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WHEN DO THREE LONGEST PATHS HAVE A COMMON VERTEX?

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Abstract. It is well known that any two longest paths in a connected graph share a vertex. It is also known that there are connected graphs where 7 longest paths do not share a common vertex. It was conjectured that any three longest paths in a connected graph have a vertex in common. In this note we prove the conjecture for outerplanar graphs and give sufficient conditions for the conjecture to hold in general.

1. Introduction

Gallai asked in 1966 whether every connected graph has a vertex that appears in all its longest paths. Zamfirescu [6] found a graph with 12 vertices in which there is no common vertex to all longest paths. See Voss, [5] for related problems. It is well-known that every two longest paths in a connected graph have a common vertex. Skupien [4] obtained, for $k \geq 7$, a connected graph in which some k longest paths have no common vertex, but every $k-1$ longest paths have a common vertex. Klavzar et al. [3] showed that in a connected graph G, all longest paths have a vertex in common if and only if for every block B of G all longest paths in G which use at least one edge of B have a vertex in common. Thus, if every block of a graph G is Hamilton-connected, almost Hamilton-connected, or a cycle, then all longest paths in G have a vertex in common. It was also proved in [3] that in a split graph all longest paths intersect. Balister et al. [1] showed that all longest path of a circular arc graphs share a common vertex. Still, the following conjecture remains open in general:

Conjecture 1 *For any three longest paths in a connected graph, there is a vertex which belongs to all three of them.*

In this note we prove the conjecture for special classes of graphs, more specificaly for triples of longest paths whose union belongs to special classes of graphs. One of our main results is the following.

Theorem 1. *Let G be a connected graph and P_0, P_1, P_2 be its longest paths. If $P_0 \cup P_1 \cup P_2$ forms an outerplanar graph then there is a vertex $v \in V(P_0) \cap V(P_1) \cap V(P_2)$.*

Note that using Kuratowski theorem and the above result, we see that if the union of three longest paths in a connected graph does not contain K_4 or $K_{2,3}$ as a topological minor, or as a minor then there is a vertex common to all three of these paths.

2. Definitions and Results

We say that a family F, of graphs is monotone if for any $G \in F$, and any $e \in E(G)$, $G - e \in F$. For a path $P = x_0, x_1, \ldots, x_k$, we say that path $P' = x_i, x_{i+1}, \ldots, x_j$, $j > i$, is a segment of P; we say that P' is an end-segment of P if $i = 0$ or $j = k$. The addition of indeces will be taken modulo 3. For a path P, $|P|$ denotes its lengths. The following definition describes two configurations forbidden in the union of three longest path with no common vertex, see Figure 1.

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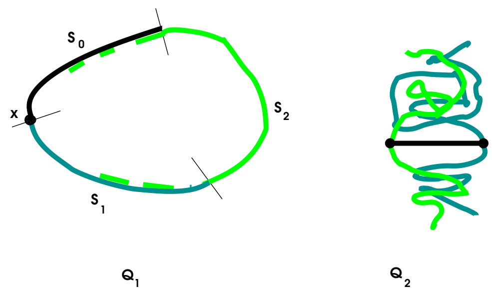

FIGURE 1. Configurations impossible in the union of three longest paths of a connected graph.

Definition 1.

Configuration Q1 is a cycle which is the union of internally disjoint segments S_0, S_1, S_2 of P_0, P_1, P_2, such that a) interior of S_0 and interior of S_2 do not contain any vertices of P_1 and b) interior of S_1 and interior of S_2 do not contain any vertices of P_0.

Configuration Q2 is defined by a segment P of P_0 with end-points x, y, such that
a) $x \in V(P_1)$, $y \in V(P_2)$,
b) internal vertices of P belong to P_0 only, and
c) $P_1 - \{x\}$ is the union of two paths P_1', P_1''; $P_2 - \{y\}$ is the union of two paths P_2', P_2''; such that $V(P_1' \cup P_2') \cap V(P_1'' \cup P_2'') = \emptyset$ or $V(P_1' \cup P_2'') \cap V(P_1'' \cup P_2') = \emptyset$.

Lemma 1. *Let G be the union of its longest paths P_0, P_1, P_2. Then G does not contain either configuration Q_1 or Q_2.*

Proof. Assume that G contains configuration Q_1. Let the length of S_i be ℓ_i, $i = 0, 1, 2$. Replacing S_0 with $S_1 \cup S_2$ in path P_0 requires that $\ell_0 \geq \ell_1 + \ell_2$. Similarly, replacing S_1 with $S_0 \cup S_2$ gives $\ell_1 \geq \ell_0 + \ell_2$. Adding these inequalitied gives $\ell_2 \leq 0$, a contradition to the fact that S_2 is a segment of positive length.

Assume that G has a configuration Q_2, let $c = |P|$. Let y split P_2 into segments of lengths ℓ_1 and ℓ_2. Let x split P_1 into segments of lengths ℓ_1' and ℓ_2', respectively, so $\ell = \ell_1 + \ell_2 = \ell_1' + \ell_2' = |P_i|$, $i = 0, 1, 2$. Then, without loss of generality, we have paths of length $\ell_1 + c + \ell_1'$ and $\ell_2 + c + \ell_2'$. So, $c + \ell_1 + \ell_1' \leq \ell_1 + \ell_2$, $c + \ell_2 + \ell_2' \leq \ell_1' + \ell_2'$. By adding these last two inequalities, we get that $c = 0$, a contradiction. □

Lemma 2. *For a monotone family F, let $G \in F$ be a connected graph with smallest $|V(G)| + |E(G)|$ having three longest paths with no common vertex. Then G has exactly one nontrivial block.*

Proof. Assume that G has at least two nontrivial blocks and there are three longest paths P_0, P_1, P_2 which do not have a vertex in common. By minimality we have then that $G = P_0 \cup P_1 \cup P_2$ since otherwise we could delete edges not in $P_0 \cup P_1 \cup P_2$. If there are two nontrivial blocks each containing the vertices of all three paths, then clearly there is a cutvertex contained in all these paths. Thus there is a nontrivial block, say B, containing vertices of only two paths, say of P_0 and P_1. Thus P_2 is in a component, X, of $G - B$. Since P_2

intersects P_0 and P_1, X must contain vertices of P_0 and P_1, in particular, X contains one endpoints of P_0 and one endpoint of P_1. Thus G has a cut-vertex, $u \in V(B)$, such that $G - \{u\}$ has disjoint graphs with vertex sets X_1', X_2', where $X_1 = X_1' \cup \{u\}$ has vertices of P_0 and P_1 only, and $X_2 = X_2' \cup \{u\}$ has vertices from all three paths, satisfying an additional property that one endpoint of P_j is in X_1 and another endpoint of P_j in in X_2, $j = 0, 1$.

Let $\ell_1 = |P_0[X_1]|$ and $\ell_2 = |P_1[X_1]|$. Then $\ell_1 = \ell_2$, otherwise, if say $\ell_2 > \ell_1$, then $P_1[X_1] \cup P_0[V \setminus X_1]$ is a path of length greater than the maximum length, a contradiction. Thus one can replace $G[X_1]$ with a single path $P_0[X_1]$. The resulting graph has less edges and it has three longest paths P_0, P_2 and $P_1' = P_1[V - X_1] \cup P_0[X_1]$. Moreover, P_0, P_1', P_2 do not share a common vertex. In addition, if G is from a monotone family F then the resulting graph is from F too. A contradiction to minimality of G. If G has no nontrivial blocks, i.e., G is a tree, then the result follows instantly since each longest path must pass through the center. □

Lemma 3. *Let G be a connected graph which is the union of its longest paths, P_0, P_1, P_2. If $P_1 \cup P_2$ has at most one cycle then there is a vertex common to all three paths P_0, P_1, P_2.*

Proof. The path P_0 intersect both P_1 and P_2. Thus, there is a segment of P_0 with one endpoint, x, in P_1 and another endpoint, y, in P_2 such that all other vertices on this segment belong only to P_0. If $P_1 \cup P_2$ is acyclic then we have forbidden configuration Q_2. So we assume that $P_1 \cup P_2$ has a unique cycle C. If $x, y \in V(C)$, or $x, y \notin V(C)$ then we have a forbidden configuration Q_2. If $x \in V(C), y \notin V(C)$, we have a forbidden configuration Q_1. □

Now, we are ready to prove the main theorem.

Proof of Theorem 1. Throughout the proof we shall say that an edge e is in the face F, or that F has an edge e, if e is an edge on the boundary of F. Assume that there is a graph with three longest paths forming an outerplanar graph and such that these paths do not have a vertex in common. Let G be the union of these paths, P_0, P_1, P_2 and assume further that G is minimum such graph. By Lemma 2, G has exactly one nontrivial block, B. Consider the outer-planar embedding of G. Let C be the cycle of the unbounded face. We refer to the edges of G which are not in C but have endpoints in C as chords. Clearly, the vertices of each chord form a cutset of G, and any two bounded faces are separated by some chord. Each chord e splits G into two new outerplanar graphs, $G(e)'$, $G(e)''$, such that $G(e)' \cup G(e)'' = G$ and $G(e)' \cap G(e)'' = e$. Observe also that for each $i \in \{0, 1, 2\}$ there is a chord which belongs to P_i, otherwise, if, for example, there is no chord from P_0, we have that $P_1 \cup P_2$ has at most one cycle, a contradiction to Lemma 3. We call a face using edges of all three paths P_0, P_1, P_2 *three-colored*.

Claim 1 There is a bounded three-colored face.
Assume that there is no bounded three-colored face. We have from above observation that for each $i \in \{0, 1, 2\}$ there is a bounded face containing edges from P_i. Then, there are two bounded faces which share a chord e such that, without loss of generality, one of these faces has edges from $P_0 \cup P_1$ and another has edges from $P_1 \cup P_2$. Then $e = \{x, y\} \in E(P_1)$. Moreover, $P_2 \subseteq G(e)'$ and $P_0 \subseteq G(e)''$. Since P_2 and P_0 intersect, we have that $V(P_0) \cap V(P_2) \subseteq \{x, y\}$, but $x, y \in V(P_1)$, so there is a vertex which belongs to all three paths, a contradiction.

Claim 2 For every chord e, either $G(e)'$ or $G(e)''$ has edges from at most two paths, P_i, P_j, $i \in \{0, 1, 2\}$.
Assume that $e = \{x, y\} \in E(P_0)$, a chord such that $G(e)'$ and $G(e)''$ both have edges of all three paths P_0, P_1, P_2. We have then that P_1 and P_2 must pass through $\{x, y\}$. Because no vertex belongs to all three paths, we have that $x \in V(P_1)$ and $y \in V(P_2)$, which gives a forbidden configuration Q_2.

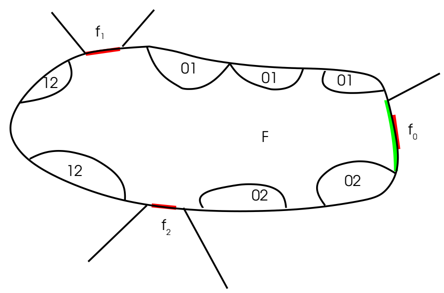

FIGURE 2. Three-colored face F with chords of types $\{0,1\}$, $\{1,2\}$, $\{2,0\}$ on its boundary.

This implies that there is exactly one bounded three-colored face, call it F. Let F have chords $e_0, e_1, \dots, e_{k-1}$, $k \geq 3$ on its boundary, in order. Then let $G(e_i)'$, $i = 0, \dots, k-1$ not contain F. Then, $G(e_i)'$ uses vertices of only two paths, P_{i_1}, P_{i_2}, $i_1, i_2 \in \{0, 1, 2\}$, by Claim 2. We say that e_i has *type* $\{i_1, i_2\}$, see Figure 2.

Claim 3 For each $i_1, i_2 \in \{0, 1, 2\}$ there is an e_i of type $\{i_1, i_2\}$ for soem $i \in \{0, 1, \dots, k-1\}$.
Assume that there is no e_i of type $\{0,1\}$. Since there are chords from all three paths, we have that there are e_j, e_l, $j, l \in \{0, 1, \dots, k-1\}$ such that e_j is of type $\{1,2\}$ and e_l is of type $\{0,2\}$. Let, without loss of generality $l = 1, j = 2$. Thus $G'(e_1) \subseteq P_0 \cup P_2$ and $G'(e_2) \subseteq P_1 \cup P_2$. Then there is an edge $\tilde{e}$ on $G(e_1)' - G(e_2)'$ path such that $\tilde{e} \in E(P_2) \setminus E(P_0 \cup P_1)$. Thus $C \not\subseteq P_0 \cup P_1$. Therefore $P_0 \cup P_1$ is acyclic, a contradiction to Lemma 3.

Claim 4 The chords e_i's of the same type appear consecutively along C.
Assume that e_{i_1}, e_{i_3} are of type $\{0,1\}$, e_{i_2} is or type $\{0,2\}$, and e_{i_4} is of type $\{1,2\}$, for some $i_1 < i_2 < i_3 < i_4$, in cyclic order on C. Let vertex x be incident to e_{i_1} and vertex y be incident to e_{i_3}. We have that $x, y \in V(P_0) \cup V(P_1)$, moreover, $\{x, y\}$ is a cutset of G, and $G'(e_{i_2})$ and $G'(e_{i_4})$ are in the different components of $G - \{x, y\}$. Since $G'(e_{i_2})$ and $G'(e_{i_4})$ contain edges of P_2, it implies that P_2 is disconnected, a contradiction.

To conclude the proof, let $e_0, e_2, \dots, e_S$ be of type $\{0,1\}$, $e_{s+1}, \dots, e_t$ be of type $\{1,2\}$ and $e_{t+1}, \dots, e_{k-1}$ be of type $\{0,2\}$. There is an edge $f_1 \in E(C)$ between $G(e_s)'$ and $G(e_{s+1})'$, so that $f_1 \in E(P_1) \setminus E(P_0 \cup P_2)$. Similarly, there is $f_2 \in E(C) \cap E(P_2) \setminus E(P_0 \cup P_1)$, $f_0 \in E(C) \cap E(P_0) \setminus E(P_2 \cup P_1)$, see Figure 2. Consider the longest segment of P of P_0 containing f_0 such that all its vertices except for the endvertices are not in $V(P_2) \cup V(P_1)$, one endvertex is in $V(P_1)$ and another endvertex is in $V(P_2)$. Then this segment defines configuration Q_2, a contradiction.

□

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