Chromatic number of ordered graphs with forbidden ordered subgraphs.

Maria Axenovich, Jonathan Rollin, and Torsten Ueckerdt

Department of Mathematics, Karlsruhe Institute of Technology

Abstract

It is well-known that the graphs not containing a given graph H as a subgraph have bounded chromatic number if and only if H is acyclic. Here we consider *ordered graphs*, i.e., graphs with a linear ordering $\prec$ on their vertex set, and the function

$$f_{\prec}(H) = \sup\{\chi(G) \mid G \in \mathrm{Forb}_{\prec}(H)\},$$

where $\mathrm{Forb}_{\prec}(H)$ denotes the set of all ordered graphs that do not contain a copy of H.

If H contains a cycle, then as in the case of unordered graphs, $f_{\prec}(H) = \infty$. However, in contrast to the unordered graphs, we describe an infinite family of ordered forests H with $f_{\prec}(H) = \infty$. An ordered graph is crossing if there are two edges uv and $u'v'$ with $u \prec u' \prec v \prec v'$. For connected crossing ordered graphs H we reduce the problem of determining whether $f_{\prec}(H) \neq \infty$ to a family of so-called monotonically alternating trees. For non-crossing H we prove that $f_{\prec}(H) \neq \infty$ if and only if H is acyclic and does not contain a copy of any of the five special ordered forests on four or five vertices, which we call bonnets. For such forests H, we show that $f_{\prec}(H) \leqslant 2^{|V(H)|}$ and that $f_{\prec}(H) \leqslant 2|V(H)| - 3$ if H is connected.

1 Introduction

What conclusions can one make about the chromatic number of a graph knowing that it does not contain certain subgraphs? Let H be a graph on at least two vertices, $\mathrm{Forb}(H)$ be the set of all graphs not containing H as a subgraph, and $f(H) = \sup\{\chi(G) \mid G \in \mathrm{Forb}(H)\}$. If H has a cycle of length ℓ, then for any integer χ there is a graph G of girth at least $\ell+1$ and chromatic number χ, see [11], implying that $f(H) = \infty$. On the other hand, if H is a forest on k vertices and G is a graph of chromatic number at least k, then G contains a k-critical subgraph G', that in turn has minimum degree at least $k-1$. Thus a copy of H can be found as a subgraph of G' by a greedy embedding. Therefore $G \notin \mathrm{Forb}(H)$, implying that $f(H) \leqslant k-1$. So, we see that $f(H)$ is finite if and only if H is acyclic.

A similar situation holds for directed graphs, with a similarly defined function $f_{\mathrm{dir}}(H)$ being finite if and only if the underlying graph of H is acyclic. A result of Addalirio-Berry *et al.* [1], see also [4], implies that $f_{\mathrm{dir}}(H) \leqslant k^2/2-k/2-1$ whenever H is a directed k-vertex graph whose underlying graph is acyclic.

Here, we consider the behavior of the chromatic number of ordered graphs with forbidden ordered subgraphs. An *ordered graph* G is a graph (V, E) together with a linear ordering $\prec$ of its vertex set V. An *ordered subgraph* H of an ordered graph G is a subgraph of the (unordered) graph (V, E) together with the linear ordering of its vertices inherited from G. An ordered subgraph H is a *copy of an ordered graph* H' if there is an order preserving isomorphism between H and H'. For an ordered graph H on at least two vertices[1] let $\mathrm{Forb}_\prec(H)$ denote the set of all ordered graphs that do not contain a copy of H. We consider the function $f_\prec$ given by

$$f_\prec(H) = \sup\{\chi(G) \mid G \in \mathrm{Forb}_\prec(H)\}.$$

We show that it is no longer true that $f_\prec(H)$ is finite if and only if H is acyclic. When H is connected, we reduce the problem of determining whether $f_\prec(H) \neq \infty$ to a well behaved class of trees, which we call monotonically alternating trees. We completely classify so-called "non-crossing" ordered graphs H for which $f_\prec(H) = \infty$. In case of "non-crossing" H with finite $f_\prec(H)$, we provide specific upper bounds on this function in terms of the number of vertices in H. Note that $f_\prec(H) \geqslant |V(H)| - 1$ for any ordered graph H, since a complete graph on $|V(H)| - 1$ vertices is in $\mathrm{Forb}_\prec(H)$.

We need some formal definitions before stating the main results of the paper. We consider the vertices of an ordered graph laid out along a horizontal line according to their ordering $\prec$ and say that for $u \prec v$ the vertex u is to the *left of* v and the vertex v is to the *right of* u. We write $u \preceq v$ if $u \prec v$ or $u = v$. For two sets of vertices U and U' we write $U \prec U'$ if all vertices in U are left of all vertices in U'. Two edges uv and $u'v'$ ***cross*** if $u \prec u' \prec v \prec v'$ and an ordered graph H is called *crossing* if it contains two crossing edges. Otherwise, H is called *non-crossing.* Two distinct ordered graphs G and H *cross each other* if there is an edge in G crossing an edge in H.

An ordered graph is a ***bonnet*** if it has 4 or 5 vertices $u_1 \prec u_2 \preceq u_3 \prec u_4 \preceq u_5$ and edges u_1u_2, u_1u_5, u_3u_4, or if it has vertices $u_1 \preceq u_2 \prec u_3 \preceq u_4 \prec u_5$ and edges u_1u_5, u_4u_5, u_2u_3. See Figure 1 (first two rows). An ordered path $P = u_1, \ldots, u_n$ is a ***tangled path*** if for a vertex u_i, $1 < i < n$, that is either leftmost or rightmost in P there is an edge in the subpath $u_1, \ldots, u_i$ that crosses an edge in the subpath $u_i, \ldots, u_n$. See Figure 1 (last row, left and middle). Note that there are crossing paths which are not tangled, see for example Figure 1 (right).

Theorem 1. *If an ordered graph H contains a cycle, a bonnet, or a tangled path, then $f_\prec(H) = \infty$.*

[1] If H has only one vertex, then $\mathrm{Forb}_\prec(H)$ consists only of the graph with empty vertex set and one can think of $f_\prec(H)$ as being equal to 0. However, we will avoid this pathologic case throughout.

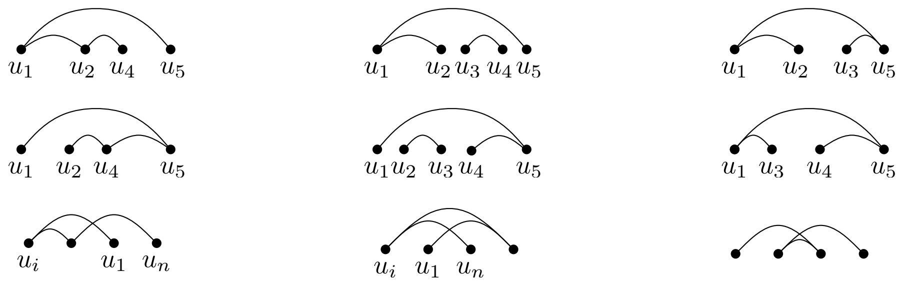

Figure 1: All bonnets (first two rows), two tangled paths (last row, left and middle) and a crossing path that is not tangled (last row, right).

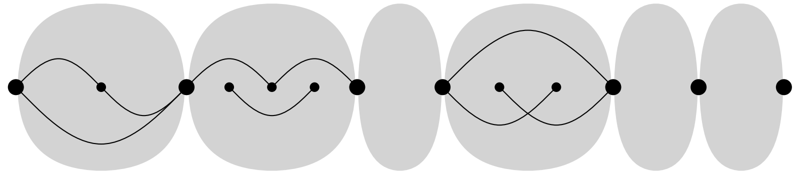

Figure 2: Segments of an ordered graph. The bold vertices are either inner cut-vertices or left-, rightmost vertices.

A vertex v of an ordered graph G is called *inner cut vertex*, if there is no edge uw with $u \prec v \prec w$ in G and v is not leftmost or rightmost in G. An *interval* in an ordered graph G is a set I of vertices such that for all vertices $u, v \in I$, $x \in V(G)$ with $u \prec x \prec v$ we have $x \in I$. A ***segment*** of an ordered graph G with $|V(G)| \geqslant 2$ is an induced subgraph H of G such that $|V(H)| \geqslant 2$, $V(H)$ is an interval in G, the leftmost and rightmost vertices in H are either inner cut vertices of G or leftmost respectively rightmost in G, and all other vertices in H are not inner cut vertices in G. So, G is the union of its segments, any two segments share at most one vertex and the inner cut vertices of G are precisely the vertices contained in two segments of G. In particular, the number of inner cut vertices of G is exactly one less than the number of its segments. See Figure 2.

The *length* of an edge xy is the number of vertices v such that $x \preceq v \prec y$. A shortest edge among all the edges incident to a vertex x is referred to as a *shortest edge incident to x*. Note that there is either 1 or 2 shortest edges incident to a given vertex in a connected graph on at least two vertices. Let U be a vertex set in an ordered tree T, such that each vertex in U has exactly one shortest edge incident to it. For such a set U, let $S(U)$ be the set of edges e_u such that e_u is a shortest edge incident to u, $u \in U$. We call an ordered tree T ***monotonically alternating*** if there is a partition $V(T) = L \dot\cup R$, with $L \prec R$, such that L and R are independent sets in T, $E = S(L) \cup S(R)$, and neither $S(L)$ nor $S(R)$ contains a pair of crossing edges.

Theorem 2. *An ordered tree T contains neither a bonnet nor a tangled path if and only if each segment of T is monotonically alternating. In particular if $f_{\prec}(H) \neq \infty$ for some connected ordered graph H, then each segment in H is a monotonically alternating tree.*

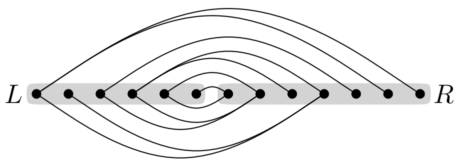

Figure 3: A monotonically alternating tree. Each edge on top is the shortest edge incident to a vertex in R and each edge at the bottom is the shortest edge incident to a vertex in L.

Recall that an ordered graph is non-crossing if it does not contain any crossing edges. Note that a non-crossing graph does not contain tangled paths.

Theorem 3. *Let T be a non-crossing ordered graph on k vertices. Then $f_{\prec}(T) \neq \infty$ if and only if T is a forest that does not contain a bonnet.*

Moreover, if $f_{\prec}(T) \neq \infty$ then $k-1 \leqslant f_{\prec}(T) \leqslant 2^k$. If, in addition T is connected, then $f_{\prec}(T) \leqslant 2k-3$. Finally, for each $k \geqslant 4$ there is an ordered non-crossing tree T with $k \leqslant f_{\prec}(T) \neq \infty$, while for $k = 2, 3$ we have $f_{\prec}(T) = k-1$.

For certain classes of ordered forests we prove better upper bounds on $f_{\prec}$. A *k-nesting* is an ordered graph T on vertices $u_1 \prec \cdots \prec u_k \prec v_k \prec \cdots \prec v_1$ and edges u_iv_i, $1 \leqslant i \leqslant k$. A *$k$-crossing* is an ordered graph T on vertices $u_1 \prec \cdots \prec u_k \prec v_1 \prec \cdots \prec v_k$ and edges u_iv_i, $1 \leqslant i \leqslant k$. We may omit the parameter k if it is not important. A *generalized star* is a union of a star and isolated vertices.

The following theorem summarizes several results on trees which are either not covered by Theorem 3 or improve the upper bound from Theorem 3 significantly.

One of the known classes of such graphs is a special family of star forests, or, in other words, tuple matchings. For positive integers m and t and a permutation π of $[t]$, an *m-tuple t-matching* $M = M(t, m, \pi)$ is an ordered graph with vertices $v_1 \prec \cdots \prec v_{t(m+1)}$, where each edge is of the form $v_iv_{t+j+m(\pi(i)-1)}$ for $1 \leqslant i \leqslant t$, $1 \leqslant j \leqslant m$. I.e., an m-tuple t-matching is a vertex disjoint union of t stars on m edges each, where $v_1, \ldots, v_t$ are the centers of the stars that are to the left of all leaves and the leaves of each star form an interval in M, so that these intervals are ordered according to the permutation π. The third item in the following theorem is an immediate corollary of a result by Weidert [19] who provides a linear upper bound on the the extremal function for M. The other results are based on linear upper bounds for the extremal functions of nestings due to Dujmovic and Wood [10], on the extremal function of crossings due to Capoyleas and Pach [5] and lower bounds for ordered Ramsey numbers due to Conlon *et al.* [7], see also Balko et al. [2]. See Section 3 for a more detailed description of extremal functions and ordered Ramsey numbers.

Theorem 4. *Let T be an ordered forest on k vertices.*

- *If each segment of T is either a generalized star, a 2-nesting, or a 2-crossing, then $f_{\prec}(T) = k-1$.*

- *If each segment of T is either a nesting, a crossing, a generalized star, or a non-crossing tree without bonnets, then $k-1 \leqslant f_{\prec}(T) \leqslant 2k-3$.*
- *If T is a tuple matching, then $k-1 \leqslant f_{\prec}(T) \leqslant 2^{10k\log(k)}$.*
- *There is a positive constant c such that for each even positive integer $k \geqslant 4$ there is a matching M on k vertices with $f_{\prec}(M) \geqslant 2^{c\frac{\log(k)^2}{\log\log(k)}}$.*

The paper is organized as follows. In Section 2 we introduce all missing necessary notions. In Section 3 we summarize the known results on extremal functions and Ramsey numbers for ordered graphs and show how they could be used in determining $f_{\prec}$. In Section 4 we prove some structural lemmas and provide several reductions that are used in the proofs of the main results and that might be of independent interest. Section 5 contains the proofs of Theorems 1–4. We summarize all known results for forests with at most three edges in Section 6. Finally, Section 7 contains conclusions and open questions.

2 Definitions

Let K_n denote a complete graph on n vertices. For a positive integer n and an ordered graph H, let $\mathrm{ex}_{\prec}(n,H)$ denote the *ordered extremal number*, i.e., the largest number of edges in an ordered graph on n vertices in $\mathrm{Forb}_{\prec}(H)$. For an ordered graph H the *ordered Ramsey number* $R_{\prec}(H)$ is the smallest integer n such that in any edge-coloring of an ordered K_n in two colors there is a monochromatic copy of H. Recall that an interval in an ordered graph G is a set I of vertices such that for all vertices $u,v \in I$, $x \in V(G)$ with $u \prec x \prec v$ we have $x \in I$. The *interval chromatic number* $\chi_{\prec}(G)$ of an ordered graph G is the smallest number of intervals, each inducing an independent set in G, needed to partition $V(G)$. An inner cut vertex v of an ordered graph G *splits G into ordered graphs G_1 and G_2* if G_1 is induced by all vertices u with $u \preceq v$ in G and G_2 is induced by all vertices u with $v \preceq u$. A vertex of degree 1 is called a *leaf.* A vertex in an ordered graph G is called *reducible*, if it is a leaf in G, is leftmost or rightmost in G and has a common neighbor with the vertex next to it. We call an edge uv in a graph G *isolated* if u and v are leaves in G. A graph G is *t-degenerate* if each subgraph of G has a vertex of degree at most t. A vertex v is *between* vertices u and w if $u \preceq v \preceq w$. The *reverse* $\overline{G}$ of an ordered graph G is the ordered graph obtained by reversing the ordering of the vertices in G. A *u-v-path* P is a path starting with u and ending with v, i.e., a path $v_1,\ldots,v_k$ with $u=v_1$, $v=v_k$. Given a path $P=v_1,\ldots,v_k$ let $v_iP=v_i,\ldots,v_k$ and $Pv_i=v_1,\ldots,v_i$. Similarly for a neighbor $v \notin V(P)$ of v_1 let $vP=v,v_1,\ldots,v_k$. If $U \subseteq V(G)$, $F \subseteq E(G)$ let $G[U]$, $G-U$ and $G-F$ denote the graphs $(U, E(G) \cap \binom{U}{2})$, $(V(G) \setminus U, E(G) \cap \binom{V(G)-U}{2})$, and $(V(G), E(G) \setminus F)$, respectively. In particular if $u,v \in V(G)$ then $G-\{u,v\}$ is the graph obtained by removing u and v from G, not the edge uv only. If $u \in V(G)$ let $G-u = G-\{u\}$. The definitions of tangled paths, bonnets, crossing edges and subgraphs, intervals, segments, inner cut-vertices, and monotonically alternating trees are given before the statements of the main theorems in the introduction. We shall typically denote a general ordered graph by H, a tree or a forest by T, and a larger ordered graph by G. For all other undefined graph theoretic notions we refer the reader to West [20].

3 Connections to known results

There are connections between the extremal number $\mathrm{ex}_{\prec}(n, H)$ and the function $f_{\prec}(H)$. If there is a constant c such that $\mathrm{ex}_{\prec}(n, H) < c\,n$ for every n, then

$$f_{\prec}(H) \leqslant 2c, \tag{1}$$

so $f_{\prec}(H)$ is finite. Indeed, if $\mathrm{ex}_{\prec}(n, H) < c\,n$ then any $G \in \mathrm{Forb}_{\prec}(H)$ has less than $c\,|V(G)|$ edges, and hence has a vertex of degree less than $2c$. Thus if $G \in \mathrm{Forb}_{\prec}(H)$, then each subgraph of G is in $\mathrm{Forb}_{\prec}(H)$, so each subgraph has a vertex of degree less than $2c$, so G is $(2c-1)$-degenerate. Therefore $\chi(G) \leqslant 2c$.

Ordered extremal numbers are studied in detail in [17]. Recall that $\chi_{\prec}(G)$ is the smallest number of intervals, each inducing an independent set, needed to partition the vertices of an ordered graph G. Pach and Tardos [17] prove that for each ordered graph H

$$\mathrm{ex}_{\prec}(n, H) = \left(1 - \frac{1}{\chi_{\prec}(H) - 1}\right)\binom{n}{2} + o(n^2).$$

For ordered graphs with interval chromatic number 2, Pach and Tardos find a tight relation between the ordered extremal number and pattern avoiding matrices. For an ordered graph H with $\chi_{\prec}(H) = 2$ let $A(H)$ denote the 0-1-matrix where the rows correspond to the vertices in the first color and the columns to the vertices in the second color of a proper interval coloring of H in 2 colors and let $A(H)_{u,v} = 1$ if and only if uv is an edge in H. A 0-1-matrix B *avoids* another 0-1-matrix A if there is no submatrix in B which becomes equal to A after replacing some ones with zeros. For a 0-1-matrix A let $\mathrm{ex}(n, A)$ denote the largest number of ones in an $n \times n$ matrix avoiding A. In [17] it is shown that for each ordered graph H with $\chi_{\prec}(H) = 2$ there is a constant c such that $\mathrm{ex}(\lfloor \frac{n}{2} \rfloor, A(H)) \leqslant \mathrm{ex}_{\prec}(n, H) \leqslant c\,\mathrm{ex}(n, A(H)) \log n$. Thus, when $\mathrm{ex}(n, A(H))$ is linear in n, one can guarantee that $\mathrm{ex}_{\prec}(n, H) = O(n \log n)$, but this is not enough to claim that $f_{\prec}(H) \neq \infty$.

In addition, we see that there is no direct connection between $f_{\prec}(H)$ and $\mathrm{ex}_{\prec}(n, H)$ because there are dense ordered graphs avoiding H for some ordered graphs H with small $f_{\prec}(H)$. A specific example for such a graph H is an ordered path $u_1u_2u_3u_4$, with $u_1 \prec u_2 \prec u_3 \prec u_4$. One can see from Theorem 4 that $f_{\prec}(H) = 3$, but a complete bipartite ordered graph G with all vertices of one bipartition class to the left of all other vertices does not contain H and has $|V(G)|^2/4$ edges. However, for some ordered graphs H with interval chromatic number 2, one can show that $\mathrm{ex}_{\prec}(n, H)$ is linear. This in turn, implies that $f_{\prec}(H)$ is finite.

Some of the extensive research on forbidden binary matrices and extremal functions for ordered graphs can be found in [3, 12, 14, 15, 16].

There are also connections between the Ramsey numbers $R_{\prec}(H)$ for ordered graphs and the function $f_{\prec}(H)$. If the edges of K_n, $n = R_{\prec}(H) - 1$, are colored in two colors without monochromatic copies of H, then both color classes form ordered graphs G_1 and G_2 not containing H as an ordered subgraph. Then one of the G_i's has chromatic number at least $\sqrt{n}$, since a product of proper colorings of G_1 and G_2 yields a proper coloring of K_n. Therefore $f_{\prec}(H) \geqslant \sqrt{R_{\prec}(H) - 1}$. Ordered Ramsey numbers were recently studied by Conlon *et al.* [7] and Balko *et al.* [2]. Other research on ordered graphs includes characterizations of classes of graphs by forbidden ordered subgraphs [8, 13] and the study of perfectly ordered graphs [6].

4 Structural Lemmas and Reductions

In this section we first analyze the structure of ordered trees without bonnets and tangled paths. This leads to a proof of Theorem 2 in Section 5. Afterwards we establish several cases when $f_{\prec}(H)$ can be upper bounded in terms of $f_{\prec}(H')$ for a subgraph H' of H. This allows us to reduce the problem of whether $f_{\prec}(H) \neq \infty$ to the problem of whether $f_{\prec}(H') \neq \infty$. These *reductions* are the crucial tools in the proof of Theorem 3 in Section 5.

Lemma 4.1. *Let T be an ordered tree that does not contain a tangled path and let $u \prec v \prec w$ be vertices in T. If uw is an edge in T, then all vertices of the path connecting u and v in T are between u and w.*

Proof. Let P be the path in T that starts with v and ends with the edge uw. Let ℓ denote the leftmost vertex in P. Assume for the sake of contradiction that $\ell \prec u$. Then the path $vP\ell$ contains neither u nor w and therefore crosses the edge uw. Hence the paths $P\ell$ and ℓP cross and P is tangled, a contradiction. Therefore $\ell = u$. Due to symmetric arguments w is the rightmost vertex in P. Hence all vertices in P are between u and w. □

Lemma 4.2. *Let T be an ordered tree that contains neither a bonnet nor a tangled path and that has only one segment . Deleting any leaf from T yields an ordered tree that contains neither a bonnet nor a tangled path and that has only one segment.*

Proof. Let uv be an edge in T incident to a leaf u and let $T' = T - u$. Then clearly T' is an ordered tree that contains neither a bonnet nor a tangled path. For the sake of contradiction assume that T' has at least two segments and let x be an inner cut vertex in T'. Then $x \neq u$,v and is between u and v in T, since x is not an inner cut vertex in T. By reversing T if necessary we may assume that $v \prec x \prec u$. Let P be the v-x-path in T'. All vertices in P are between v and u by Lemma 4.1 applied to u, v and x. In addition no vertex in P is to the right of x since x is an inner cut vertex in T'. So all vertices in P are between v and x. Let vw denote the first edge of P and let xy denote an edge in T' with $x \prec y$. Such an edge xy exists since the inner cut vertex x is not rightmost in T' and T' is connected. If $u \prec y$, then $uvPxy$ is a tangled path in T. If $y \prec u$, then u, v, w, x and y form a bonnet in T. In both cases we have a contradiction and hence T' has only one segment. □

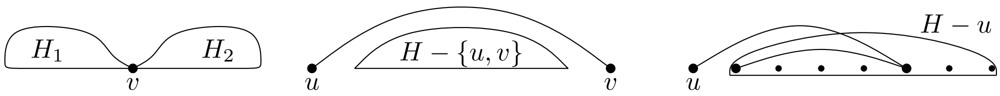

Figure 4: An inner cut vertex v splitting an ordered graph into ordered graphs H_1 and H_2 (left), an isolated edge uv in an ordered graph H (middle), and a reducible vertex u (right).

Lemma 4.3. *If T is an ordered tree that contains neither a bonnet nor a tangled path and that has only one segment, then $\chi_\prec(T) \leqslant 2$.*

Proof. We prove the claim by induction on $k = |V(T)|$. If $k \leqslant 2$, then clearly $\chi_\prec(T) \leqslant 2$. So assume that $k \geqslant 3$. Let u denote a leaf in T, v its neighbor in T, and let $T' = T - u$. Then T' has only one segment and contains neither a bonnet nor a tangled path due to Lemma 4.2. Inductively $\chi_\prec(T') \leqslant 2$, i.e., there is a partition $L\dot{\cup}R = V(T')$, with $L \prec R$, such that all edges in T' are between L and R. By reversing T if necessary we assume that $v \in L$. For the sake of contradiction assume that $\chi_\prec(T) > 2$. Then $u \prec \ell$ for the rightmost vertex ℓ in L, possibly $\ell = v$. Let $w \in R$ denote one fixed neighbor of v in T'. Then all vertices of the path connecting ℓ and v in T' are between v and w due to Lemma 4.1. In particular ℓ is incident to an edge ℓx, $x \in R$, with $x \preceq w$. Hence $u \prec v$, since otherwise there is a bonnet on vertices v, u, ℓ, x, and w in T. If there is a vertex y, $u \prec y \prec v$, then all vertices of the path connecting y and u in T are between u and v due to Lemma 4.1. But this is not possible since $y, v \in L$ and all the neighbors of y are in R. Hence u is immediately to the left of v in T. Note that u is not leftmost in T, since otherwise v is an inner cut vertex in T. Consider the path P connecting a vertex left of u to ℓ in T. This path contains distinct vertices $p, q \in L$, $r \in R$, such that pr and rq are edges in P and $p \prec u \prec v \preceq q \prec r$. Hence there is a bonnet, a contradiction. This shows that $\chi_\prec(T) \leqslant 2$. □

We now present several reductions. Let us mention that some of the following arguments are similar to reductions used for extremal numbers of matrices [17, 18].

Recall, that an inner cut vertex v of an ordered graph H splits H into ordered graphs H_1 and H_2, where H_1 is induced by all vertices u with $u \preceq v$ in H and H_2 is induced by all vertices u with $v \preceq u$. See Figure 4 (left).

Reduction Lemma 1. *If an inner cut vertex v splits an ordered graph H into ordered graphs H_1 and H_2 with $f_\prec(H_1), f_\prec(H_2) \neq \infty$, then*

$$f_\prec(H) \leqslant f_\prec(H_1) + f_\prec(H_2).$$

Proof. Consider an ordered graph $G \in \mathrm{Forb}_\prec(H)$. Let V_1 denote the set of vertices in G that are rightmost in some copy of H_1 in G. Further let $V_2 = V(G) \setminus V_1$. Then $G[V_2] \in \mathrm{Forb}_\prec(H_1)$ by the choice of V_1. Moreover $G[V_1] \in \mathrm{Forb}_\prec(H_2)$, since otherwise the leftmost vertex u in a copy of H_2 in $G[V_1]$ is also a rightmost vertex in a copy of H_1 and hence plays the role of v in a copy of H in G. Thus $\chi(G) \leqslant \chi(G[V_1]) + \chi(G[V_2]) \leqslant f_\prec(H_2) + f_\prec(H_1)$ and since $G \in \mathrm{Forb}_\prec(H)$ was arbitrary we have $f_\prec(H) \leqslant f_\prec(H_1) + f_\prec(H_2)$. □

Reduction Lemma 2. *If v is an isolated vertex in an ordered graph H with $|V(H)| \geqslant 3$ and $f_\prec(H-v) \neq \infty$, then $f_\prec(H) \leqslant 2\, f_\prec(H-v)$.*

Proof. Consider an ordered graph $G \in \mathrm{Forb}_\prec(H)$. If v is not leftmost or rightmost in H, then let V_1 denote a set of every other vertex in G and let $V_2 = V(G) \setminus V_1$. Then $G[V_1], G[V_2] \in \mathrm{Forb}_\prec(H-v)$, since for any two vertices $u \prec w$ in V_i there is a vertex $v \in V_{3-i}$ with $u \prec v \prec w$, $i = 1, 2$. Hence $\chi(G) \leqslant \chi(G[V_1]) + \chi(G[V_2]) \leqslant 2f_\prec(H-v)$. If v is the leftmost or the rightmost in H, assume without loss of generality the former. Then clearly $G - u \in \mathrm{Forb}_\prec(H-v)$ for the leftmost vertex u of G. Thus $\chi(G) \leqslant 1 + \chi(G-u) \leqslant 1 + f_\prec(H-v) \leqslant 2f_\prec(H-v)$. Since $G \in \mathrm{Forb}_\prec(H)$ was arbitrary we have $f_\prec(H) \leqslant 2f_\prec(H-v)$ in both cases. □

Reduction Lemma 3. *Let u and v be the leftmost and rightmost vertices in an ordered graph H, $|V(H)| \geqslant 4$. If uv is an isolated edge in H and $f_\prec(H-\{u,v\}) \neq \infty$, then*

$$f_\prec(H) \leqslant 2\, f_\prec(H - \{u,v\}) + 1.$$

Proof. See Figure 4 (middle). Let $H' = H - \{u,v\}$ and consider an ordered graph $G \in \mathrm{Forb}_\prec(H)$. If G does not contain a copy of H', then $\chi(G) \leqslant f_\prec(H') \leqslant 2f_\prec(H') + 1$. So, assume that G contains a copy of H'. Let $V_1 \dot\cup \cdots \dot\cup V_p$ denote a partition of $V(G)$ into disjoint intervals with $V_1 \prec \cdots \prec V_p$, v_i being the leftmost vertex in V_i, $1 \leqslant i \leqslant p$, such that $G[V_i] \in \mathrm{Forb}_\prec(H')$, $1 \leqslant i \leqslant p$, and $G[V_i \cup \{v_{i+1}\}]$ contains a copy of H', $1 \leqslant i < p$. Note that one can find such a partition greedily by iteratively choosing a largest interval from the left that does not induce any copy of H' in G. If $p \geqslant 3$, there are no edges xy with $x \in V_i$ and $v_{i+2} \prec y$, since otherwise xy together with a copy of H' in $G[V_{i+1} \cup \{v_{i+2}\}]$ forms a copy of H, $1 \leqslant i \leqslant p-2$.

Choose a set Φ of $2\, f_\prec(H') + 1$ distinct colors. Let $\Phi_1, \ldots, \Phi_p \subset \Phi$ denote subsets of colors such that $|\Phi_i| = f_\prec(H')$, $1 \leqslant i \leqslant p$, $\Phi_i \cap \Phi_{i+1} = \emptyset$, $1 \leqslant i < p$, and, if $p \geqslant 3$, $\Phi_{i+2} \setminus (\Phi_i \cup \Phi_{i+1}) \neq \emptyset$, $1 \leqslant i \leqslant p-2$. Note that such sets Φ_i can be chosen greedily from Φ. Since $G[V_i] \in \mathrm{Forb}_\prec(H')$ we can color $G[V_i]$ properly with colors from Φ_i, $1 \leqslant i \leqslant p$, such that, if $i \geqslant 3$, v_i is colored with a color in $\Phi_i \setminus (\Phi_{i-1} \cup \Phi_{i-2})$. This yields a proper coloring of G using colors from the set Φ only. Hence $\chi(G) \leqslant 2\, f_\prec(H') + 1$. Since $G \in \mathrm{Forb}_\prec(H)$ was arbitrary we have $f_\prec(H) \leqslant 2\, f_\prec(H - \{u,v\}) + 1$. □

Recall, that a vertex in an ordered graph H is called *reducible*, if it is a leaf in H, is leftmost or rightmost in H and has a common neighbor with the vertex next to it. See Figure 4 (right).

Reduction Lemma 4. *Let H denote an ordered graph with $|V(H)| \geqslant 3$. If u is a reducible vertex in H and $f_\prec(H-u) \neq \infty$, then*

$$f_\prec(H) \leqslant 2\, f_\prec(H-u).$$

Moreover, for each $G \in \mathrm{Forb}_\prec(H)$ there is $G' \subseteq G$ such that G' is 1-degenerate and deleting the edges of G' from G yields a graph from $\mathrm{Forb}_\prec(H-u)$.

Proof. By reversing H if necessary we may assume that the reducible vertex u is leftmost in H. Let $G \in \mathrm{Forb}_{\prec}(H)$. Let E denote the set of edges in G consisting for each vertex w in G of the longest edge to the left incident to w in G, if such an edge exists.

Assume that there is a copy H' of $H - u$ in $G - E$. Let v denote the vertex in H' corresponding to the vertex immediately to the right of u in H and let w denote the vertex in H' corresponding to the neighbor of u in H. Then v is leftmost in H' and there is an edge between v and w in H'. Thus, there is an edge xw in E incident to w in G with $x \prec v$. Hence H' extends to a copy of H in G with the edge xw, a contradiction. This shows that $G - E \in \mathrm{Forb}_{\prec}(H - u)$.

Finally observe that the graph G' with the edge-set E is 1-degenerate and hence 2-colorable. This shows that $\chi(G) \leqslant \chi(G')\chi(G - E) \leqslant 2f_{\prec}(H - u)$ and since $G \in \mathrm{Forb}_{\prec}(H)$ was arbitrary we have $f_{\prec}(H) \leqslant 2f_{\prec}(H - u)$. ☐

Having Reduction Lemma 4 at hand, we are now ready to prove that every non-crossing monotonically alternating tree T satisfies $f_{\prec}(T) \neq \infty$.

Lemma 4.4. *If T is a non-crossing monotonically alternating tree with $|V(T)| \geqslant 2$, then*

$$f_{\prec}(T) \leqslant 2|V(T)| - 3.$$

Proof. Let $k = |V(T)|$ and $G \in \mathrm{Forb}_{\prec}(T)$. We shall prove that G can be edge-decomposed into $(k - 2)$ 1-degenerate graphs by induction on k.

If $k = 2$, then T consists of a single edge only. Hence G has an empty edge-set and there is nothing to prove.

So consider $k \geqslant 3$ and assume that the induction statement holds for all smaller values of k. Assume for the sake of contradiction that the leftmost vertex u and the rightmost w in T are of degree at least 2. Then the longest and the shortest edge incident to w do not coincide. Let e be the longest edge incident to w. Since in a monotonically alternating tree each edge is the shortest edge incident to its left or right endpoint, e is the shortest edge incident to its left endpoint. In particular, $e \neq uw$ because u is incident to another edge e', shorter than uw. Thus e and e' cross since $\chi_{\prec}(T) \leqslant 2$, a contradiction. Hence the leftmost or the rightmost vertex is a leaf in T.

By reversing T if necessary we assume that u is of degree 1. We shall show that u is a reducible leaf. To do so, we need to show that the vertex x that is immediately to the right of u is adjacent to the neighbor v of u. Assume for the sake of contradiction that x is not adjacent to v. Note that v is adjacent to a leaf, so it is not a leaf itself. Let e'' be an edge incident to v, $e'' \neq uv$. Then an edge incident to x crosses either uv or e'' since $\chi_{\prec}(T) \leqslant 2$, a contradiction. Thus x is adjacent to v and u is a reducible leaf in T.

Therefore, by Reduction Lemma 4, there is a 1-degenerate subgraph G' of G such that removing the edges of G' from G yields a graph $G'' \in \mathrm{Forb}_{\prec}(T - u)$. Observe that the tree $T - u$ is non-crossing and monotonically alternating with $k > |V(T - u)| = k - 1 \geqslant 2$. Hence G'' can be edge-decomposed into $(k - 3)$

1-degenerate graphs $G_1, \dots, G_{k-3}$ by induction. Thus the graphs $G_1, \dots, G_{k-3}, G'$ decompose G into $(k-2)$ 1-degenerate graphs, proving the induction step.

If $k = 2$, we know that G has no edges and $\chi(G) = 1 \leqslant 2|V(T)| - 3$. So assume that $k \geqslant 3$. Singe G is a union of $(k-2)$ 1-degenerate graphs, each subgraph of G is a union of $(k-2)$ 1-degenerate graphs, so each subgraph G^* of G on at least one vertex that has at most $(k-2)(|V(G^*)| - 1)$ edges, and thus has a vertex of degree at most $2(k-2)-1$. Therefore G is $(2(k-2)-1)$-degenerate, so $\chi(G) \leqslant 2(k-2) \leqslant 2|V(T)|-3$. Since $G \in \mathrm{Forb}_{\prec}(H)$ was arbitrary we have $f_{\prec}(H) \leqslant 2|V(T)| - 3$. □

Reduction Lemma 5. *Let T denote an ordered matching on at least 2 edges. If uv is an edge in T and u and v are consecutive and $f_{\prec}(T - \{u, v\}) \neq \infty$, then*

$$f_{\prec}(T) \leqslant 3\, f_{\prec}(T - \{u, v\}).$$

Proof. Let $G \in \mathrm{Forb}_{\prec}(T)$ with vertices $v_1 \prec \cdots \prec v_n$. We shall prove that $\chi(G) \leqslant 3\, f_{\prec}(T - \{u, v\})$ by induction on $n = |V(G)|$. If $n \leqslant 3\, f_{\prec}(T - \{u, v\})$, then the claim holds trivially. So assume that $n > 3\, f_{\prec}(T - \{u, v\}) \geqslant 3$. If there are two consecutive vertices x, y in G that are not adjacent, then let G' denote the graph obtained by identifying x and y. Then $G' \in \mathrm{Forb}_{\prec}(T)$ and $\chi(G) \leqslant \chi(G')$. Hence $\chi(G) \leqslant \chi(G') \leqslant 3\, f_{\prec}(T - \{u, v\})$ by induction. If each pair of consecutive vertices in G forms an edge, then consider a partition $V(G) = V_0 \dot\cup V_1 \dot\cup V_2$ such that $V_i = \{v_j \in V(G) \mid j \equiv i \pmod 3\}$. Observe that for each pair of vertices $x, y \in V_i$ there are at least two adjacent vertices from $V(G) \smallsetminus V_i$ between x and y. Hence $G[V_i] \in \mathrm{Forb}_{\prec}(T - \{u, v\})$, $i = 0, 1, 2$, since any copy of $T - \{u, v\}$ in $G[V_i]$ extends to a copy of T in G. Hence $\chi(G) \leqslant 3\, f_{\prec}(T - \{u, v\})$ and since $G \in \mathrm{Forb}_{\prec}(H)$ was arbitrary we have $f_{\prec}(H) \leqslant 3\, f_{\prec}(T - \{u, v\})$. □

5 Proofs of Theorems

5.1 Proof of Theorem 1

We will prove that if an ordered graph H contains a cycle, a tangled path or a bonnet then for each positive integer k there is an ordered graph $G \in \mathrm{Forb}_{\prec}(H)$ with $\chi(G) \geqslant k$.

First assume that H contain a cycle of length ℓ. Fix a positive integer k and consider a graph G of girth at least $\ell+1$ and chromatic number at least k that exists by [11]. Then no ordering of the vertices of G gives an ordered subgraph isomorphic to H. This shows that for any positive integer k, $f_{\prec}(H) \geqslant k$ and hence $f_{\prec}(H) = \infty$.

A tangled path is minimal if it does not contain a proper subpath that is tangled. Next we shall show that for each minimal tangled path P and each $k \geqslant 1$ there is an ordered graph $G_k \in \mathrm{Forb}_{\prec}(P)$ with $\chi(G_k) \geqslant k$.

By reversing P if necessary we assume that in P the paths Pu and uP cross for the rightmost vertex u in P. We will prove the claim by induction on k. If

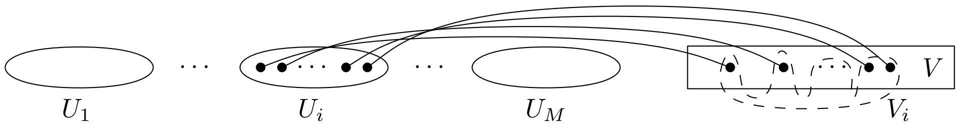

Figure 5: A graph G_k obtained by Tutte's construction from a graph G_{k-1}. Here $G_k[U_i] = G_{k-1}$, $1 \leqslant i \leqslant M$.

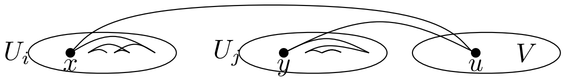

Figure 6: A path in G_k with rightmost vertex $u \in V$ is not tangled if Pu and uP are not tangled.

$k \leqslant 3$ let $G_k = K_k$ that has no crossing edges and thus no tangled paths. Consider $k \geqslant 4$ and let G_{k-1} denote an n-vertex graph of chromatic number at least $k-1$ that does not contain a copy of P. Such a graph exists by induction. The following construction is due to Tutte (alias Blanche Descartes) for unordered graphs [9]. Let $N = (k-1)(n-1)+1$ and $M = \binom{N}{n}$. Consider pairwise disjoint sets of vertices $U_1, \dots, U_M, V$ such that $|U_i| = n$, $i = 1, \dots, M$, $|V| = N$ and $U_1 \prec \dots \prec U_M \prec V$. Let $V_1, \dots, V_M$ be the n-element subsets of V. Let each U_i, $i = 1, \dots, M$, induce a copy of G_{k-1}. Finally let there be a perfect matching between U_i and V_i such that the j^{th} vertex in U_i is matched to the j^{th} vertex in V_i, $i = 1, \dots, M$. See Figure 5.

First we shall show that $\chi(G_k) \geqslant k$. If there are at most $k-1$ colors assigned to the vertices of G_k, then by Pigeonhole Principle there are n vertices of V of the same color, i.e., there is a set V_i with all vertices of the same color, say color 1. Since each vertex of U_i is adjacent to a vertex in V_i, no vertex in U_i is colored 1, so if the coloring is proper, then $G[U_i]$ uses at most $k-2$ colors. Hence the coloring is not proper, since $\chi(G[U_i]) = \chi(G_{k-1}) \geqslant k-1$. Therefore $\chi(G_k) \geqslant k$.

Now, we shall show that G_k does not contain a copy of P. Assume that there is such a copy P' of P in G_k with rightmost vertex u of P'. Let x and y be the neighbors of u in P', i.e., P' is a union of paths $P'yu$ and uxP'. Then $u \in V$ and $x, y \notin V$, since $G[U_i]$ does not contain a copy of P and there are no edges in $G_k[V]$. Let $x \in U_i$ and $y \in U_j$. Note that $i \neq j$ because the edges between U_i and V form a matching. The path uxP' is a proper subpath of P' and hence is not tangled. Recall that for each edge zw with $z \in U_i$, $w \in V$, and $w \prec u$, we have $z \prec x$ due to the construction of the matching between U_i and V_i. Hence the path uxP' does not contain any vertex $w \in V$ with $w \prec u$, since otherwise the path $uxP'w$ has a vertex left of x contradicting Lemma 4.1 applied to u, x and w. Hence $V(xP') \subseteq U_i$, because there are no edges between U_i's and u is rightmost in P'. See Figure 6. Similarly, all vertices of $P'y$ are contained in U_j. Thus $P'u$ and uP' do not cross. However, P' is a copy of P with respective subpaths crossing, a contradiction. Hence $G_k \in \text{Forb}_\prec(P)$.

Now, if an ordered graph H contains a tangled path, then it contains a minimal tangled path. Thus $f_\prec(H) = \infty$.

Now, let B be a bonnet. By reversing B if necessary, we assume that B has vertices $u \prec v \preceq x, y \preceq w$ and edges uv, uw, xy. A shift graph $S(n)$ is defined on vertices $\{(i,j) \mid 1 \leqslant i < j \leqslant n\}$ and edges $\{\{(i,j),(j,t)\} \mid 1 \leqslant i < j < t \leqslant n\}$. We will show that some ordering of $S(n)$ does not contain B. Let $G = S(n)$ be a shift graph with vertices ordered lexicographically, i.e., $(x_1, x_2) \prec (y_1, y_2)$ if and only if $x_1 < y_1$, or $x_1 = y_1$ and $x_2 < y_2$. Assume that G contains vertices $u = (u_1, u_2)$, $v = (v_1, v_2)$, $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $w = (w_1, w_2)$ that form a copy of B with $u \prec v \preceq x, y \preceq w$ and edges uv, uw, xy. Then $u_2 = v_1$, $u_2 = w_1$, $x_2 = y_1$. Thus $v_1 = w_1$. However, since $v \preceq x, y \preceq w$, we have that $v_1 \leqslant x_1, y_1 \leqslant w_1$, so $x_1 = y_1 = v_1 = w_1$. But $x_2 = y_1$, thus $x_2 = x_1$, a contradiction. Thus $G \in \mathrm{Forb}_\prec(B)$. We claim that $\chi(G) \geqslant \log(n) \geqslant \log c|V(G)|$. Indeed consider a proper coloring ϕ of G using $\chi(G)$ colors and sets of colors $\Phi_i = \{\phi(i,j) \mid i < j \leqslant n\}$, $1 \leqslant i \leqslant n$. Then $\phi(i,j) \notin \Phi_j$, since a vertex (i,j) is adjacent to all vertices (j,t), $j < t \leqslant n$. Therefore $\Phi_i \neq \Phi_j$ for all $j < i$. Hence all the sets of colors are distinct. This shows that $2^{\chi(G)} \geqslant n$, since there are at most $2^{\chi(G)}$ distinct subsets of colors. This proves that $\chi(G) \geqslant \log(n)$. Thus, for any k, there is an ordered graph of chromatic number at least k in $\mathrm{Forb}_\prec(B)$. So, if an ordered graph H contains a bonnet, then $f_\prec(H) = \infty$. □

5.2 Proof of Theorem 2

Let T' be a segment of an ordered tree that does not contain a bonnet or a tangled path. We shall prove that T' is monotonically alternating by induction on $k = |V(T')|$. Every ordered tree on at most two vertices is monotonically alternating. So suppose $k \geqslant 3$. We have $\chi_\prec(T') = 2$ due to Lemma 4.3.

Claim. *The leftmost or the rightmost vertex in T' is of degree* 1.

Proof of Claim. For the sake of contradiction assume that both the leftmost vertex u and the rightmost vertex v in T' are of degree at least 2. If u and v are adjacent then the edge uv, another edge incident to u and another edge incident v form a tangled path since $\chi_\prec(T') = 2$, a contradiction. If u and v are not adjacent let P denote the path in T' connecting u and v. It uses at most one of the edges incident to u. Then any other edge zu incident to u crosses the edge in P that is incident to v since $\chi_\prec(T') = 2$. Hence zP forms a tangled path, a contradiction. This shows that at least one of u or v is a leaf in T'. △

By reversing T' if necessary we assume that the leftmost vertex u is a leaf in T'. The ordered tree $T' - u$ is monotonically alternating by induction and Lemma 4.2. Consider the partition $V(T') = L \dot\cup R$, with $L \prec R$ and L and R being independent sets. Such a partition is unique since T' is connected. Let v be the neighbor of u in T'. Since $\chi_\prec(T') = 2$, $v \in R$. Since T' is connected, $k \geqslant 3$ and u is leftmost in T', the edge uv is not the shortest edge incident to v. Hence $uv \notin S(R)$ and therefore $S(R)$ has no crossing edges by induction. Clearly $uv \in S(L)$ since uv is the only edge incident to u and thus it is the shortest incident to u edge. If uv crosses some edge xy in T', $x \prec y$, then all vertices in the path connecting v and x are between x and v due to Lemma 4.1 applied to x, y and v. Therefore xy is not the shortest edge incident to x and hence $xy \notin S(L)$. This shows that $S(L)$ has no crossing edges and thus T' is monotonically alternating.

The other way round assume that each segment of an ordered tree T is monotonically alternating. We need to show that each segment contains neither a bonnet nor a tangled path. Let T' denote a segment of T, $V(T') = L \cup R$, $L \prec R$ and $E(T') = S(L) \cup S(R)$, so each edges is either a shortest edge incident to a vertex in R or a shortest edge incident to a vertex in L. Then $\chi_\prec(T') \leqslant 2$ and hence T' does not contain a bonnet. We will prove that T' does not contain a tangled path by induction on $k = |V(T')|$. If $k \leqslant 3$, then there are no crossing edges in T' and hence no tangled path. Suppose $k \geqslant 4$.

Assume that the leftmost vertex u and the rightmost vertex w in T' are of degree at least 2. If $uw \in E(T')$ then $uw \notin S(L)$ and $uw \notin S(R)$, a contradiction. So, $uw \notin E(T')$. Consider the longest edge xw incident to w. Then $x \neq u$ and since $xw \notin S(R)$, $xw \in S(L)$. Then the shortest edge incident to u crosses xw, a contradiction since $S(L)$ does not contain crossing edges. Hence the leftmost or the rightmost vertex is a leaf in T'.

By reversing T' if necessary we assume that the leftmost vertex u is a leaf. We see that $T' - u$ is monotonically alternating, thus by induction it does not contain a tangled path. Hence if T' has a tangled path P, then P contains an edge uv crossing some other edge in P, where v is the neighbor of u in T'. Then the rightmost vertex r in P is of degree 2 and to the right of v, since P is tangled and u is leftmost and of degree 1 in T'. Let x and y, $x \prec y$, be neighbors of r in P. Then xr is the shortest edge incident to x, since any shorter edge forms a tangled path with r and y in $T' - u$. This is a contradiction since uv and xr cross and T' is monotonically alternating. Thus T' has no tangled path.

Finally we prove the last statement of the theorem. If H is a connected ordered graph with $f_\prec(H) \neq \infty$, then H is a tree that contains neither a bonnet nor a tangled path due to Theorem 1. Hence each segment of H is a monotonically alternating tree. □

5.3 Proof of Theorem 3

Let T be a non-crossing ordered graph such that $f_\prec(T) \neq \infty$. Then T is acyclic, contains no tangled path and no bonnet by Theorem 1. Hence T is a non-crossing ordered forest with no bonnet.

On the other hand let T be a non-crossing forest with no bonnet. Recall that $f_\prec(H) \geqslant k-1$ for each ordered k-vertex graph H because $K_{k-1} \in \mathrm{Forb}_\prec(H)$. We shall prove that $f_\prec(T) \neq \infty$. Let $k = |V(T)|$ and consider any ordered graph $G \in \mathrm{Forb}_\prec(T)$. We will prove by induction on k that $\chi(G) \leqslant 2^k$ and $\chi(G) \leqslant 2k-3$ if T is a tree. If $k = 2$, then clearly $\chi(G) = 1$. So consider $k \geqslant 3$.

If T is a tree, then each segment of T is a monotonically alternating tree, by Theorem 2. If there is only one segment in T, then $f_{\prec}(T) \leqslant 2k-3$ by Lemma 4.4. If there is more than one segment in T, then there is an inner cut vertex splitting T into two trees T_1 and T_2 that are clearly also non-crossing and contain no bonnet. Thus by Reduction Lemma 1 and induction we have $f_{\prec}(T) \leqslant f_{\prec}(T_1) + f_{\prec}(T_2) \leqslant 2|V(T_1)| - 3 + 2|V(T_2)| - 3 = 2(|V(T)| + 1) - 6 = 2k - 4$.

If T is a forest we consider several cases. If T has more than one segment, then there is an inner cut vertex splitting T into two forests T_1 and T_2 that are clearly also non-crossing and contain no bonnet. Thus by Reduction Lemma 1 and induction we have $f_{\prec}(T) \leqslant f_{\prec}(T_1) + f_{\prec}(T_2) \leqslant 2^{|V(T_1)|} + 2^{|V(T_2)|} = 2^t + 2^{k+1-t} \leqslant 2^k$ with $t = |V(T_1)| \geqslant 2$. If T has an isolated vertex u, then by Reduction Lemma 2 and induction we have $f_{\prec}(T) \leqslant 2f_{\prec}(T-u) \leqslant 2 \cdot 2^{k-1} = 2^k$. Finally, if T has no isolated vertices and exactly one segment, then consider the leftmost and rightmost vertices u and v of T. Since u and v are not isolated in this case, and T is non-crossing with no inner cut vertices, uv is an edge. If uv is isolated, then $k \geqslant 4$ (since there is no isolated vertex) and by Reduction Lemma 3 and induction we have $f_{\prec}(T) \leqslant 2 \cdot f_{\prec}(T - \{u,v\}) + 1 \leqslant 2 \cdot 2^{k-2} + 1 \leqslant 2^k$. If uv is not isolated, then either u or v, say u, is a leaf of T, since T is non-crossing and does not contain a bonnet. Let xv denote the longest edge incident to v in $T-u$. Note that x exists since the edge uv is not isolated. Then there is no other vertex between u and x, since such a vertex would be isolated in the non-crossing forest T without bonnets. Thus, u is a reducible vertex, so by Reduction Lemma 4 and induction we have $f_{\prec}(T) \leqslant 2f_{\prec}(T-u) \leqslant 2 \cdot 2^{k-1} = 2^k$.

Next, we provide a k-vertex non-crossing tree with no bonnet such that $\infty \neq f_{\prec}(T) \geqslant k$. Let T be a monotonically alternating path on $k \geqslant 4$ vertices with leftmost vertex of degree 1, as in Figure 7 (right). Further let G denote a graph on vertices $u \prec x_1 \prec \cdots \prec x_{k-2} \prec y_1 \prec \cdots \prec y_{k-2} \prec x \prec y$ such that xy is an edge and $\{u, x_1, \ldots, x_{k-2}\}$, $\{u, y_1, \ldots, y_{k-2}\}$, $\{x, x_1, \ldots, x_{k-2}\}$, and $\{y, y_1, \ldots, y_{k-2}\}$ induce complete graphs on $k-1$ vertices each. See Figure 7 (left).

We shall show that $G \in \mathrm{Forb}_{\prec}(T)$ and $\chi(G) \geqslant k$. Consider a proper vertex coloring of G using colors $1, \ldots, k-1$. Without loss of generality u has color 1. Then all colors $2, \ldots, k-1$ are used on the vertices $x_1, \ldots, x_{k-2}$ as well as on $y_1, \ldots, y_{k-2}$. Hence both x and y are of color 1, a contradiction. Thus $\chi(G) \geqslant k$.

Assume that there is a copy P of T in G. Let v be the leftmost and w be the rightmost vertex in P. Note that vw is an edge and that there are k vertices between v and w. Therefore vw is one of the edges uy_i, $1 \leqslant i \leqslant k-2$, $x_j x$, $1 \leqslant j \leqslant k-2$, or $y_1 y$. In the first case $V(P) \subseteq \{u, y_1, \ldots, y_{k-2}\}$, in the second case $V(P) \subseteq \{x_1, \ldots, x_{k-2}, x\}$ and in the last case either $P = y_1, y, x$ or $V(P) \subseteq \{y, y_1, \ldots, y_{k-2}\}$. Since T has at least 4 vertices, $P \neq y_1, y, x$. So in any case P has at most $k-1$ vertices, a contradiction since T has k vertices. Hence $G \in \mathrm{Forb}_{\prec}(T)$.

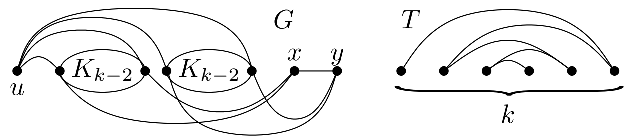

Figure 7: An ordered graph G with chromatic number k not containing a non-crossing and ordered tree T on k vertices without bonnets on the right, $k = 6$.

Finally it is easy to see that $f_\prec(T) = k - 1$ for any ordered tree T on at most 3 vertices using Reduction Lemmas 1 and 4. □

5.4 Proof of Theorem 4

- Let T be an ordered forest on k vertices where each segment is a generalized star, a 2-nesting, or a 2-crossing. Let $T_1, \ldots, T_s$ denote the segments of T and $k_i = |V(T_i)|$, $1 \leqslant i \leqslant s$. Let T' be a segment of T. If T' is a generalized star on k' vertices, then the center of the star is leftmost (or rightmost) in T'. Let $G \in \text{Forb}_\prec(T')$. Then each vertex in G has at most $k' - 2$ neighbors to the right (or to the left). Thus each such graph can be greedily colored from right to left (or left to right) with at most $k' - 1$ colors. This shows that $f_\prec(T') \leqslant |V(T')| - 1$. If T' is a 2-nesting, then $f_\prec(T') = 3 = |V(T')| - 1$ due to [10] (Lemma 9). If T' is a 2-crossing, then $f_\prec(T') = 3 = |V(T')| - 1$, since any graph not containing T' is outerplanar and outerplanar graphs have chromatic number at most 3. We apply Reduction Lemma 1 and the results above which yield $f_\prec(T) \leqslant \sum_{i=1}^{s} f_\prec(T_i) \leqslant \sum_{i=1}^{s}(k_i - 1) = k - 1$.

- Let T be an ordered forest on k vertices where each segment is a generalized star, a non-crossing tree without bonnets, a crossing or a nesting. Let $T_1, \ldots, T_s$ denote the segments of T and $k_i = |V(T_i)| \geqslant 2$. Let T' be a segment of T. If T' is a k'-nesting or a k'-crossing, $k' \geqslant 2$, then $f_\prec(T') \leqslant 4(k' - 1) \leqslant 2|V(T')| - 3$ due to equation (1), since any graph $G \in \text{Forb}_\prec(T')$ contains less than $2(k' - 1)|V(G)|$ edges due to Dujmovic and Wood [10] (for nestings), respectively Capoyleas and Pach [5] (for crossings). Further $f_\prec(T') \leqslant 2|V(T')| - 3$ if T' is a non-crossing tree without bonnets due to Theorem 3. Hence Reduction Lemma 1 yields $f_\prec(T) \leqslant \sum_{i=1}^{s} f_\prec(T_i) \leqslant \sum_{i=1}^{s}(2k_i - 3) \leqslant 2k - 3$.

- Let $T = M(t, m, \pi)$ for some positive integers m and t and a permutation π of $[t]$. If $t = 1$, then $f_\prec(T) = m$ due to the results above, since $M(1, m, \pi)$ is a star on $m + 1$ vertices. Weidert [19] proves that $\text{ex}_\prec(n, M(t, 1, \pi)) \leqslant \text{ex}_\prec(n, M(t, 2, \pi)) \leqslant 11t^4\binom{2t^2}{2t}n < t^4(2t^2)^{2t}n$ for any positive integer $t \geqslant 2$ and any permutation π of $[t]$. Moreover if $m \geqslant 2$, then

$$\text{ex}_\prec(n, M(t, m, \pi)) \leqslant 2^{t(m-2)}\text{ex}_\prec(n, M(t, 2, \pi))$$

due to a reduction by Tardos [18]. Therefore $\text{ex}_\prec(n, M(t, m, \pi)) < 2^{tm}t^{4+4t}n$.

Thus, using the fact that $|V(T)| = k = tm + t$ and equation (1) we have that $f_\prec(M(t,m,\pi)) \leqslant 2^{tm+9t\log(t)} \leqslant 2^{10k\log k}$.

- Conlon *et al.* [7] and independently Balko *et al.* [2] prove that that there is a positive constant c such that for any sufficiently large positive integer k there is an ordered matchings on k vertices with ordered Ramsey number at least $2^{c\frac{\log(k)^2}{\log\log(k)}}$. If, for some ordered graph H, the edges of a complete ordered graph G on $N = R_\prec(H) - 1$ vertices are colored in two colors without monochromatic copies of H, then both color classes form ordered graphs G_1 and G_2 in $\mathrm{Forb}_\prec(H)$. Then one of the G_i's has chromatic number at least $\sqrt{N}$, since a product of proper colorings of G_1 and G_2 yields a proper coloring of G using $\chi(G_1)\chi(G_2) \geqslant \chi(G) = N$ colors. This shows that there is a positive constant c' such that for all positive integers k and ordered matchings H on k vertices with $f_\prec(H) \geqslant 2^{c'\frac{\log(k)^2}{\log\log(k)}}$. □

6 Small Forests

Let P_k denote a path on k vertices, M_k a matching on k edges and S_k a star with k leaves (note that $M_1 = S_1 = P_2$ and $P_3 = S_2$). Further let $G + H$ denote the vertex disjoint union of graphs G and H. Then the set of all forests without isolated vertices and at most 3 edges is given by

$$\{P_2, S_2, M_2, S_3, P_4, S_2 + P_2, M_3\}.$$

Let G denote a graph on n vertices and a automorphisms. Then the number $\mathrm{ord}(G)$ of non-isomorphic orderings of G equals $\mathrm{ord}(G) = \frac{n!}{a}$. Hence

$$\mathrm{ord}(P_2) = \tfrac{2!}{2} = 1,\ \mathrm{ord}(S_2) = \tfrac{3!}{2} = 3,\ \mathrm{ord}(M_2) = \tfrac{4!}{8} = 3,\ \mathrm{ord}(S_3) = \tfrac{4!}{3!} = 4,$$
$$\mathrm{ord}(P_4) = \tfrac{4!}{2} = 12,\ \mathrm{ord}(S_2 + P_2) = \tfrac{5!}{2\cdot 2} = 30,\ \mathrm{ord}(M_3) = \tfrac{6!}{6\cdot 4\cdot 2} = 15.$$

Recall that the reverse $\overline{T}$ of an ordered graph T is the ordered graph obtained by reversing the ordering of the vertices in T. Note that $f_\prec(T) = f_\prec(\overline{T})$ for any ordered graph T since $G \in \mathrm{Forb}_\prec(T)$ if and only if $\overline{G} \in \mathrm{Forb}_\prec(\overline{T})$. Table 8 shows all ordered forests T without isolated vertices and at most 3 edges and their $f_\prec$ values, where only one of T and $\overline{T}$ is listed. So when T and $\overline{T}$ are not isomorphic ordered graphs the entry in the table represents two graphs. Such cases are marked with an $*$. For example there are only two instead of three entries for S_2 and similarly for the other graphs.

7 Conclusions

In this paper, we consider the function $f_\prec(H) = \sup\{\chi(G) \mid G \in \mathrm{Forb}_\prec(H)\}$ for ordered graphs H on at least 2 vertices. We prove that in contrast to unordered and

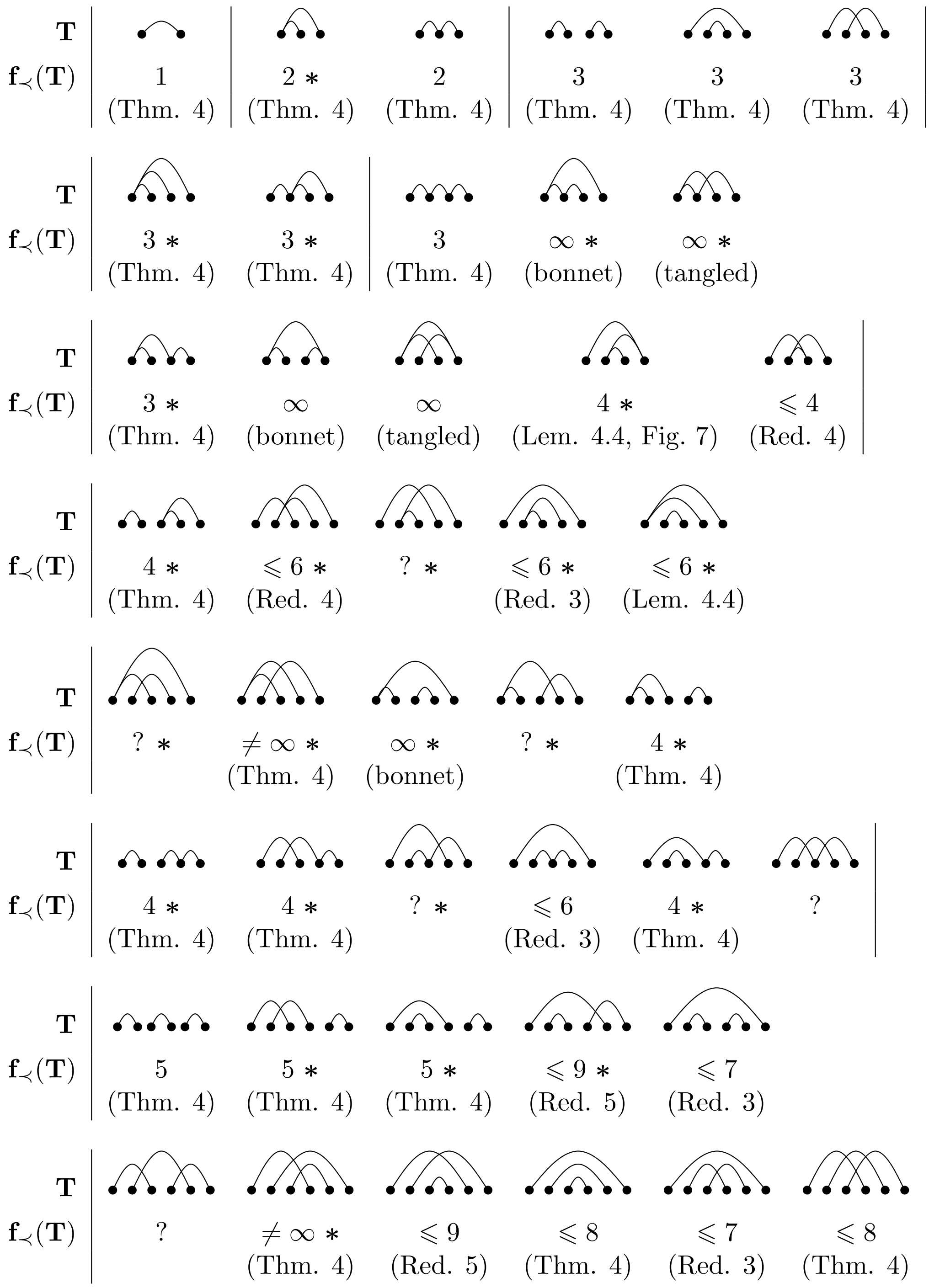

Figure 8: All ordered forests T on at most 3 edges without isolated vertices and their $f_\prec$ value.

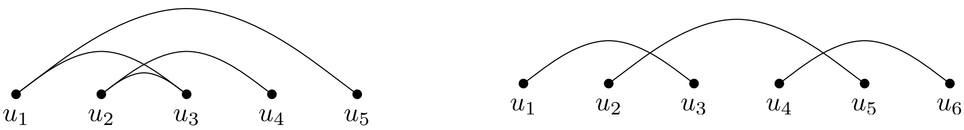

Figure 9: Ordered graphs H for which we don't know whether $f_\prec(H) = \infty$.

directed graphs, $f_\prec(H) = \infty$ for some ordered forests H. To this end we explicitly describe several infinite classes of minimal ordered forests H with $f_\prec(H) = \infty$. A full answer to the following question remains open.

Question 1. *For which ordered forests H does $f_\prec(H) = \infty$ hold?*

We completely answer Question 1 for non-crossing ordered graphs H. Suppose that H is a non-crossing ordered k-vertex graph with $f_\prec(H) \neq \infty$. We prove that, if H connected, then $k - 1 \leqslant f_\prec(H) \leqslant 2k - 3$ and, if H is disconnected, then $k-1 \leqslant f_\prec(H) \leqslant 2^k$. In addition, we give infinite classes of graphs for which $f_\prec(H) = |V(H)|-1$, as well as infinite classes of graphs for which $|V(H)| \leqslant f_\prec(H) \neq \infty$. Note that we do not know whether $f_\prec(H) \neq \infty$ for the matchings in the last statement of Theorem 4. For crossing connected ordered graphs, we reduce Question 1 to monotonically alternating trees:

Question 2. *For which monotonically alternating trees H does $f_\prec(H) = \infty$ hold?*

We do not have an answer to Question 2 even for some monotonically alternating paths. A smallest unknown such path is $u_5u_1u_3u_2u_4$, where $u_1 \prec \cdots \prec u_5$. See Figure 9 (left). The situation becomes even more unclear for crossing disconnected graphs. We do not know the value of $f_\prec(H)$ for some ordered matchings H. A smallest such matching has edges u_1u_3, u_2u_5 and u_4u_6 where $u_1 \prec \ldots \prec u_6$. See Figure 9 (right). Note that Reduction Lemmas 1, 2, 3 and 4 apply to crossing ordered graph as well. We find a more precise version of Reduction Lemma 2 and other types of reductions, similar to reductions for matrices in [18], but none of these lead to significantly better upper bounds in Theorems 3 and 4 or a new class of forests with finite $f_\prec$. The following question remains open, even when restricted to non-crossing graphs.

Question 3. *For $k \geqslant 4$, what is the value of the function*

$$f_\prec(k) = \max\{f_\prec(H) \mid \quad |V(H)| = k,\ f_\prec(H) \neq \infty\}?$$

References

[1] L. Addario-Berry, F. Havet, C. L. Sales, B. Reed, and S. Thomassé. Oriented trees in digraphs. *Discrete Math.*, 313(8):967–974, 2013.

[2] M. Balko, J. Cibulka, K. Král, and J. Kynčl. Ramsey numbers of ordered graphs. *Electronic Notes in Discrete Mathematics*, 49:419 – 424, 2015. The Eight European Conference on Combinatorics, Graph Theory and Applications, EuroComb 2015.

[3] P. Brass, G. Károlyi, and P. Valtr. A Turán-type extremal theory of convex geometric graphs. In *Discrete and computational geometry*, volume 25 of *Algorithms Combin.*, pages 275–300. Springer, Berlin, 2003.

[4] S. A. Burr. Subtrees of directed graphs and hypergraphs. In *Proceedings of the Eleventh Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1980), Vol. I*, volume 28 of *Congr. Numer.*, pages 227–239, 1980.

[5] V. Capoyleas and J. Pach. A Turán-type theorem on chords of a convex polygon. *J. Combin. Theory Ser. B*, 56(1):9–15, 1992.

[6] V. Chvátal. Perfectly ordered graphs. In *Topics on perfect graphs*, volume 88 of *North-Holland Math. Stud.*, pages 63–65. North-Holland, Amsterdam, 1984.

[7] D. Conlon, J. Fox, C. Lee, and B. Sudakov. Ordered Ramsey numbers. *J. Combin. Theory Ser. B.* to appear.

[8] P. Damaschke. Forbidden ordered subgraphs. In *Topics in combinatorics and graph theory (Oberwolfach, 1990)*, pages 219–229. Physica, Heidelberg, 1990.

[9] B. Descartes. A three colour problem. *Eureka*, 21, 1947.

[10] V. Dujmovic and D. R. Wood. On linear layouts of graphs. *Discrete Math. Theor. Comput. Sci.*, 6(2):339–358, 2004.

[11] P. Erdős. Graph theory and probability. *Canad. J. Math.*, 11:34–38, 1959.

[12] Z. Füredi and P. Hajnal. Davenport-Schinzel theory of matrices. *Discrete Math.*, 103(3):233–251, 1992.

[13] M. Ginn. Forbidden ordered subgraph vs. forbidden subgraph characterizations of graph classes. *J. Graph Theory*, 30(2):71–76, 1999.

[14] M. Klazar. Extremal problems for ordered (hyper) graphs: applications of Davenport-Schinzel sequences. *European J. Combin.*, 25(1):125–140, 2004.

[15] M. Klazar. Extremal problems for ordered hypergraphs: small patterns and some enumeration. *Discrete Appl. Math.*, 143(1-3):144–154, 2004.

[16] A. Marcus and G. Tardos. Excluded permutation matrices and the Stanley-Wilf conjecture. *J. Combin. Theory Ser. A*, 107(1):153–160, 2004.

[17] J. Pach and G. Tardos. Forbidden paths and cycles in ordered graphs and matrices. *Israel J. Math.*, 155:359–380, 2006.

[18] G. Tardos. On 0-1 matrices and small excluded submatrices. *J. Combin. Theory Ser. A*, 111(2):266–288, 2005.

[19] C. Weidert. Extremal problems in ordered graphs. Master's thesis, Simon Fraser University, 2009. arXiv:0907.2479.

[20] D. B. West. *Introduction to graph theory.* Prentice Hall, Inc., Upper Saddle River, NJ, 1996.