n, H)$ in the proposition above can be improved for various classes of trees H. Next, we consider the case when H is a cycle. This case turns out to be far more tricky than we expected, and we do not know too much in this case. It is not even clear what the growth rate of $f(n, C_k)$ and $g(n, C_k)$ should be. We are, however, able to find some lower bounds on $f(n, C_k)$ and $g(n, C_k)$ and determine $f(n, C_k)$ either asymptotically or exactly for $k = 3, 4$.

First, we give some lower bounds on $f(n, C_k)$ and $g(n, C_k)$.

Proposition 3.2 *For fixed k and sufficiently large n, there exists a coloring of $E(K_n)$ with minimum color degree at least $\lfloor (k-1)/2 \rfloor$ that contains no properly colored cycle of length at least k. In particular, we have*

$$f(n, C_k) \geq \lfloor (k-1)/2 \rfloor.$$

Proof. Let $G = K_n$. Let $m = \lfloor (k-1)/2 \rfloor$. Let $A = \{a_1, \cdots, a_m\}$ be a set of m vertices in G. Let $B = V(G) - A$. We define a coloring c of $E(G)$ as follows. First color the edges between A and B by assigning color i to all of those incident to a_i. Then we use a new set of colors to color the edges within A such that those edges all have different colors. Finally, we assign a new color to all the edges within B.

It is easy to check that c has minimum color degree at least $\lfloor (k-1)/2 \rfloor$. We show that it contains no properly colored cycle of length at least k. Consider a properly colored cycle L in c. Clearly, L cannot lie completely in B since all edges in B have the same color. If L lies completely in A, then it has length at most $\lfloor (k-1)/2 \rfloor < k$. So we may assume that L contains vertices in both A and B. Observe that since L is properly colored, each vertex on L must have at least one of its two neighbors on L lie in A. From this one can easily deduce that at least half of the vertices on L lie in A. Hence, in particular, L has length at most $2|A| = 2m < k$. ■

Given positive integers m, n with $m \geq \lceil \log_2 n \rceil$, by the n smallest binary m-tuples, we mean the m-tuples that are binary representations of $1, 2, \cdots, n$ (with preceding 0s if necessary). For example, the 5 smallest binary 3-tuples are $001, 010, 011, 100, 101$.

Proposition 3.3 *For fixed k and sufficiently large n, we have*

$$g(n, C_k) \geq \lceil \log_2 n \rceil - 1.$$

Proof. Let $m = \lceil \log_2 n \rceil$. We define a coloring of $E(K_n)$ with minimum color degree at least $m-1$ that contains no rainbow cycles at all. Label the vertices of K_n distinctly using the n smallest binary m-tuples. For every pair $u, v \in V(K_n)$, let $c(uv)$ be the lowest position that the two tuples differ. It is easy to see that the color degree of any vertex u is at least $m-1$, and that the coloring c contains no rainbow cycles. ■

Next, we give a general upper bound on $g(n, C_k)$. We make an easy observation first, which appeared in several previous papers.

Lemma 3.4 *Let n, k be positive integers with $n \geq k$. Let c be a coloring of $E(K_n)$ that has a rainbow cycle C of length $2k-2$ then c also contains a rainbow cycle of length k.*

Proof. Let u, v denote two vertices on C at distance $k-1$. One of the u, v-portions of C avoids color $c(uv)$ (since C is rainbow), thus completing a C_k with uv. ■

Proposition 3.5 *For fixed k and sufficiently large n, we have $g(n, C_k) \leq n/2 + o(n)$.*

Proof. For even k, this follows from Theorem 2.3. For odd k, we have by Lemma 3.4 $g(n, C_k) \leq g(n, C_{2k-2}) \leq n/2 + o(n)$. ■

Obviously, the lower and upper bounds in Propositions 3.5 and 3.3 are not satisfactory in general. They even do not give the order of magnitude for $g(n, C_k)$. However, as we shall see from the next few results that the order of magnitude of $g(n, C_k)$ most likely varies with the value of k. We shall show that $g(n, C_k)$ is sublinear in n if $k = 3$ or 4 and that it is linear in n if $k \equiv 2 \pmod 4$). For $k = 3$, by a result of Erdős and Tuza ([11], Theorem 1), we have $g(n, C_3) \leq \lceil \log_2 n \rceil + O(1)$. By Proposition 3.3, we have $g(n, C_3) \geq \lceil \log_2 n \rceil - 1$. Hence

Proposition 3.6 $f(n, C_3) = g(n, C_3) = (1 + o(1)) \log_2 n$.

For $k = 4$ we are able to improve the general upper bound to $g(n, C_4) = O(n^{2/3})$. In our proof below, we drop ceiling and floor signs whenever they are not crucial, and we do not attempt to optimize absolute constants.

Theorem 3.7 *Let n be a sufficiently large positive integer. Then every edge-coloring c of $K = K_n$ with minimum color degree at least $4n^{2/3}$ contains a rainbow cycle of length 4.*

Proof. Define a color used in c to be *sparse* if it appears on at most $n^{4/3}$ edges of K, other colors used in c will be called *dense*. Clearly, we have

$$\# \text{ dense colors} \leq n^2/n^{4/3} = n^{2/3}. \tag{1}$$

Let H denote the subgraph of K consisting of edges using sparse colors. Let c_H denote c restricted to H. Let q denote the minimum color degree of c_H. Clearly, by (1) we have

$$q \geq 4n^{2/3} - n^{2/3} = 3n^{2/3}. \tag{2}$$

Let S denote a largest monochromatic star in H. Suppose S is centered at x and has s leaves, each connected to x with an edge of color 1. Note that $|U| = s \leq n - q$ since c_H has minimum color degree q. Let T denote a rainbow star on $q/4$ edges centered at x, such that color 1 is not used in T and $V(T)$ is disjoint from U. Since H has minimum color degree q, such T clearly exists.

Let V denote the set of leaves of T. Suppose $V = \{v_1, v_2, \cdots, v_p\}$, where $p = q/4$. For each $i = 1, 2, \cdots, p$, let S_i denote a rainbow star with $q/2$ edges in H centered at v_i such that (1) S_i does not use color 1 or any of colors used in T, and (2) its set of leaves L_i is disjoint from $V(T)$. Since H has minimum color degree at least q, such stars S_i's exist. Let $W = \cup_{i=1}^{p} L_i$. Note that W is disjoint from $V(T)$ but it may overlap with U.

Let $F = \bigcup_{i=1}^{p} S_i$. Note that by definition, F does not use color 1 or any of the colors used in T, and $e(F) = \Sigma_{i=1}^{p}|S_i| \geq |V|q/2 = q^2/8$. Suppose $W = \{w_1, w_2, \cdots, w_t\}$. If some w_j is incident to two edges, say v_iw_j and $v_{i'}w_j$ of different colors in F, then these two edges complete a rainbow cycle of length 4 with v_ix and $v_{i'}x$. Hence we may assume that for each $j = 1, \cdots, t$, the edges in F incident to w_j all have the same color α_j.

Call a vertex $w_j \in W$ *bad* if it is incident to at most one edge of F. Let B denote the set of bad vertices in W. Let $W^* = W - B$ and $F^* = F - B$. Note that $e(F^*) \geq e(F) - |B| \geq q^2/8 - n$. Note also that each $w_j \in W^*$ is incident to at least two edges of F (with color α_j). We consider two cases.

Case 1. There exists w_j that sends an edge w_ju of K to some vertex $u \in U$ such that $c(w_ju) \notin \{1, \alpha_j\}$.

Suppose v_iw_j and $v_{i'}w_j$ are two edges (with color α_j) of F incident to w_j. Since T is rainbow one of v_ix and $v_{i'}x$ does not use color $c(w_ju)$. Suppose without loss of generality that v_ix does not use color $c(w_ju)$. Note by our construction that $c(w_ju) \notin \{1, \alpha_j\}$. Now, xv_iw_jux is rainbow cycle of length 4.

Case 2. For each $w_j \in W^*$, all the edges of K from w_j to U use colors either 1 or α_j.

For each $w_j \in W^*$, let a_j denote the number of edges of F incident to w_j (note that they all have color α_j), and let b_j denote the number of edges of K with color 1 from w_j to U (note that w might lie in U). By our assumption, the other edges of K between w_j and U all have color α_j; there are at least $|U| - 1 - b_j$ of them. Since V is disjoint from U, w_j is then incident to at least $a_j + |U| - 1 - b_j = |U| + (a_j - b_j - 1)$ edges of K with color α_j. Since we have chosen S to be a largest monochromatic star in a sparse color, and α_j is also a sparse color, we must have $|U| + (a_j - b_j - 1) \leq |U|$. Hence $a_j \leq b_j + 1$ for all $w_j \in W^*$. Now, we have

$$q^2/8 - n \leq e(F^*) = \Sigma_{w_j \in W^*} a_j \leq \Sigma_{w_j \in W^*}(b_j + 1) \leq n^{4/3} + n, \qquad (3)$$

where the last inequality follows from the fact that color 1 is a sparse color. By (2) and (3), we have

$$9n^{4/3} \leq q^2 \leq 8(n^{4/3} + 2n),$$

a contradiction. This completes our proof. ■

Although we have sublinear upper bounds for C_3 and C_4, in general, however, $g(n, C_k)$ can be linear in n for certain values of k, as indicated by the next Proposition.

Proposition 3.8 *Suppose $k > 4$ is a positive integer with $k \equiv 2 \pmod 4$. Then*

$$g(n, C_k) \geq \lfloor n/4 \rfloor.$$

Proof. Let $K = K_n$. Suppose the vertices of K are $x_1, \cdots, x_n$. Let V_0, V_1, V_2, V_3 be a partition of $V(K)$ into almost equal subsets, each of size $\lfloor n/4 \rfloor$ or $\lceil n/4 \rceil$. Let C_n^4 denote the spanning subgraph of K containing all the edges between V_i and V_{i+1}, for all $i = 0, \cdots, 3$ (with subscripts taken modulo 3). Next, we define an orientation $D(C_n^4)$ of C_n^4 by orienting all the edges between V_i and V_{i+1} from V_i to V_{i+1} for $i = 0, 1, 2, 3$.

We now define a coloring c of $V(K_n)$ in two steps. Step 1, we color the edges in $D(C_n^4)$ by assigning, for each vertex x_i, color i to all edges with tail x_i, where colors $1, \cdots, n$ are all different. Step 2, we assign a new color 0 to all the remaining edges in K. Clearly, c has minimum color degree at least $\lfloor n/4 \rfloor$. To complete the proof it suffices to show that c contains no rainbow cycles of length $2 \pmod 4$. Let C be a rainbow cycle in c. We consider two cases. In each case, we show that C has length $0, 1$, or $3 \pmod 4$.

Case 1. C does not use color 0.

Then C lies completely in $D(C_n^4)$. Observe that C has to be a directed cycle in $D(C_n^4)$; otherwise C contains a vertex x_j of out-degree two in $D(C_n^4)$ restricted to C. These two out-edges both receive color j by our definition of c, contradicting C being rainbow. Now, since C is a directed cycle in $D(C_n^4)$, clearly it has length $0 \pmod 4$.

Case 2. Color 0 appears on an edge uv in C.

Since C is rainbow, uv is the only edge on C with color 0. Note that since uv has color 0, we have either u, v belong to the same V_i or $u \in V_i$ and $v \in V_j$ where $j \equiv i + 2 \pmod 4$. Let $P = C - uv$. Then P lies completely in $D(C_n^4)$. Since P is rainbow, by a similar argument as in Case 1, either P is a directed path in $D(C_n^4)$ or P contains a unique sink w. We consider two subcases.

Subcase 2.1 P is a directed path in $D(C_n^4)$.

Then P has length either $0 \pmod 4$ if u, v belong to the same V_i or $2 \pmod 4$ if u, v belong to V_i, V_j, respectively, with $j = i + 2 \pmod 4$. So, C has length either $1 \pmod 4$ or $3 \pmod 4$.

Subcase 2.2 P contains a unique sink w.

Let P_1 denote the portion of P from u to w, and let P_2 denote the portion of P from v to w. Let l, l_1, l_2 denote the length of P, P_1, P_2, respectively. If u, v belong to the same V_i, then $l_1 \equiv l_2 \pmod 4$. Thus, $l \equiv 2l_1 \pmod 4$, implying that $l \equiv 0$ or $2 \pmod 4$, and hence C has length 1 or $3 \pmod 4$. If u, v belong to V_i, V_j, respectively, with $j \equiv i + 2 \pmod 4$, then $l_1 \equiv l_2 + 2 \pmod 4$, and so $l \equiv 2l_2 + 2 \equiv 0$ or $2 \pmod 4$. Again, C has length 1 or $3 \pmod 4$. ■

We now summarize our bounds on $g(n, C_k)$. The upper bound comes from Theorem 2.3

Proposition 3.9 *Let $k \geq 3$ be a fixed integer and n a sufficiently positive large integer. We have*

$$\lceil \log_2 n \rceil - 1 \leq g(n, C_k) \leq n/2 + o(n).$$

Furthermore, $g(n, C_3) = (1 + o(1)) \log_2 n$, $g(n, C_4) = O(n^{2/3})$, and if $k \equiv 2 \pmod 4$ then $g(n, C_k) \geq \lfloor n/4 \rfloor$.

In particular, we know the order of magnitude of $g(n, C_k)$, when $k = 3$ and when $k \equiv 2 \pmod 4$. We also have an upper bound on the order $n^{2/3}$ for C_4. In general, we suspect that the order of magnitude of $g(n, C_k)$ may vary with the value of k.

Let us now go back to $f(n, C_k)$. It appears that forcing properly colored cycles is considerably easier than forcing rainbow cycles. Our speculation is that $f(n, C_k)$ may be bounded from above by a constant depending only on k (independent of n). We are able to determine the exact value of $f(n, C_4)$, which is surprisingly small.

First, let us consider the following coloring of $E(K_n)$. Let x_1, x_2 be two vertices of K_n. Let $X = \{x_1, x_2\}$ and $Y = V(K_n) - Y$. Color the edges of K_n as follows: for $i = 1, 2$, assign color i to all the edges between x_i and Y. Then assign color 1 to all the edges within Y and color 3 to all the edges within X. It is easy to see that the coloring defined above has minimum color degree two and contains no properly colored C_4. Thus $f(n, C_4) \geq 2$. We show next that raising the minimum color degree to be at least 3 would ensure a properly colored C_4 and hence $f(n, C_4) = 2$.

Theorem 3.10 *For $n \geq 4$ every coloring of $E(K_n)$ with minimum color degree at least 3 contains a properly colored C_4. In particular, we have $f(n, C_4) = 2$.*

Proof. Let c be an edge-coloring of K_n with minimum color degree at least 3. Let G be a minimal complete subgraph of K_n such that c restricted to G has minimal color degree at least 3. We show that G contains a properly colored C_4. For convenience, we henceforth use c to denote c restricted to G. If $n(G) = 4$ then c is a proper coloring of $E(G)$ and the claim holds trivially. So we may assume that $n(G) \geq 5$.

Suppose G does not contain a properly colored C_4, we derive a contradiction. By our choice of G, for each vertex u in G, c restricted to $G - u$ has minimum color degree 2. Let $A(u)$ denote the set of vertices with color degree 2 in c restricted to $G - u$. Create a digraph D on $V(G)$ with edge set $\bigcup_{u \in V(G)} \{uv : v \in A(u)\}$. By the definition of D and the assumption that c contains no properly colored C_4 we can make the the following observations about D.

(1) Every vertex u in D has out-degree at least 1.

(2) If $uv \in E(D)$, then v is incident to exactly three colors in G and uv is the only edge incident to v with color $c(uv)$.

(3) Every vertex v in D has in-degree at most 2.

(4) If uv and xy are two independent edges of D then $c(uv) \neq c(xy)$.

(5) If D' is a sub-digraph of D with maximum out-degree at most 1, then the edges of D' all have different colors.

Observations (1) and (2) follow immediately from the definition of D. Observation (3) follows from (2) and the fact that $n(D) \geq 4$. Observation (4) holds since suppose $c(uv) = c(xy) = \alpha$ then $C = uvxyu$

would be a properly colored C_4 (noting that by observation (2), $c(vx) \neq c(uv) = \alpha$ and $c(yu) \neq c(xy) = \alpha$). Finally, observations (2) and (4) together imply observation (5).

Observation (1) implies that D contains either a directed cycle or a double edge (joining two vertices in both directions). We consider three cases. In each case, we derive a contradiction.

Case 1. D has a directed cycle C of length at least 4.

By observations (2) and (5), the edges on C all have different colors and each vertex on C is incident to exactly three colors in G. Since G contains no properly colored C_4, C has length at least 5. Suppose $C = u_0u_1 \cdots u_m u_0$, where $m \geq 4$. For each $i = 0, \cdots, m$, suppose $u_{i-1}u_i$ uses color i (where subscripts are taken modulo m), by our discussion above, $0, 1, \cdots, m$ are all different colors, and for $i = 0, \cdots, m$, $u_{i-1}u_i$ is the only edge incident to u_i with color i.

Since $u_0u_1u_2u_3$ is a properly colored P_4 using colors $1, 2, 3$ in order and $c(u_0u_3) \neq a_3$, to avoid a properly colored C_4, we must have $c(u_0u_3) = 1$. This also implies that the three colors incident to u_3 in G are $1, 3, 4$. Similarly, we have $c(u_1u_4) = 2$. Now, since $c(u_1u_3) \in \{1, 3, 4\}$ and $c(u_1u_3) \neq 1$ or 3, we have $c(u_2u_4) = 4$.

Since $u_0u_3u_1u_4$ is a properly colored P_4 using colors $1, 4, 2$ in order, to avoid a properly colored C_4 we must have $c(u_0u_4) \in \{1, 2\}$. If $c(u_0u_4) = 2$ then $u_0u_1u_2u_4u_0$ is a properly colored C_4, noting that $c(u_2u_4) \neq 2$, a contradiction. So, $c(u_0u_4) = 1$, thus C has length at least 6 otherwise we would have two edges in C having the same color. Moreover, in this case u_4 is incident to four different colors $1, 2, 4, 5$, again a contradiction.

Case 2. D has a cycle T of length 3 and no longer cycle.

Let $T = u_1u_2u_3$ be a directed triangle in D. We first prove that there exists a set S of at most four vertices containing $V(T)$ such that no vertex in $D - S$ sends an edge into S in D. If no vertex outside T sends an edge into T, we let $S = V(T)$. Otherwise, let w be a vertex that sends an edge into T. Without loss of generality, suppose $wv_1 \in E(D)$, let $S = V(T) \cup \{w\}$. Suppose there exists a vertex w' outside S that sends an edges $w'x$ into S, where $x \in S$, we will derive a contradiction.

By observations (2) and (5), the colors used on edges $u_1u_2, u_2u_3, u_3u_1, wu_1, w'x$ are distinct, let them be $1, 2, 3, 4, 5$, respectively. Each of u_1, u_2, u_3 is incident to exactly three colors in G. Since $wu_1u_2u_3$ is a properly colored P_4 using colors $4, 1, 2$ in order and $c(wu_3) \neq c(u_2u_3) = 2$, we must have $c(wu_3) = 4$ to avoid a properly colored C_4.

If $x = u_1$ then we clearly also have $c(w'u_3) = 5$, in which case u_3 is incident to four colors $2, 3, 4, 5$, a contradiction. If $x = u_3$, then u_3 is incident to four colors $2, 3, 4, 5$, a contradiction. If $x = u_2$, then we switch the roles of w and w' and obtain a similar contradiction. This leaves the only possibility that $x = w$. In this case, $c(w'u_1) = 1$ since $c(w'u_1) \in \{1, 3, 4\}$ and $c(w'u_1) \neq 3$ or 4. Now, $w'u_1u_3ww'$ is a properly colored C_4 using colors $1, 3, 4, 5$ in order, again a contradiction.

We have shown that $V(D) - S$ sends no edge into S. Since D has minimum out-degree at least 1, so does $D - S$. Hence $D - S$ contains either a directed triangle or a double edge. We consider the two cases, assuming the colors used on u_1u_2, u_2u_3, u_3u_1 are $1, 2, 3$, respectively.

Subcase 2.1. $D - S$ contains a directed triangle $T' = \{v_1v_2v_3\}$.

By observations (2) and (5) the colors used on v_1v_2, v_2v_3, v_3v_1 are distinct and different from colors on T, say they are $5, 6, 7$ respectively. Each of $u_1, u_2, u_3, v_1, v_2, v_3$ is incident to exactly three colors in G.

Now, since $u_3u_1v_3v_2$ is a properly colored P_4 (noting that $c(u_1v_3) \notin \{3, 6\}$ by observation (2)), we must have $c(u_3v_2) \in \{3, 6\}$ to avoid a properly colored C_4. By symmetry, we may assume that $c(u_3v_2) = 3$. Now, since $u_1u_2u_3v_2$ is a properly colored P_4 using colors $1, 2, 3$ in order and $c(u_1v_2) \neq c(u_3u_1) = 3$, we must have $c(u_1v_2) = 1$. But then v_2 is incident to four colors $1, 3, 5, 6$, a contradiction.

Subcase 2.2 $D - S$ contains a double edge $xy \cup yx$.

Suppose the colors used on xy in G is 5. As before, colors $1, 2, 3, 5$ are all different, and each of u_1, u_2, u_3, x, y is incident to exactly three colors in G. Since $c(u_2y) \notin \{1, 5\}$, u_1u_2yx is a properly colored P_4. This forces $c(u_1x) = 1$ (noting that $c(u_1x) \neq 5$) in order to avoid a properly colored C_4. By similar arguments, we have $c(u_2x) = 2$ and $c(u_3x) = 3$. But then x is incident to four colors $1, 2, 3, 5$, a contradiction.

Case 3. D contains a double edge $uv \cup vu$ and no directed cycles.

If D contains another double edge $xy \cup yx$ which is independent of $uv \cup vu$, then by observation (2), one can easily see that $uvxyu$ is a properly colored C_4. So we may assume that D has no double edge independent of D.

Let D' be a maximal subgraph of D containing uv such that the underlying graph G' of D' is a tree and all edges in D' other than uv are oriented towards $\{u, v\}$. By the definition of D', $D - V(D')$ sends no edge into $V(D')$. If $V(D) - V(D') \neq \emptyset$, then $D - V(D')$ has minimum outdegree at least 1, which then contains a double edge independent of $uv \cup vu$, a contradiction. So D' spans D. In particular, D' has at least 5 vertices.

Note that D' contains no vertex of in-degree more than two by observation (3). By symmetry, we may assume that there exist vertices x, y such that $xy, yu \in E(D')$. Clearly, D' contains another edge wz where $w \notin \{u, v, x, y\}$ and $z \in \{u, v, x, y\}$. Suppose the colors used on xy, yu, uv, wz are $1, 2, 3, 4$, respectively. By observations (2) and (5), colors $1, 2, 3, 4$ are all different, and each of u, v, y is incident to exactly three colors in G.

Suppose first that $z = x$ then $c(wu) = 4$ by an argument like before. By observation (2), $c(xu) \notin \{2, 3, 4\}$ otherwise $vuxyv$ forms a properly colored C_4. This forces u to be incident to four different colors, a contradiction. Next, suppose $z = y$, then $c(wv) = 4, c(xv) = 1$. Now by observation (2), $c(yv) \notin \{1, 3, 4\}$, this again forces v to be incident to four different colors. Finally, suppose $z = v$. In this case we have $c(xv) = 1$ and $c(yv) \notin \{1, 3, 4\}$, forcing v to be incident to four different colors. This completes our proof. ■

We now summarize our bounds on $f(n, C_k)$. The general upper bounds follow from those on $g(n, C_k)$.

Proposition 3.11 *Let $k \geq 3$ be a fixed integer and n a sufficiently positive large integer. We have*

$$\lfloor (k-1)/2 \rfloor \leq f(n, C_k) \leq n/2 + o(n).$$

Furthermore, $f(n, C_3) = (1 + o(1))n/2$, and $f(n, C_4) = 2$.

Any improvement of these bounds for $k \geq 5$ would be interesting. Finally, we consider a related problem by relaxing our requirement. We are now interested in the threshold on the minimum color degree that forces properly colored cycles of length at least k. Our bounds for this threshold are much better than for f. Let $\theta(n, k)$ denote the maximum l such that there is an l-good coloring with no properly colored cycle of length at least k.

Using the construction from Proposition 3.2 we have $\theta(n, k) \geq \lfloor (k-1)/2 \rfloor$. On the other hand, we have

Theorem 3.12 *For fixed n, k, if c is a coloring of K_n with minimum color degree at least $3k/2$, then c contains a properly colored cycle of length at least k. Hence,*

$$\lfloor (k-1)/2 \rfloor \leq \theta(n, k) \leq 3k/2.$$

Proof. Let P be a longest properly colored path in c. Let u, v denote the two endpoints of P. Orient P from u to v. For a vertex x on P, we use x^+ and x^- to denote its successor and predecessor, respectively. Let $a_1 = c(uu^+)$, $b_1 = c(vv^-)$. Let $a_2, \cdots, a_p$, where $p \geq 3k/2$, denote other colors incident to u in c. For each $i = 2, \cdots, p$, let ux_i be an edge with $c(ux_i) = a_i$. By our choice of P, $x_2, \cdots, x_p$ all lie on P. Suppose without loss of generality that $x_2, \cdots, x_p$ are at increasing distance from u on P. Define $b_1, \cdots, b_q$ and $y_2, \cdots, y_q$ similar to $a_2, \cdots, a_p$ and $x_2, \cdots, x_p$ for the other endpoint v of P, here $q \geq 3k/2$.

Now consider the edges ux_k and vy_k. If $c({x_k}^-, x_k) \neq a_k$ then $P[u, x_k] \cup ux_k$ is properly colored cycle of length at least k. So we may assume that $c({x_k}^-, x_k) = a_k$, and hence $c(x_k, x_k^+) \neq a_k$. Similarly, we may assume that $c(y_k, y_k^+) = b_k$ and $c(y_k^-, y_k) \neq b_k$. We consider two cases.

Case 1. x_k lies between y_k and v in P.

In this case, $P[u, y_k] \cup y_k v \cup P[x_k, v] \cup ux_k$ is a properly colored cycle of length at least k.

Case 2. x_k lies between u and y_k in P.

Let $m = \lfloor k/2 \rfloor$. Since $C' = P[x_m, y_m] \cup x_m y_m$ is cycle of length at least k, we may assume that either $c(x_m, y_m) = c(x_m, x_m^+)$ or $c(x_m, y_m) = c(y_m^-, y_m)$, otherwise C' is properly colored. By symmetry we may assume that $c(x_m, y_m) = c(x_m, x_m^+)$. Then $c(x_m, y_m) \neq c(x_m^-, x_m)$. If $c(x_m, y_m) \neq c(y_m^-, y_m)$ then $P[u, x_m] \cup x_m y_m \cup P[x_k, y_m] \cup ux_k$ is a properly colored cycle of length at least k. Otherwise, $P[u, x_m] \cup x_m y_m \cup P[y_m, v] \cup vy_k \cup P[x_k, y_k] \cup ux_k$ is a properly colored cycle of length at least k. ■

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