ON A GENERALIZED ANTI-RAMSEY PROBLEM

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ABSTRACT. For positive integers p, q_1, q_2, a coloring of $E(K_n)$ is called a (p, q_1, q_2)-coloring if the edges of every K_p receive at least q_1 and at most q_2 colors. Let $R(n, p, q_1, q_2)$ denote the maximum number of colors in a (p, q_1, q_2)-coloring of K_n. Given (p, q_1) we determine the largest q_2 such that all (p, q_1, q_2)-colorings of $E(K_n)$ have at most $O(n)$ colors and we determine $R(n, p, q_1, q_2)$ asymptotically when it is of order equal to n^2. We give several bounds and constructions.

1. INTRODUCTION

The classical Ramsey problem asks for the minimum n such that every k-coloring of the edges of K_n yields a monochromatic K_p. For each n below this threshold, there is a k-coloring such that every K_p receives at least 2 colors. We may study the same problem by fixing n and asking for the minimum k such that $E(K_n)$ can be k-colored with each p-clique receiving *at least* 2 colors. For integers n, p, q, a (p, q)-coloring of K_n is a coloring in which the edges of every K_p-subgraph receive *at least* q colors and we denote the minimum number of colors in a (p, q)-coloring of K_n by $f(n, p, q)$. This function was first studied in this form by Elekes, Erdős, and Füredi (as described in Section 9 of [5]). Using the Lovász Local Lemma, Erdős and Gyárfás [6] proved that $f(n, p, q) = O(n^{c_{p,q}})$, where $c_{p,q} = (p-2)\big/\left(\binom{p}{2} - q + 1\right)$. They also determined, for each p, the smallest q such that $f(n, p, q)$ is linear in n and the smallest q such that $f(n, p, q)$ is quadratic in n. Axenovich, Füredi and Mubayi [2] generalized this notion by defining an (H, q)-coloring of the edges of a graph G to be any coloring assigning at least q colors to every copy of H in G. They gave several general bounds and determined the asymptotic behavior of the minimum number of colors in an (H, q)-coloring of G for the case when G and H are complete bipartite graphs. These type of problems can be summarily called *Generalized Ramsey Problems.*

Much earlier, Erdős, Simonovits and Sós [8] defined the *Anti-Ramsey number* $f(n, H)$ to be the maximum number of colors on the edges of K_n such that every copy of H receives at most $|E(H)| - 1$ colors. They established a close relationship between this problem and Turán-type problems and they studied the case when H is a path or a cycle in detail. In [1] Ahlswede, Cai and Zhang introduced the concept of a *rich coloring* of certain hypergraphs. One problem they address is that of finding the maximum number of colors in a coloring of $E(K_n)$, such that every K_p contains *at most* q colors, which they denote by $N(n, p, 2, q)$. They prove that for $n \geq p \geq 3$ we have $N(n, p, 2, p-1) = n-1$ and $N(n, p, 2, t(p, k) + q') = t(n, k) + q'$ for $1 \leq k$ and $1 \leq q' \leq \lfloor \lfloor \frac{p}{k} \rfloor /2 \rfloor$. Here the *Turán number* $t(n, k)$ is the number of edges in the *Turán graph* $T(n, k)$, that is the complete k-partite graph on n vertices with all partition sets of size $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$. This problem is strongly related to the following Turán-type problem studied by Griggs, Simonovits and Thomas [12]: Find the maximum number of edges in a graph on n vertices, such that every k-set contains at most r edges. If we denote this parameter by $\mathrm{ex}(n, k, r)$ then $\mathrm{ex}(n, p, q-1) + 1 \leq N(n, p, 2, q) \leq \mathrm{ex}(n, p, q)$. Indeed, multicoloring a graph achieving $\mathrm{ex}(n, p, q-1)$ and coloring its complement with a new color gives the lower bound, whereas the upper bound follows by considering one edge from each color in a coloring achieving $N(n,p,2,q)$. In many cases the lower bound is tight.

In this paper we introduce the notion of a (p,q_1,q_2)-coloring to combine the Ramsey- and Turán-type problems mentioned above into a common framework.

For a coloring $c : E(K_n) \to C$, not necessarily a proper edge-coloring in the usual sense, and a subgraph H of K_n we denote the set of colors appearing on the edges of H by $c(H)$. If $A \subset V(K_n)$, then we let $c(A) := c(K_n|_A)$ be the set of colors in the subgraph induced by A.

Definition 1. Let p, q_1, q_2 be integers such that $p \geq 3$ and $1 \leq q_1 \leq q_2 \leq \binom{p}{2}$. A coloring of $E(K_n)$ is called a (p,q_1,q_2)*-coloring* if $q_1 \leq |c(A)| \leq q_2$ for every $A \subset V(K_n)$ with $|A| = p$.

For example $f(n,p,q)$ is the minimum number of colors $|c(K_n)|$ in a $(p,q,\binom{p}{2})$-coloring c of K_n. In [3] Chung and Graham determine the minimum number of colors in a $(3,2,2)$- and a $(4,3,3)$-coloring and give various other results on what they call precise colorings. Similarly $N(n,p,2,q)$ is the maximum number of colors in a $(p,1,q)$-coloring. In general we have the following

Definition 2. The maximal number of colors in a (p,q_1,q_2)-coloring of K_n is called $R(n,p,q_1,q_2)$.

So $R(n,p,1,q) = N(n,p,2,q)$. The problem of determining $R(n,p,q_1,q_2)$ can be considered as a *Generalized Anti-Ramsey Problem.* We immediately get

Observation 1. If $q_1' \leq q_1 \leq q_2 \leq q_2'$ then every (p,q_1,q_2)-coloring is a (p,q_1',q_2')-coloring and $R(n,p,q_1,q_2) \leq R(n,p,q_1',q_2')$.

Definition 3. A coloring $c : E(K_n) \to \{c_1 < c_2 < ... < c_{n-1}\}$ is called *canonical* if there is an ordering of the vertices of K_n, $v_1, \ldots, v_n$ such that $c(v_iv_j) = \max\{c_i, c_j\}$. A coloring is called *monochromatic* if $c(uv) = c(xy)$ for all $u,v,x,y \in V(K_n)$, and a *multicoloring* or *rainbow* coloring if $c(uv) \neq c(xy)$ for $uv \neq xy$.

The following result is due to Erdős and Rado [7].

Theorem 1 (Canonical Ramsey Theorem). *There is a constant C_p such that every coloring of $E(K_n)$, $n > C_p$, contains a K_p that is monochromatic, rainbow or canonically colored.*

Theorem 1 is an important tool and Lefmann and Rödl [15] have shown that $2^{cp^2} \leq C_p \leq 2^{c'p^2 \log p}$ for some constants c, c'. Theorem 1 essentially settles the question for which parameters (p,q_1,q_2)-colorings exist.

Proposition 2. *If $n > C_p$ then a (p,q_1,q_2)-coloring of $E(K_n)$ exists if and only if $q_1 = 1$, $q_2 = \binom{p}{2}$ or $q_1 \leq p-1 \leq q_2$.*

Proof. The necessity follows from Theorem 1 since for $n > C_p$ every coloring of K_n contains a p-set with 1, $p-1$ or $\binom{p}{2}$ colors. The conditions are sufficient, since coloring K_n monochromatically , canonically or rainbow, respectively, yields a (p,q_1,q_2)-coloring. □

Trivially, $R(n,p,q_1,\binom{p}{2}) = \binom{n}{2}$, so that we will assume from now on that $q_2 < \binom{p}{2}$ and $q_1 < p$. In Section 2 we asymptotically determine $R(n,p,q_1,q_2)$ when it is quadratic in n. In Sections 3 and 4 we will determine when $R(n,p,q_1,q_2) = \mathrm{O}(n)$ and give constructions and rough bounds. Finally in Section 5 we collect some bounds in the case $p = 3, 4$.

2. The quadratic case

The case $q_1 = 1$ has been solved in a paper of Erdős, Simonovits and Sós [8] but since the proof follows easily along the lines of our proof for $q_1 > 1$ we include it for completeness. Before we determine the asymptotic value of $R(n, p, q_1, q_2)$ we need a technical lemma:

Lemma 1. *For any complete k-partite graph H with partite sets $A_1, \ldots, A_k$ there is a complete k-partite graph H' with the following property: If a coloring c of K_n contains a rainbow copy of H', then it contains a rainbow copy of H with the additional property that the colors in $c(A_i)$ are not in $c(H)$, that is*

$$(c(A_1) \cup \cdots \cup c(A_k)) \cap c(H) = \emptyset.$$

Proof. Let $a_{i,0} = |A_i|$ and recursively define for $1 \leq j \leq k$ that $a_{j,j} = a_{j,j-1}$ and $a_{i,j} = a_{i,j-1} + \binom{a_{j,j}}{2}$ for $i \neq j$. H' will be a complete k-partite graph with parts of size $a_{i,k}$, $1 \leq i \leq k$. Now suppose a coloring c contains a rainbow H'. We will find a sequence of complete k-partite graphs $H' = H_k \supset H_{k-1} \supset \cdots \supset H_1 \supset H_0$, such that the partite sets of each H_j, denoted by A_i^j $(1 \leq i \leq k)$, are of size $a_{i,j}$ and for $0 \leq j \leq k$

$$(c(A_{j+1}^j) \cup c(A_{j+2}^j) \cup \cdots \cup c(A_k^j)) \cap c(H_j) = \emptyset.$$

Hence H_0 will be the desired copy of H. To obtain H_{j-1} from H_j observe that in H_j we have $|c(A_j^j)| \leq \binom{a_{j,j}}{2}$ and it suffices to remove these colors from $c(H_j)$. Every one of these colors appears on at most one edge in H_j and every such edge has at least one endpoint not in A_j^j. Thus removing one such vertex for every color in $c(A_j^j)$, and removing some extra vertices if necessary, we obtain a subgraph H_{j-1} with part sizes $a_{i,j-1}$ and the colors in $c(A_i^{j-1})$, $j \leq i \leq k$, do not appear in $c(H_{j-1})$. □

Proposition 3. *For $2 \leq k \leq p$*

$$R(n, p, 1, t(p, k)) \leq t(n, k-1) + o(n^2) \text{ if } k < p, \text{ and}$$
$$R(n, p, 2, t(p, k) + p - k - 1) \leq t(n, k-1) + o(n^2).$$

Proof. Let H be the complete k-partite graph with each part of size $C_p + 1$ and let H' be the supergraph of H obtained from Lemma 1. We claim that H' can not be multicolored in any (p, q_1, q_2)-coloring under consideration. If this is the case, then picking one edge from every color we obtain a graph containing no H' and thus by a theorem of Erdős, Simonovits and Stone [9, 10] we have at most $t(n, k-1) + \mathrm{o}(n^2)$ colors.

To prove the claim suppose that there is a coloring with a multicolored H' subgraph. Then by Lemma 1 we obtain a multicolored subgraph H such that the edges within the A_i do not use any color from H. In the case $q_1 = 1$ this yields a contradiction immediately, since we can pick a Turán subgraph $T(p, k) \subset H$ and have at least one extra color in one of its partite sets for a total of $t(p, k) + 1$ colors. In the case $q_1 = 2$ we can apply Theorem 1 to the A_i's and we will obtain that each one of them contains a canonical K_p. Again picking a Turán subgraph of H, this time restricting ourselves to vertices from the K_p, we obtain a p-set containing exactly $t(p, k) + p - k$ colors. □

Now we are in the position to prove the main result of this section:

Theorem 4. *If $k = \min\{k' : \ t(p, k') \geq q_2\} < p$, then*

$$R(n, p, 1, q_2) = t(n, k-1) + o(n^2).$$

If $2 \leq q_1 \leq p-1 \leq q_2 < \binom{p}{2}$ *and* $k = \min\{k' : \ t(p,k') + p - k' - 1 \geq q_2\}$, *then*

$$R(n,p,q_1,q_2) = t(n,k-1) + o(n^2).$$

Proof. In the case $q_1 = 1$ we get from Remark 1 and Proposition 3

$$R(n,p,1,q_2) \leq R(n,p,1,t(p,k)) \leq t(n,k-1) + \mathrm{o}(n^2).$$

A construction achieving this can be found by taking a rainbow colored $T(n,k-1)$ and giving the remaining edges the same new color, so that each p-set has at most $t(p,k-1)+1 \leq t(p,k)$ colors.

In the second case the upper bound follows similarly. If $t(p,k-1)+p-k < q_2 \leq t(p,k)+p-k-1$ then

$$R(n,p,q_1,q_2) \leq R(n,p,2,t(p,k)+p-k-1) \leq t(n,k-1) + \mathrm{o}(n^2).$$

The construction achieving this is a canonical coloring of K_n where the edges of a $T(n,k-1)$-subgraph get recolored into a multicoloring using new colors. Now each p-set has at least $p-1$ colors and at most $t(p,k-1)+p-k+1 \leq q_2$ colors. □

3. The linear case

In this section we will determine the triples (p,q_1,q_2) for which $R(n,p,q_1,q_2) = \mathrm{O}(n)$.

Theorem 5. $R(n,p,1,q) = O(n)$ *exactly when* $1 \leq q \leq p-1$.

Proof. It follows from [1] that for $q \leq p-1$

$$R(n,p,1,q) \leq R(n,p,1,p-1) = n-1.$$

Furthermore there are several constructions of graphs with girth $p+1$ and at least $cn^{1+\varepsilon}$ edges, the best of which can be found in [14]. Multicoloring such a graph, and coloring its complement with one new color results in a $(p,1,p)$-coloring with a super-linear number of colors. □

Proposition 6. *If* $q \leq 2p-4$ *then* $R(n,p,2,q) \leq C_p(n-p)+q$

Proof. We shall prove the statement by induction on n, where the case $n = p$ is trivial. For $n > p$ consider $x \in V$ and a set of vertices $x_1, \ldots, x_k \in V - x$ such that $c(xx_i)$ are distinct colors not in $c(V-x)$. If $k > C_p$ then $x_1, \ldots, x_k$ contains a canonically colored K_p and thus a canonically colored $K_{p-1} = (V', E')$. But then $x \cup V'$ spans $(p-1)+(p-2) = 2p-3$ colors, a contradiction. Thus x has at most C_p new colors, finishing the proof. □

The bound of $q \leq 2p-4$ is optimal, since we will show that $R(n,p,2,2p-3)$ is superlinear. The obvious construction would be to multicolor a girth $p+1$ graph and color the remaining edges canonically. However we can show that multicoloring a graph G and then coloring the remaining edges canonically with new colors can only result in a $(n,p,p-1,2p-3)$-coloring if G has at most $\mathrm{O}(n)$ edges (and no chordless cycle of length at least 4).

So we need to come up with a different construction. Our construction will consist of multicoloring a girth $\geq p$ subgraph of a hypercube, and coloring the remaining edges pseudo-canonically.

Lemma 2. *The* k*-dimensional hypercube* Q_k *contains a subgraph of girth at least* g *with at least* $ck^{(g-2)/(2g-6)}2^k$ *edges.*

Proof. We use the deletion method to get a simple probabilistic proof of this result. First observe that every cycle of length $2j$ is contained in some subhypercube Q_j. Indeed, since every edge is a swap of one coordinate, and every coordinate needs to be swapped back eventually in a cycle, we may have changed at most j coordinates. So if Q_j has c_j $2j$-cycles, then Q_k contains at most $\binom{k}{j}2^{k-j}c_j$ such cycles.

Hence if we pick a subgraph of Q_k with edge-probability $p > \sqrt{2/k}$, then the number of cycles of length at most $2r$ we expect is bounded by

$$\sum_{j=2}^{r}\binom{k}{j}2^{k-j}c_jp^{2j} \le c_r2^k\sum_{j=2}^{r}\frac{1}{j!}(kp^2/2)^j \le c_r2^k(kp^2/2)^r.$$

After deleting one edge from every short cycle we expect to have at least $pk2^{k-1} - c_r2^k(kp^2/2)^r$ edges left. If we take

$$p = \left(2^{r-2}/c_r\right)^{\frac{1}{2r-1}} k^{-\frac{r-1}{2r-1}},$$

then for k sufficiently large $p > \sqrt{2/k}$, and

$$pk2^{k-1} - c_r2^k(kp^2/2)^r = \left(2^{r-2}/c_r\right)^{\frac{1}{2r-1}} k^{\frac{r}{2r-1}}2^{k-2}.$$

□

Definition 4. For a given labeling of the vertices of K_n with distinct integral k-tuples we can define the following coloring: for all $u, v \in V(K_n)$, let $c(uv) = (u_1, ..., u_{i-1}, \max\{u_i, v_i\})$, where $(u_1, ..., u_k)$ and $(v_1, ..., v_k)$ are the labels of u and v respectively and $i := \min\{j : \ u_j \neq v_j\}$ is the *length* of the edge uv (with respect to this coloring). We call this coloring, and every coloring that is merely a relabeling of such a coloring, a *k-canonical coloring* (determined by the vertex representation).

Hence the canonical colorings are exactly the 1-canonical colorings.

Lemma 3. *For every p-set A, $1 \le p \le n$, of a k-canonical coloring c, $|c(A)| = p - 1$.*

Proof. By induction on p, with the result being trivial for $p = 1$. Let A be a set of $p \ge 2$ vertices and set

$$i = \min\{j : \exists u, v \in A, u_j \neq v_j\}.$$

Then A can be partitioned into non-empty sets $A_j = \{u \in A : u_i = j\}$, $j \in J$, for some index set $J \subset \mathbb{Z}$ of at least 2 elements. Since all vertices start with the same $i-1$ coordinates $(u_1, u_2, \dots u_{i-1})$ the edges joining A_j and $A_{j'}$ all have color $(u_1, u_2, \dots u_{i-1}, \max\{j, j'\})$. These $|J| - 1$ colors are distinct from the colors in $c(A_j)$, $j \in J$. Also the sets of colors $c(A_j)$ and $c(A_{j'})$ are disjoint for $j \neq j'$, so that

$$|c(A)| = |J| - 1 + \sum_{j\in J}|c(A_j)| = |J| - 1 + \sum_{j\in J}(|A_j| - 1) = |A| - 1.$$

□

Theorem 7. *$R(n, p, p-1, 2p-3) \ge cn\log^{(p+\varepsilon-2)/(2p+2\varepsilon-6)} n \ge cn\sqrt{\log n}$, if n is sufficiently large, and $\varepsilon = 0$ when p is even and $\varepsilon = 1$ otherwise.*

Proof. Let k be the minimal integer such that $n \le 2^k$. We are going to construct a coloring of K_n as follows. Find a subgraph G_0 of Q_k with girth at least $g = p + \varepsilon$ as in Lemma 2, and label its vertices by $\{0,1\}$-vectors of length k in the usual sense. We can pick an n-vertex subgraph G with at least

$e(G_0)\binom{n}{2}/\binom{2^k}{2} \geq (c/2)n\log^{(g-2)/(2g-6)} n$ edges. Our coloring will now consist of a rainbow G and the remaining edges are colored k-canonically as determined by the $\{0,1\}$-vectors of $V(G)$.

By Lemma 3 each p-set A has at least $p-1$ colors and we have to show that it spans at most $2p-3$ colors. The k-canonical coloring contributes at most $p-1$ colors to $c(A)$. If $e(G|_A) \leq p-2$ then $|c(A)| \leq 2p-3$ and we are done. If not, and $G|_A$ is acyclic, then $G|A$ is a tree and we will show that one of the $p-1$ colors in this underlying k-canonical coloring is not in $c(A)$.

Notice that if uv is an edge of maximum length among edges in A, then no other edge in A has the same color in the underlying k-canonical coloring. Furthermore, there is a u,v-path in $G|_A$, possibly the edge uv itself. Since uv is of length i, we have $u_i \neq v_i$, and there is an edge xx' on the path such that $u_i = x_i \neq x'_i = v_i$. Because the path is a subgraph of Q_k this is the only coordinate where x and x' differ, so that the length of xx' is i as well. But then the color on xx' was unique in the underlying k-canonical coloring and is not used in $c(A)$.

Finally, if $G|_A$ contains a cycle, then $\varepsilon = 0$ and $G|_A$ is a cycle on p vertices. In this case there are 2 disjoint uv-paths in $G|_A$ for every pair of vertices u, v. But then by the same argument as above 2 colors from the underlying k-canonical coloring are not used in $c(A)$, finishing the proof. □

Corollary 8. *Let* $2 \leq q_1 \leq p-1 \leq q_2 < \binom{p}{2}$. $R(n,p,q_1,q_2) = O(n)$ *exactly when* $q_2 \leq 2p-4$.

Proof. The necessity follows from Theorem 7 and the sufficiency from Proposition 6. □

Theorem 7 still leaves the question if there is an $\varepsilon > 0$, such that $R(n,p,q_1,2p-3) \geq n^{1+\varepsilon}$. The following upper bound is the best we currently have in this case.

Proposition 9. *If* $p \geq 4$ *then* $R(n,p,2,2p-3) = O(n^{5/3})$.

Proof. Let G be a rainbow subgraph with the maximum number of edges in a $(p,2,3(p-3)-1)$-coloring of K_n. First we notice that G does not contain a $K_{3,p-3}$. By a result of Kövari, Sós and Turán [13] the maximal size of a graph without $K_{3,t}$, $3 \leq t$, is at most $Cn^{5/3}$. Thus the bound follows immediately for $p \geq 7$ since

$$R(n,p,2,2p-3) \leq R(n,p,2,3(p-3)-1) \leq |E(G)| \leq Cn^{5/3}.$$

A $(6,2,9)$-coloring does not contain a rainbow $K_{3,4}$ with vertex set $\{u_1,u_2,u_3\} \cup \{v_1,\ldots,v_4\}$, since at most one edge u_iv_j has color $c(u_1u_2)$, so that $V(K_{3,4}) - v_j$ spans at least 10 colors. A similar case analysis shows that a $(5,2,7)$-coloring does not contain a rainbow $K_{3,5}$.

Finally, a $(4,2,5)$-coloring does not contain a rainbow $K_{3,5}$ with partite sets $U = \{u_1,u_2,u_3\}$ and $V = \{v_1,\ldots,v_5\}$.
Case 1. U induces a monochromatic triangle, say of color α. We may assume that U is joined to $V - v_5$ by colors different from α. Since there is no monochromatic K_4, we can also assume that $c(v_1v_2) \neq \alpha$. At most one u_iv_j has color $c(v_1v_2)$, say $i = 3$, so that $\{u_1,u_2,v_1,v_2\}$ induces a rainbow K_4, a contradiction.
Case 2. U induces at least two edges of different colors. In this case there are at least two vertices in V, say v_1 and v_2 which are not joined to U by edges with colors in $c(U)$. But then one of the sets $U - u_i + \{v_1,v_2\}$ induces a rainbow K_4. □

4. Further linear bounds

Our best linear lower bound is similar to a result of Gol'berg and Gurvich [11] on the Turán numbers $\mathrm{ex}(n,p,r)$ for $r \le p-2$. They prove that in this case

$$\mathrm{ex}(n,p,r) = r + \left\lfloor \frac{u-1}{u}(n-p+1) \right\rfloor, \tag{1}$$

where $u = \lfloor (p-1)/(p-1-r) \rfloor$. We can apply their result straight away: multicolor the edges in a subgraph of K_n that shows that equality in (1) can hold and then color the remaining edges canonically with new colors. This way one can prove that, for $q_2 \le 2p-4$,

$$R(n,p,p-1,q_2) \ge n-1+\mathrm{ex}(n,p,q_2-p+1)-p+3.$$

However, we can improve this by working with the structure of the optimal graphs directly

Proposition 10. *If $p \le q_2 \le 2p-4$, then*

$$R(n,p,p-1,q_2) \ge q_2 - 1 + \left\lfloor \frac{2u-1}{u}(n-p+1) \right\rfloor, \text{ where } u = \left\lfloor \frac{p-2}{2p-3-q_2} \right\rfloor.$$

Proof. Let $q = q_2$. Observe that $v = p-2-u(2p-4-q) \ge u \ge 1$. First let G be the n-vertex graph consisting of pairwise vertex disjoint stars as follows: one star on v vertices, as many stars on u vertices as possible, and one remainder star on at most u vertices. Next we label the vertices, in increasing order, so that the center of each star comes before its leaves, and the stars come one after the other. To obtain our construction, multicolor G and color the remaining edges canonically $c(xy) = \min\{x,y\}$.

Now let A be a p-set of vertices. We observe that A has at least $p-1$ colors, and it contains vertices from at least $2p-q-2$ stars. So every p-set has at most $p-(2p-q-2) = q+2-p$ edges from G, and thus contains at most

$$(q+2-p)+(p-1) = q+1$$

colors. However for this to be achieved we have to use exactly $2p-q-2$ stars, and we observe that even the smallest star must have at least $p-v-u(2p-q-4) = 2$ vertices. But then the last star (in the labeling) has its center s in A, and all vertices with higher labels are leaves of this star, so that the canonical color $c = \min\{s,u\}$ is never used.

Finally to count the number of colors in this coloring, we observe that

$$\mathrm{e}(G) = (v-1) + \left\lfloor \frac{u-1}{u}(n-v) \right\rfloor = n-1-\left\lceil \frac{n-v}{u} \right\rceil.$$

Also if u divides $n-v-1$, then the last star is just one vertex, so that the canonical coloring contributes $n-1$ colors, otherwise it contributes only $n-2$ colors. In either case this adds up to at least

$$2n-3-\left\lceil \frac{n-v-1}{u} \right\rceil = q-1+\left\lfloor \frac{n(2u-1)-2u-uq+v+1}{u} \right\rfloor = q-1+\left\lfloor \frac{2u-1}{u}(n-p+1) \right\rfloor.$$

□

The gap between the upper bound in Proposition 6 and the constructions in Proposition 10 is considerable. More specifically we have not found any constructions on more than $2n$ colors, and it is possible that there are no such constructions. The best general upper bound we have is

Theorem 11. *Let $c = q_2 - q_1 + \lfloor \frac{q_1+q_2-2}{p-1} \rfloor + 1$. If $p \geq 4$, $q_1 < p$ and $c \leq p$, then*

$$R(n, p, q_1, q_2) \leq (c-1)(n-p) + q_2.$$

Proof. Since the statement is trivial for $n = p$, we may assume that $n > p$. If there is any vertex v_0 such that $|c(V) - c(V - v_0)| < c$, then the result follows by induction. So, aiming for a contradiction, we take a vertex v_0 with at least c vertices $v_1, \dots, v_c$, so that the colors $c(v_0v_i)$ are distinct and appear only on edges incident to v_0. Next pick vertices $v_{c+1}, \dots, v_p$ and let $A = \{v_0, \dots, v_p\}$, so that $|c(A - v_0)| \geq q_1$. We will count $a = \sum_{i=1}^{p} |c(A - v_i)|$ in two ways to obtain a contradiction. Certainly $a \leq pq_2$. On the other hand, $a \geq (p-1)c + (p-2)q_1$, since every one of the colors $c(v_0v_i)$ is in all but at most one of the sets $A - v_i$, and every one of the colors in $c(A - v_0)$ is in all but at most two. Actually

$$a \geq (p-1)c + (p-2)q_1 + 2.$$

Indeed, if $|c(A - v_0)| > q_1$, then this follows immediately since $p \geq 4$. If however $|c(A - v_0)| = q_1$, then since $q_1 \leq p - 1 < \binom{p}{2}$ at least one color is repeated and we gain at least 1. But for even this to be sharp it would imply that every color occurs only once, except for one color which may form a star. This is not possible, since the number of edges in a complete graph spanned by $A - v_0$ is $q_1 + p - 2 \leq 2(p-1) - 1 < \binom{p}{2}$. Hence we get

$$pq_2 \geq a \geq (p-1)c + (p-2)q_1 + 2,$$

or equivalently $c \leq q_2 - q_1 + (q_1 + q_2 - 2)/(p-1)$, a contradiction. □

Although this simple bound is a significant improvement, and it gives the correct answer when $q_1 = q_2 = p - 1$, it is still far from the truth when $q_2 - q_1$ is large. With more effort we can prove

Theorem 12. *If $p > 8$, then $\lfloor (p+2)n/(p+1) \rfloor \leq R(n, p, 2, p) < 2n$.*

Proof. First we give a construction establishing the lower bound. When n is divisible by $p + 1$ we partition the vertex set into $n/(p+1)$ cycles C^i of size $p + 1$. We will obtain our coloring by rainbow coloring the cycles, coloring all other edges induced by C^i with color 1, and edges joining C^i and C^j with color $\max\{i, j\}$. This coloring has the desired properties. If n is not divisible by $p + 1$, then pick the smallest $n' > n$ divisible by $p + 1$, consider the coloring for n' and delete $n' - n$ vertices from C^1.

We will show the upper bound by induction on n with the result trivially being true for $n = p$. For $n > p$ let c be a $(p, 2, p)$-coloring. Construct a directed graph G by selecting one edge from each color class forming a star and orienting it from the center of the star to the leaf. For stars consisting of a single edge, that is if it is the unique edge of its color, orient it in both directions. If we can show that G has a vertex of out-degree at most 2 we can remove it and proceed by induction.

So, aiming for a contradiction, we will suppose that G has minimum out-degree $\delta^+(G) \geq 3$. First we are going to prove that G has an induced P_{p-1}. Suppose that G does not have an induced P_{p-1}. Consider a subgraph H of G on p vertices containing a maximum number of arcs.

Notice that H has at most one component which is a tree. Suppose H has components H_1 and H_2 which are trees and consider a leaf x of H_1. Since $\delta^+(x) \geq 3$ there is an arc xy going outside of H. Add y to H and delete some leaf from H_2. Continuing this process we enlarge one tree and make the other smaller until we either delete H_2 completely or create a cycle in H_1, in either case creating a graph with more arcs. Without loss of generality we assume that the only tree-component is H_1. Also H can not contain a component H_q of G. Indeed, since $\delta^+(G) \geq 3$, we get $|E(H_q)| \geq (3/2)|V(H_q)| \geq |V(H_q)| + 2$.

But then the total number of colors on the edges in H is at least

$$|V(H_q)| + 2 + |V(H_1)| - 1 + \sum_{i \neq 1,q} |V(H_i)| \geq |V(H)| + 1 = p + 1.$$

Furthermore we can assume that H does not have a tree component since removing leaves and adding edges to a component that contains a cycle we can eventually eliminate the tree component. Therefore $|E(H)| \geq |V(H)|$. If $|E(H)| \geq |V(H)|+1 = p+1$, then the coloring is not a $(p, 2, p)$-coloring. Thus $|E(H)| = |V(H)| = p$ and all other edges induced by vertices of H have the colors of H. Also, since there is no tree component, every component has precisely one cycle.

If H has two vertices of degree 1, then removing one of them and adding in a new neighbor of the other will eventually either introduce a cycle or result in a path. Iterating the procedure we end up with a union of cycles with at most one pendant path. Next we can reduce the number of components in H. Choose two vertices in different components which are not vertices of attachment of a pendant path, say u and v. Then, as we noticed above, the color of uv is the color of some edge of H, without loss of generality, $c(uv) = c(uw)$ for some edge uw of H. Therefore removing uw and adding uv we obtain a graph with fewer components. Repeating the above procedure we end up with a cycle with at most one pendant path. Therefore H contains an induced P_{p-1}.

Let $P = \{x_1, \ldots, x_{p-2}\}$ be such an induced path, not necessarily directed, of length $(p-2)$ in G. Then, since every vertex has outdegree at least three, there are vertices $y_0, y_1, y_2, \ldots, y_{p-2}, y_{p-1}$ such that the arcs $x_i y_i$, $x_1 y_0$ and $x_p y_{p-1}$ are in G and all of them have distinct colors not used on P. Let $Y = \{y_0, y_1, y_2, \ldots, y_{p-2}, y_{p-1}\}$.
If $y_i = y_j$ for some $i \neq j$, then consider $P \cup \{y_i, y_k\}$ where $y_k \neq y_i$ (such a vertex exists since $y_0 \neq y_1$). These vertices induce at least $p + 1$ colors since $y_i y_k$ has a color different from the colors on P and the colors of $x_i y_i$, $x_j y_j$ and $x_k y_k$. So we may assume that $|Y| = p$.

Now $Y' = \{y_i, y_j, y_k\}$ induces a monochromatic K_3 for $0 \leq i < j < k \leq p-1$ if $1 < i$ or $k < p-2$. Otherwise $P \setminus \{x_{p-2}\} \cup Y'$ or $P \setminus x_1 \cup Y'$ induces a K_p with at least $p+1$ colors, since the colors of $c(Y')$ are not in $c(P + x_i y_i + x_j y_j + x_k y_k)$ (or, respectively in $c(P + x_1 y_0 + x_j y_j + x_k y_k)$ or $c(P + x_i y_i + x_j y_j + x_{p-2} y_{p-1})$). So if we let $Q = \{(y_0, y_{p-2}), (y_0, y_{p-1}), (y_1, y_{p-2}), (y_1, y_{p-1})\}$, then we can conclude that all edges induced by Y, except possibly those in Q are of the same color, say a.

Since there is no monochromatic K_p we may assume that some edge in Q, say y_0, y_{p-1}, is of color $b \neq a$. Therefore $T = P + y_0 + y_1 + y_{p-1}$ has $p + 1$ vertices and induces at least $p + 2$ colors. If $c(x_3 x_5) \notin c(P) \cup \{b\}$, then $T - y_1$ induces $p + 1$ colors, and if $c(x_3 x_5) = b$, then $T - y_{p-1}$ induces at least $p + 1$ colors. Thus the only possible colors of $x_3 x_5$ are $c(x_3 x_4)$, $c(x_4 x_5)$, $c(x_2 x_3)$ and $c(x_5 x_6)$.

If $c(x_3 x_5) \in \{c(x_3 x_4), c(x_4 x_5)\}$ then $T - x_4$ contains at least $p + 1$ colors. Similarly if $c(x_3 x_5) = c(x_2 x_3)$ then $T - x_2$ or if $c(x_3 x_5) = c(x_5 x_6)$ then $T - x_6$ respectively induce at least $p+1$ colors. Thus in any case we have either a monochromatic K_p or a K_p inducing at least $p + 1$ colors. □

5. Chart for Small values

We will use the previous results to give a chart summarizing the known results for $p = 3, 4$ for all choices of q_1, q_2 that are possible by Proposition 2.

p	q_1	q_2	$R(n,p,q_1,q_2)$	Proof
p	q_1	$\binom{p}{2}$	$\binom{n}{2}$	obvious
p	$\leq p-1$	$p-1$	$n-1$	[1]
p	q_1	$\leq \lfloor p/2 \rfloor$	q_2	[1]
4	1	5	$\lfloor n^2/4 \rfloor + 1$	[1]
4	2, 3	5	$cn \log n \leq R = \mathrm{O}(n^{5/3})$	Thm. 7, Prop. 9
4	1	4	$Cn^{3/2} \leq R = \mathrm{o}(n^2)$	see Rem. 13 below
4	2	4	$\lfloor 3(n-1)/2 \rfloor \leq R \leq 3n-8$	Prop. 10, Thm. 11
4	3	4	$\lfloor 3(n-1)/2 \rfloor \leq R \leq 2n-4$	Prop. 10, Thm. 11

Remark 13. The upper bound in the table entry for $R(n,4,1,4)$ follows immediately from the first part of Proposition 3 with $k=2$, since $t(n,1)=0$. Examining the proof of Proposition 3 this can be improved to $\mathrm{O}(n^{1-\varepsilon})$. The lower bound follows from the proof of Theorem 5: multicolor the edges of a girth 5 graph G with the maximum number of edges, and give all remaining edges the same new color. By a 1938 result of Erdős [4], G has $\Theta(n^{3/2})$ edges.

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