Splitting Planar Graphs of Girth 6 into Two Linear Forests with Short Paths.

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Abstract

Recently, Borodin, Kostochka, and Yancey (On 1-improper 2-coloring of sparse graphs. *Discrete Mathematics*, 313(22), 2013) showed that the vertices of each planar graph of girth at least 7 can be 2-colored so that each color class induces a subgraph of a matching. We prove that any planar graph of girth at least 6 admits a vertex coloring in 2 colors such that each monochromatic component is a path of length at most 14. Moreover, we show a list version of this result. On the other hand, for each positive integer $t \geq 3$, we construct a planar graph of girth 4 such that in any coloring of vertices in 2 colors there is a monochromatic path of length at least t. It remains open whether each planar graph of girth 5 admits a 2-coloring with no long monochromatic paths.

1 Introduction

In this paper, we consider the question of partitioning the vertex set of a planar graph into a small number of parts, also referred to as color classes, such that each part induces a graph whose connected components are short paths. The length of a path is the number of its edges. The Four Color Theorem [3, 4] implies that four parts are sufficient to guarantee such a partition with paths of length 0, i.e., on 1 vertex each. A result of Goddard [20] and Poh [26] shows that any planar graph can be vertex-colored with 3 colors such that each monochromatic component is a path. However, one can not always restrict the lengths of monochromatic paths in 3-colorings of planar graph as was shown by a specific triangulation construction of Chartrand, Geller and Hedetniemi [11]. Chappell, Gimbel, and Hartman [10] gave an explicit construction of a planar graph of girth 4 that can not be vertex colored in two colors such that each color class induces a path forest.

However, when the girth of a planar graph is sufficiently large, one can not only 3-color, but 2-color the vertices of the graph such that monochromatic components are short paths.

Borodin, Kostochka, and Yancey [9] proved that the vertices of each planar graph of girth at least 7 can be 2-colored so that each monochromatic component has at most 2 vertices, i.e., is a path of length at most 1. Note that the order of monochromatic components can not be decreased to 1 as long as the graph is not bipartite.

Chappell, Gimbel, and Hartman [10] proved that any planar graph of girth at least 6 can be 2-colored such that each monochromatic component is a path, however no bound on the sizes of these paths was given. Borodin and Ivanova [8] conjectured that there is such a coloring with monochromatic components being paths of length at most 2.

Here, we show that planar graphs of girth at least 6 can be 2-colored such that each monochromatic component is a path of length at most 14. Moreover, we prove a list version of this result. On the other hand, for each positive integer $t \geq 3$, we construct a planar graph of girth 4 such that in any coloring of vertices in 2 colors there is a monochromatic path of length at least t.

It remains open whether one can 2-color the vertices of a planar graph of girth 5 such that each monochromatic component is a short path.

Note that the problem we consider is a problem of *strong linear arboricity* or a *k-path chromatic number* introduced by Borodin *et al.* [8] and Akiyawa *et al.* [1] respectively. Here, a linear arboricity of a graph is the smallest number of parts in a vertex-partition of the graph such that each part induces a forest with path components. The k-strong linear arboricity or *k-path chromatic number* is the smallest number of colors in a vertex-coloring of the graph such that each monochromatic component is a path on at most k vertices.

Let L be a color list assignment for vertices of a graph G, i.e., $L : V(G) \to 2^{\mathbb{Z}}$. We say that c is an L-coloring if $c : V \to \mathbb{Z}$ such that $c(v) \in L(v)$ for each $v \in V(G)$.

We prove the following theorems.

Theorem 1. *For any planar graph of girth at least* 6 *and any list assignment L with lists of size* 2 *there is an L-coloring so that each monochromatic component is a path of length at most* 14.

Theorem 2. *For every positive integer t there is a planar graph G_t of girth* 4 *such that any vertex coloring of G_t in two colors results in a monochromatic path of length $t - 1$.*

Our results are a contribution to the lively and active field of *improper vertex colorings of planar graphs*, where the number of colors is strictly less than 4 but various restrictions on the monochromatic components are imposed. For standard graph theoretic notions used here, we refer to [15].

Organization of the paper. In Section 2 we give a short survey of improper colorings of planar graphs, explain the relation to our results in the present paper, and point out some open problems. In Section 3 we prove Theorem 1 and in Section 4 we prove Theorem 2. We conclude with some open questions in Section 5.

2 Improper Colorings of Planar Graphs

A proper vertex-coloring of a graph is a coloring in which each monochromatic component is a single vertex, or, equivalently, in which there are no two adjacent vertices of the same color. In this paper, a c-coloring, $c \geq 1$, of a graph is a (not necessarily proper) vertex coloring using c colors. As every planar graph has a proper 4-coloring, we focus here on 2-colorings and 3-colorings. The most studied variants of improper colorings are *defective*, *fragmented* and *P_k-free colorings*. A survey on the topic was done in the bachelor thesis of Pascal Weiner [28].

Defective colorings. For a non-negative integer k, a vertex coloring is called *k-defective* if each monochromatic component has maximum degree at most k. We define $k_d(g, c)$ to be the smallest k such that every planar graph of girth at least g admits a k-defective c-coloring. Defective colorings were introduced in 1986 by Cowen, Cowen and Woodall [13], who showed that $k_d(3, 3) = 2$, i.e., every planar graph admits a 3-coloring in which every monochromatic component has maximum degree at most 2. In fact, there is a 3-coloring of any planar graph in which every monochromatic component is a path [26]. Eaton and Hull [16], and independently Škrekovski [27], proved that $k_d(3, 2) = \infty$, i.e., there are planar graphs of girth 3 for which any 2-coloring results in a monochromatic component of arbitrarily high maximum degree. Cowen, Goddard and Jerum [14] proved that every outerplanar graph admits a 2-defective 2-coloring. Havet and Sereni [21] showed that for $c \geq 2, k \geq 0$ every graph of maximum average degree less than $c + \frac{ck}{c+k}$ admits a k-defective c-coloring. By Euler's formula a planar graph of girth g has maximum average degree less than $\frac{2g}{g-2}$. Hence, the last result implies that $k_d(5, 2) \leq 4$ and $k_d(6, 2) \leq 2$. The result of Borodin, Kostochka and Yancey [9] shows that $k_d(7, 2) = 1$.

Fragmented colorings. A c-coloring is *k-fragmented* if each monochromatic component has at most k vertices, and $k_f(g, c)$ denotes the smallest k such that every planar graph of girth at least g admits a k-fragmented c-coloring. Fragmented coloring were first introduced in 1997 by Kleinberg *et al.* in [23], where they showed that $k_f(3, 3) = \infty$, i.e., there is no k such that every planar graph admits a k-fragmented 3-coloring, a result that has been independently proven by Alon *et al.* [2]. Esperet and Joret [17] recently proved that $k_f(4, 2) = \infty$, although this already follows from the fact that $k_d(4, 2) = \infty$ [27]. Esperet and Ochem [18] proved that $k_f(6, 2) \leq 12$.

P_k-free colorings. Finally, a c-coloring is *P_k-free* if there is no monochromatic path on k vertices, and $k_p(g, c)$ denotes the smallest k such that every planar graph of girth at least g admits a P_k-free c-coloring. Such P_k-free colorings were already introduced in 1968 by Chartrand, Geller and Hedetniemi [11], who showed that $k_p(3, 3) = \infty$, i.e., there is no k such that every planar graph admits a P_k-free 3-coloring. In a different paper [12], the same authors showed that same holds for outerplanar graphs and 2 colors. More than 20 years later, the former result has been reproved by Akiyama *et al.* [1], as well as Berman and Paul [7]. Recently, Glebov and Zambalaeva [19] showed that every planar graph of girth at least 6 can be 2-colored such that every color class induces a P_6-free forest, i.e., $k_p(6, 2) \leq 6$.

We summarize the results for defective, fragmented and P_k-free colorings using 2 colors.

girth g	3	4	5	6	7
$k_d(g, 2)$	∞ [16]	∞ [27]	≥ 2 Fig. 7 ≤ 4 [21]	≤ 2 [21]	1 [9]
$k_f(g, 2)$	∞ [2, 23]	∞ [17]	≥ 3 Fig. 7	≤ 12 [18]	2 [9]
$k_p(g, 2)$	∞ [1, 7, 11]	∞ Thm. 2	≥ 4 Fig. 7	≤ 6 [19]	3 [9]

Table 1: Improper 2-coloring results for planar graphs of girth g.

Theorem 1 and Theorem 2 immediately imply the following, c.f. Table 1.

Corollary 3. *We have that* $k_f(6, 2) \leq 15$*,* $k_p(6, 2) \leq 16$*, and* $k_p(4, 2) = \infty$*.*

Let us also mention that defective and fragmented colorings have also been considered for non-planar graphs of bounded maximum degree [2, 6, 22], bounded number of vertices [24], and for minor-free graphs [29]. In natural generalizations one allows different color classes to have different defect (see for example [5, 25]), or considers list-coloring, which in fact, is the case in many of the results above.

3 Proof of Theorem 1

For a list assignment L, we call an L-coloring of a planar graph *good* if each monochromatic component is a path of length at most 14. Throughout this section we let, for the sake of contradiction, a graph G be a counterexample to Theorem 1, so that G is vertex-minimal, and among all such graphs has the largest number of edges. I.e., G has no good L-coloring, an addition of any new edge to G creates a non-planar graph or a cycle of length at most 5, and any subgraph of G with fewer vertices has a good L-coloring. To avoid a special treatment of an outer face we assume G to be embedded without crossings on the sphere and shall refer to the faced of the corresponding plane graph as faces of G. Note also that if a graph has no good L-coloring, then any of its supergraphs has no good L-coloring.

Idea of the proof. Our proof extends the ideas of Havet and Sereni [21]. We start by proving some structural properties of G, i.e., that G has minimum degree 2, all faces of G are chordless cycles of length at most 9, and proving a statement about the distribution of vertices of degree 2 around every face F in G.

If G has a path P of length at most 14 with endpoints of degree 2 and all inner vertices of degree 3, then each vertex in P has exactly one neighbor not in P. Deleting the vertices of P from G gives a graph that has a good coloring. Color each vertex of P with a color different from the color of its neighbor not in P. This gives a good coloring of G contradicting the fact that G is a minimal counterexample.

We generalize this simple argument, that uses a single path, to path systems, that is, sets of (directed) facial paths in G with all inner vertices of degree 3. Next, we consider a charge of $\deg(v) - 3$ at every vertex v and define discharging rules shifting a charge of $1/2$ from the out-endpoint of every path in X_0 to its in-endpoint, based on a specific path system X_0. The total charge on all the vertices, before as well as after the discharging, is negative, giving some vertices ending up with negative charge. We consider such a vertex w_0, build another path system based on what is "outgoing" from this vertex, and show that the corresponding subgraph of G is a reducible configuration. Here, a subgraph H is reducible if any good L-coloring of $G - V(H)$ (which exists by the minimality of G) could be extended to a good L-coloring of the whole graph G. This contradicts the assumption that G is a counterexample and hence concludes the proof.

Structural properties of G.

Lemma 4. *G is connected and has minimum degree at least* 2*.*

Proof. Indeed, if G has a vertex v of degree 1, then a good coloring of $G-v$ can be extended to a good coloring of G by choosing the color of v to be different from its neighbor in $G-v$. If G is not connected, then one of its connected components is a smaller counterexample, contradicting the definition of G. □

Lemma 5. *The boundary of each face of G forms a chordless cycle of length at most* 9*.*

Proof. First assume for the sake of contradiction there is a face F whose closed boundary walk $W = u_0, \dots, u_m$ is not a cycle. Then there is a vertex u appearing at least twice on W, say $u = u_0 = u_j$ with $j \neq 0$. As G has minimum degree 2, each of the closed walks $W_1 = u_0, u_1, \dots, u_j$ and $W_2 = u_j, u_{j+1}, \dots, u_0$ contains at least one cycle, i.e., has at least 6 vertices. Note that the vertices u_2 from W_1 and u_{j+3} from W_2 lie in distinct connected components of $G - u$. Moreover, as G has minimum degree 2, u_2 and u_{j+3} are at distance 2 and 3 from u along W, respectively. Hence any $u_2 - u_{j+3}$ path goes through u and, as there are no cycles of length at most 5, the distance between u_2 and u_{j+3} in G is 5. Thus we can add an edge u_2u_{j+3} into F, creating a planar graph with girth at least 6. A contradiction to edge-maximality of G.

Thus, the boundary of each face F forms a cycle, $C = u_0, \ldots, u_m, u_0$. Assume that C has length at least 10, i.e., that $m \geq 9$. Recall, that an *ear* E of a cycle C is a path that shares only its endpoints with the vertex set of the cycle. For $i = 0, \ldots, m$ let $G'(i)$ be obtained from G by adding an edge u_i, u_{i+5} into the face F, addition of indices modulo $m+1$. If $G'(i)$ has girth at least 6, this contradicts the edge-maximality of G. So, there is a cycle on at most 5 vertices containing edge $u_i u_{i+5}$ in $G'(i)$, denote a shortest $u_i - u_{i+5}$ path in G by $P(i, i+5)$. Its length is at most 4, less than the distance between i and $i+5$ along C (as $m \geq 9$), so there is an ear of length ℓ, $\ell \leq 4$, and ℓ is less than the distance between its endpoints along C. The *width* of an ear is the smallest distance between its endpoints along the cycle. If Q is a path or a cycle and P is a path in Q with endpoints u and v, we write $P = uQv$. We denote the length of P as $||P||$. A *k-ear* is an ear of length k.

Case 1. C has a chord.
Assume that $u_0 u_k$ is a chord, $k \geq 5$. A path $P = P(-3, 2)$ must contain u_0 or u_k. If P contains u_0, then $||u_{-3} P u_0|| \geq 3$ and $||u_0 P u_2|| \geq 2$, as otherwise $P \cup C$ contains a cycle of length at most 5. Similarly, if P contains u_k, then $||u_{-3} P u_k|| \geq 2$ and $||u_k P u_2|| \geq 3$. In any case we have that $||P|| \geq 5$, a contradiction. See Figure 1 left.

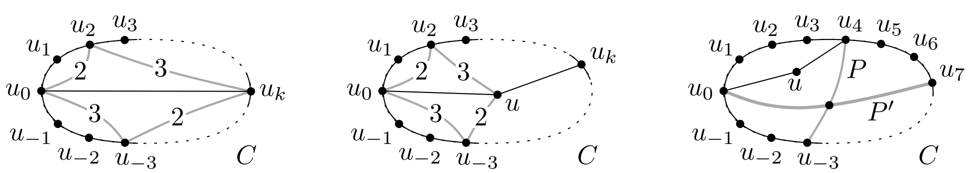


Figure 1: Illustration of Case 1 (left), Case 2 with $w \in \{u_0, u\}$ (middle) and Case 2 with $w = u_k$ (right). The face F bounded by C is shown as the outer face. The numbers indicate the minimum length of a path between the corresponding vertices.

Case 2. C has an ear of length 2 and no chords.
Let E be a 2-ear of smallest width, with vertices u_0, u, u_k, $4 \leq k \leq (m+1)/2$. A path $P = P(-3, 2)$ contains $w \in \{u_0, u, u_k\}$. If $w = u_0$, see Figure 1 center, then (as in Case 1) $||u_{-3} P u_0|| \geq 3$ and $||u_0 P u_2|| \geq 2$, and if $w = u$, then $||u_2 P u|| \geq 3$ and $||u P u_{-3}|| \geq 2$, as otherwise there is a cycle of length at most 5 in $P \cup C$. In both cases we have $||P|| \geq 5$, a contradiction. So $w = u_k$, see Figure 1 right, $||w P u_2|| \geq 2$, and $||u_{-3} P w|| \geq 2$, otherwise there is a chord. Thus each of these segments has length 2. Since $||u_k P u_2|| = 2$, $u_k P u_2$ is a subpath of C, otherwise there is a 2-ear of a smaller width. So $k = 4$. Since $||u_{-3} P u_k|| = 2$ and $m \geq 9$, $||u_{-3} C u_k|| \geq 4$, so $||C|| \geq 11$. Looking at E in the other direction along C, and taking $P' = P(2, 7)$, we see symmetrically that $u_0 P' u_7$ is an ear of length 2, that together with E and P creates a cycle of length 4.

Case 3. C has no ears of length 2 or chords.
Since each $P(i, i+5)$ results in an ear whose length is smaller than its width, we see that either there is a u_i-u_{i+5} ear of length 4 or an ear of length 3 with width between 4 and

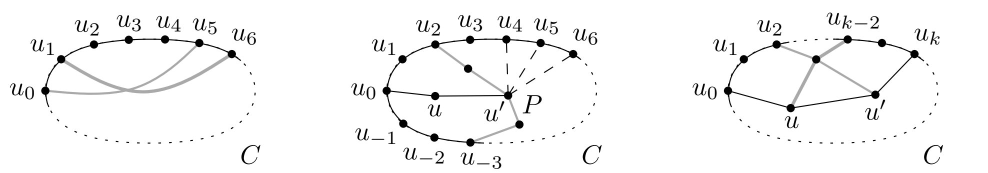


Figure 2: Illustration of Case 3 with only 4-ears (left) and Case 3 with a 3-ear (middle and right). The face F bounded by C is shown as the outer face.

6. If all such ears are of length 4, then a $u_0 - u_5$ ear and $u_1 - u_6$ ear intersect and form, together with C, a cycle of length at most 5, a contradiction, see Figure 2 left. Assume that E is a 3-ear u_0, u, u', u_k, $4 \leq k \leq 6$, and all other 3-ears have width either at most 3 or at least k, see Figure 2 center and right. A path $P = P(2, -3)$ contains a 3- or a 4-ear. Let w be a point on P and E. We have that $||u_{-3}Pw|| \geq 2$ and $||wPu_2|| \geq 2$, otherwise there is either a chord, a 2-ear, or a cycle of length at most 5. It follows that $||u_{-3}Pw|| = 2 = ||wPu_2||$. Then $w = u'$, and $k \geq 5$. Looking at E in the other direction, we see symmetrically, that there is a path of length 2 between u_{k-2} and u, implying the existence of a triangle containing u and u', a contradiction.

Thus C has length at most 9. If C has a chord, then there is a cycle of length at most 5, a contradiction. So, C is a chordless cycle. This concludes the proof the Lemma. □

Lemma 6. *Let F be any face of G incident to a vertex of degree 2. Then F is incident to a vertex of degree at least 4, and if F is incident to at least two vertices of degree 2, then there is a vertex of degree at least 4 between any two such vertices on both paths along F.*

Proof. Let C be the simple chordless cycle bounding F. First, assume for the sake of contradiction that C contains exactly one vertex v of degree 2 and all other vertices of degree 3. Consider a good L-coloring of $G' = G - V(C)$ and give each vertex u of C of degree 3 a color in $L(u)$ different from the color of its neighbor in G'. Give v a color in $L(v)$ such that C does not form a monochromatic cycle. As a result, the set of monochromatic components of G is formed by the monochromatic components of G', and paths on at most 8 vertices formed by vertices of C.

Second, assume that C contains two vertices u, v of degree 2 and a $u - v$ path P in C has no inner vertices or only inner vertices of degree 3. Consider a good L-coloring of $G' = G - V(P)$ and give the vertices of P colors from their lists, different from the colors of their unique neighbors in G'. This does not extend any connected monochromatic component of G' and every new monochromatic component is contained in P, i.e., a path on at most 8 vertices.

I.e., in both cases we have found a good L-coloring of G, a contradiction to G being a counterexample. □

Path systems. A *path system* is a set X of (not necessarily edge-disjoint) directed facial paths in G with all inner vertices being of degree 3, such that no vertex is an endpoint of one path in X and an inner vertex of another path in X. For a path $P \in X$ directed from vertex u to vertex v, we call u the *out-endvertex* and v the *in-endvertex* of P. For a path system X, the vertices that are the in-endvertices or out-endvertices of some path in X are called the *endvertices of* X, while the *inner vertices of* X are the inner vertices of some path in X. For any vertex v in G let $\text{out-deg}_X(v)$ and $\text{in-deg}_X(v)$ denote the number of paths in X with out-endvertex v and in-endvertex v, respectively. Note that for an inner vertex v of X we have $\text{out-deg}(v) = \text{in-deg}(v) = 0$. A directed path P is *occupied* by a path system X if the first or last edge of P (incident to its out-endvertex or in-endvertex) is contained in some path in X. So, if $P \in X$, then P is occupied by X. Let us emphasize that throughout the paper $\deg(v)$ and $N(v)$ always refer to the degree and neighborhood of vertex v in G, even when we consider other subgraphs of G later.

For a path system X and any two vertices u, v in G we say that u *reaches* v *in* X, denoted by $u \to_X v$, if there is a sequence $u = v_1, \ldots, v_k = v$ of vertices and a sequence $P_1, \ldots, P_{k-1}$ of paths in X such that v_i and v_{i+1} are out-endvertex and in-endvertex of P_i, respectively, $i = 1, \ldots, k-1$. Then X is *acyclic* if there are no two distinct vertices u, v with $u \to_X v$ and $v \to_X u$. For a vertex w of G, we define $X^+(w) \subseteq X$ to be the path system consisting of all paths in X whose out-endvertex is w or reachable from w in X.

A path system X is *nice* if each of the following properties **(D1)**–**(D5)** holds. A path system X with a distinguished vertex r, called root, is *almost nice* if the properties **(D1)**–**(D5)** hold for all vertices different from r.

See Figure 3 for an illustration.

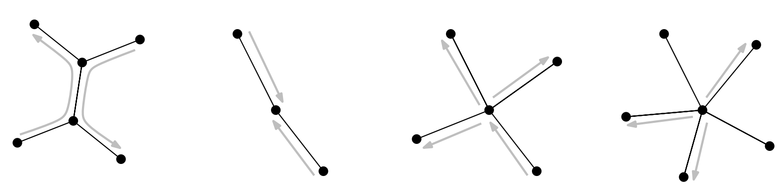
(D1) Every edge that belongs to two paths in X joins two vertices of degree 3 each.

(D2) Every vertex of degree 2 has outdegree 0 in X.

(D3) Every vertex of degree 3 has indegree 0 and outdegree 0 in X.

(D4) Every vertex of degree 4 has positive indegree in X only if it has outdegree 3 in X.

(D5) Every vertex of degree at least 5 has in-degree 0 in X.



(D1), (D3) **(D2)** **(D4)** **(D5)**

Figure 3: Illustration of properties **(D1)**–**(D5)**.

The following statements follow immediately from the definitions above.

Lemma 7. *For every path system X each of the following holds.*

(1) If no path $P \in X$ is occupied by $X - \{P\}$, then X satisfies ***(D1)****.*

(2) If $X' \subseteq X$ and X satisfies any of **(D1)**–**(D3)**, **(D5)**, *then so does X'.*

(3) If $X' \subseteq X$ and X is acyclic, then so is X'.

(4) If X is nice and w is a vertex, then $X^+(w)$ with root w is almost nice.

Discharging with respect to a path system X. Given a path system X, consider the following discharging: Put charge $ch(v) = \deg(v) - 3$ on each vertex of G. Note that $ch(v) = -1$ for a vertex of degree 2, and $ch(v) \geq 0$ for all other vertices. As all facial cycles have length at least 6, we have $6f \geq 2e$, where f denotes the number of faces of G. Together with Euler's formula $n - e + f = 2$ this implies $n - e + e/3 \geq 2$. Thus the total charge is $\sum_{v \in V(G)}(\deg(v) - 3) = 2e - 3n \leq -6$.

Define $ch'(v) = ch(v) + \frac{1}{2}(\text{in-deg}_X(v) - \text{out-deg}_X(v))$. Intuitively, for every path in X a 1/2-charge is sent from out-endvertex to in-endvertex. Thus, the total sum of charges in ch' is the same as in ch, i.e., $\sum_v ch(v) = \sum_v ch'(v)$.

Defining a path system $\mathcal{P}$. As G has girth at least 6, there is a vertex v of degree 2 in G and by Lemma 6 both faces incident to v contain a vertex of degree at least 4. So there are faces with at least two vertices of degree different from 3. For each such face F the boundary of F can be uniquely partitioned into edge-disjoint counterclockwise oriented paths with all inner vertices of degree 3 and endpoints of degree different from 3. We denote by $\mathcal{P}$ the path system consisting of all such paths with in-endvertex of degree 2 or 4 and out-endvertex of degree at least 4, for all faces F with at least two vertices of degree different from 3. So for each path in $\mathcal{P}$ the degrees d_1, d_2 of its in-endvertex and out-endvertex, respectively, satisfy $(d_1, d_2) \in \{(2,4), (2,\ell), (4,4), (4,\ell) \mid \ell \geq 5\}$.

By Lemma 5 every face of G is bounded by a simple chordless cycle of length at most 9. Thus every $P \in \mathcal{P}$ is a path on at most 8 edges. As any two paths in $\mathcal{P}$ in the boundary of the same face F are edge-disjoint, every edge of G lies in at most two paths in $\mathcal{P}$, at most one for each face incident to the edge. If an edge lies in two paths in $\mathcal{P}$, these paths have the edge oriented in opposite directions. For a vertex v in G with $\deg(v) = 3$ we have $\text{out-deg}_{\mathcal{P}}(v) = \text{in-deg}_{\mathcal{P}}(v) = 0$ by definition. Note that by Lemma 6 for every vertex v with $\deg(v) = 2$ we have $\text{out-deg}_{\mathcal{P}}(v) = 0$ and $\text{in-deg}_{\mathcal{P}}(v) = 2$. For a vertex v with $\deg(v) \geq 5$ we have $\text{in-deg}_{\mathcal{P}}(v) = 0$, i.e., $\mathcal{P}$ has properties **(D2)**, **(D3)** and **(D5)**. We provide an example illustrating these concepts in Figure 4.

Defining a path system $X_0 \subseteq \mathcal{P}$. We define $X_0 \subseteq \mathcal{P}$ selecting paths one by one, using the following procedure, where we go through the vertices in question in an arbitrary but fixed order. At all times, let X_0 denote the set of already chosen paths, initially $X_0 = \emptyset$.

1.) For every vertex v with $\deg(v) = 2$ we put a path from $\mathcal{P}$ into X_0 if its in-endpoint is v and if it is not occupied by X_0.

After step 1.) is done for all vertices of degree 2, we proceed as follows.

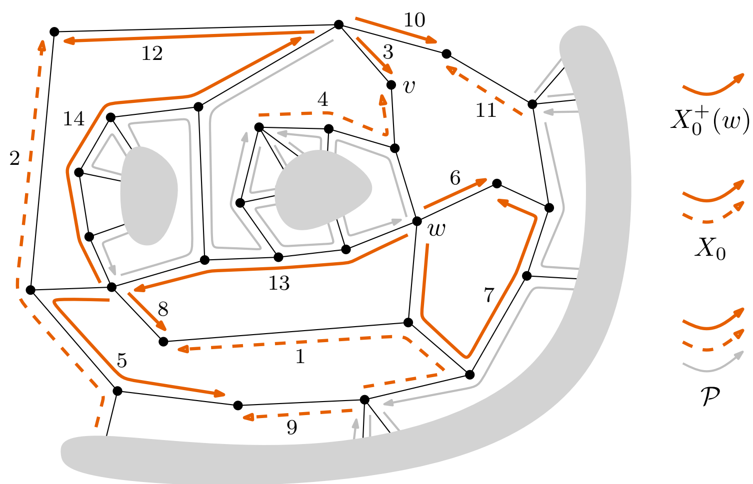


Figure 4: A (part of a) planar graph of girth 6 and the path systems in $\mathcal{P}$, X_0 and $X_0^+(w)$. The labels show the order in which paths of X_0 were selected. Paths 13 and 14 were selected in step 2.).

2.) For every vertex v with $\deg(v) = 4$ and $\text{out-deg}_{X_0}(v) = 3$, put a path from $\mathcal{P}$ into X_0 if its in-endpoint is v and if it is not occupied by X_0.

Later, we shall show that the final path system X_0 is nice and acyclic. For now, we only need to observe that **(D2)** is satisfied and $\text{in-deg}_{X_0}(u) = 2$ for every vertex u of degree 2. In fact, **(D2)** holds for $\mathcal{P}$ and thus by Lemma 7 (2) it also holds for $X_0 \subseteq \mathcal{P}$. Assume now that $\text{in-deg}_{X_0}(u) < 2$. I.e., P, one of the two paths in $\mathcal{P}$ with in-endvertex v was occupied by during step 1.). As all in-endvertices of paths chosen in step 1.) are of degree 2, and the out-endvertex of P has degree at least 4, this is impossible.

Defining the vertex w_0 based on discharging with respect to X_0. Let us apply discharging to X_0. For every vertex u with $\deg(u) = k$, we have $\text{in-deg}_{X_0}(u) \geq 0$ and $\text{out-deg}_{X_0}(u) \leq k$, i.e., u looses a charge of at most $\frac{k}{2}$. Thus if $\deg(u) = k \geq 6$, the remaining charge $ch'(u)$ is at least $k - 3 - \frac{k}{2} \geq 0$. If $\deg(u) = 3$, then $\text{out-deg}_{X_0}(u) = \text{in-deg}_{X_0}(u) = 0$ and hence $ch(u) = ch'(u) = 0$. If $\deg(u) = 2$, then $\text{in-deg}_{X_0}(u) = 2$ and $\text{out-deg}_{X_0}(u) = 0$ and hence $ch'(u) = \deg(u) - 3 + \frac{1}{2}(2 - 0) = 0$.

On the other hand we have $\sum_v ch(v) = \sum_v ch'(v)$. As $\sum_v ch'(v) \leq -6$ there is a vertex w_0 in G with $ch'(w_0) < 0$. With the above considerations we conclude that $\deg(w_0) \in \{4, 5\}$.

If $\deg(w_0) = 5$, then $0 > ch'(w_0) \geq (5 - 3) - \frac{1}{2}\,\text{out-deg}_{X_0}(w_0)$, so $\text{out-deg}_{X_0}(w_0) \geq 5$. Since $\text{out-deg}_{X_0}(w_0) \leq \deg(w_0)$, we have that $\text{out-deg}_{X_0}(w_0) = 5$. If $\deg(w_0) = 4$, then $0 > ch'(w_0) = (4 - 3) + \frac{1}{2}(\text{in-deg}_{X_0}(w_0) - \text{out-deg}_{X_0}(w_0))$, so either $\text{out-deg}_{X_0}(w_0) = 4$ or ($\text{out-deg}_{X_0}(w_0) = 3$ and $\text{in-deg}_{X_0}(w_0) = 0$). In particular, exactly one of the following must hold for the vertex w_0 with $ch'(w_0) < 0$:

Case 1: $\deg(w_0) \in \{4,5\}$ and $\text{out-deg}_{X_0}(w_0) = \deg(w_0)$.

Case 2: $\deg(w_0) = 4$, $\text{out-deg}_{X_0}(w_0) = 3$ and $\text{in-deg}_{X_0}(w_0) = 0$.

For example, in Figure 4 we see that **Case 2** applies to vertex w.

Defining rooted path systems X_1, X_2, X_3, X_4 based on w_0 and X_0. Depending on the structure of w_0 and X_0 we shall define one of four path systems X_1, X_2, X_3, X_4, each X_i with a specified vertex w_i, called the *root*, $i = 1, 2, 3, 4$. Path systems X_1, X_3 will be chosen as subsystems of X_0, X_2 as a subsystem of X_0 together with an additional path from $\mathcal{P}$, and X_4 as a subsystem of X_0 together with a subpath of a path from $\mathcal{P}$. Note that each of X_1, X_2, X_3, X_4 consists of paths of length at most 8.

Case 1: $\deg(w_0) \in \{4,5\}$ and $\text{out-deg}_{X_0}(w_0) = \deg(w_0)$.
In this case we define $X_1 = X_0^+(w_0)$ with root w_0.

Case 2: $\deg(w_0) = 4$, $\text{out-deg}_{X_0}(w_0) = 3$ and $\text{in-deg}_{X_0}(w_0) = 0$.
Consider the unique edge e at w_0 not contained in any path in X_0. As w_0 has outdegree 3, the clockwise next edge e' at w_0 after e is contained in some path in X_0 with out-endvertex w_0. The in-endvertex of this path is in the face F incident to w_0, e, and e'. See the middle part of Figure 5 for an illustration. So F has at least two vertices of degree different from 3. Thus its boundary contains a counterclockwise path P with in-endvertex w_0, using the edge e, all inner vertices of degree 3 or no inner vertices at all and out-endvertex v with $\deg(v) \neq 3$. Let e'' be the edge of P incident to v. If $\deg(v) = 2$, then from the definition of X_0, $\text{in-deg}_{X_0}(v) = 2$. Thus e'' belongs to a path in X_0 with in-endpoint v. If $\deg(v) \geq 4$ then $P \in \mathcal{P}$, and in step 2.) of the construction of X_0 the path P must have been rejected because it was occupied, i.e., e'' is contained in another path in X_0. As P has v as out-endvertex, the other path has v as in-endvertex and hence $\deg(v) = 4$. So, $\deg(v) \in \{2, 4\}$ and e'' lies in some path in X_0 with in-endvertex v. In particular it follows that $e \neq e''$, i.e., P has at least one inner vertex.

Next, we distinguish the cases when v is reachable from w_0 in X_0 or not, corresponding to the right and middle part of Figure 5, respectively. In case v is not reachable from w_0 in X_0, we define $X_2 = X_0^+(v) \cup X_0^+(w_0) \cup \{P\}$ with root v.

When v is reachable from w_0 in X_0, let $w_0, w_1, \ldots, w_{k-1}, w_k = v$, $k \geq 2$, denote the vertices of P in their order along P from its in-endvertex w_0 to its out-endvertex v. Recall that P has at least one inner vertex. Let i be the smallest index such that $w_i \neq w_0$ and w_i is contained in a path in $X_0^+(w_0)$. See the right part of Figure 5. As $v = w_k$ is reachable from w_0 in X_0, this index is well-defined. If $i = 1$, we define $X_3 = X_0^+(w_0)$ with root w_0. Otherwise we denote the directed w_{i-1}-to-w_0 subpath of P by P' and define $X_4 = X_0^+(w_0) \cup \{P'\}$ with root w_{i-1}. This is for example the case for vertex $w_0 = w$ in Figure 4.

Lemma 8.

(i) Each of X_0, X_1, X_2, X_3, X_4 is acyclic.

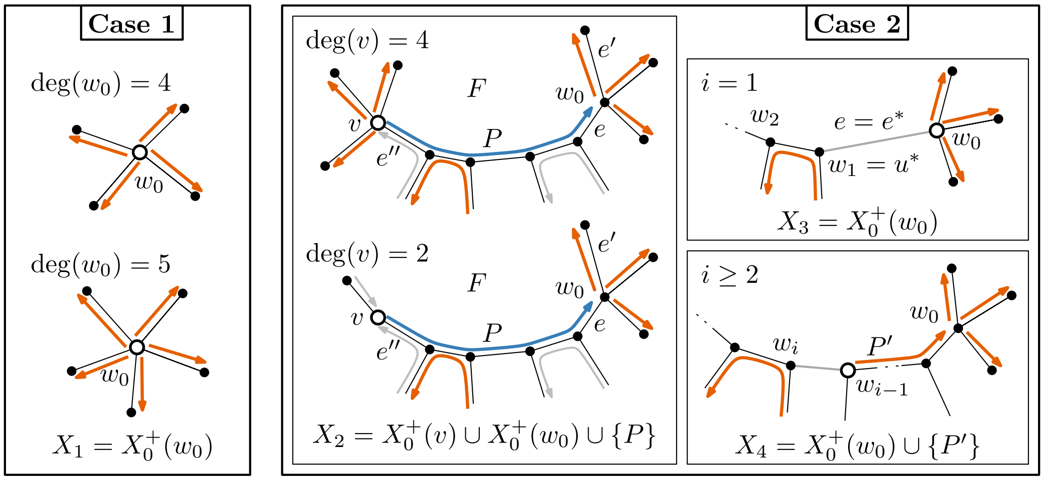


Figure 5: Illustrations of the rooted path systems X_1, X_2, X_3, X_4 with highlighted roots.

(ii) X_0 *and* X_1 *are nice.*

(iii) If the root v *of* X_2 *has degree* 4*, then* X_2 *is nice.*

(iv) If the root v *of* X_2 *has degree* 2*, then* X_2 *is almost nice with* $\text{out-deg}_{X_2}(v) = 1$.

(v) X_3 *is almost nice with* $\text{out-deg}_{X_3}(w_0) = 3$ *and* $\text{in-deg}_{X_3}(w_0) = 0$.

(vi) X_4 *is almost nice with* $\text{out-deg}_{X_4}(r) = 1$ *and* $\text{in-deg}_{X_4}(r) = 0$ *for the root* r *of* X_4.

(vii) If $j \in \{1, 2, 3, 4\}$ *then each endvertex of* X_j*, different from the root, has degree* 2 *or* 4 *in* G*, the root has degree* 2, 3, 4*, or* 5*, and each inner vertex of* X_j *has degree* 3*. Moreover, each vertex of* X_j *has at most one neighbor that is not in* X_j*.*

Proof.
(i): First, we shall show that X_0 is acyclic. Assume for the sake of contradiction that $v_0, \ldots, v_{k-1}$ and $P_0, \ldots, P_{k-1}$ are two sequences of vertices and paths in X_0 such that for every $i \in \{0, \ldots, k-1\}$ we have that v_i and v_{i+1} are out-endvertex and in-endvertex of P_i, respectively (all indices modulo k). For each $i \in \{0, \ldots, k-1\}$ we have $\deg(v_i) \in \{2, 4\}$, as we add only paths with such in-endvertices to X_0 in step 1.) and 2.). Moreover, v_i is out-endvertex of P_{i-1} and thus we have that $\deg(v_i) \neq 2$. Hence for each $i \in \{0, \ldots, k-1\}$ we have $\deg(v_i) = 4$ and P_i was put into X_0 in step 2.) because v_{i+1} was the out-endvertex of exactly three already chosen paths. Assume without loss of generality that P_0 was the path that was put into X_0 in step 2.) first among the paths $P_0, \ldots, P_{k-1}$. This means that the path P_1, whose out-endvertex is v_1, was already put into X_0. This contradicts that P_0 was the first and proves that X_0 is acyclic.

Now, $X_1, X_3 \subseteq X_0$ are acyclic by Lemma 7 (3). Moreover, $X_2 = X_0^+(v) \cup X_0^+(w_0) \cup \{P\}$ is acyclic, because $X_0^+(v), X_0^+(w_0) \subseteq X_0$, P has in-endvertex w_0 and out-endvertex v, and (in this case) v is not reachable from w_0 in X_0. Finally, $X_4 = X_0^+(w_0) \cup \{P'\}$ is acyclic, because $X_0^+(w_0) \subseteq X_0$ and $V(P') \cap \left(\bigcup_{P \in X_0} V(P)\right) = \{w_0\}$.

(ii), (v): Consider X_0. As mentioned earlier, $\mathcal{P}$ satisfies **(D2)**, **(D3)** and **(D5)** and by Lemma 7 (2) so does X_0. Moreover, by definition X_0 satisfies **(D1)** and **(D4)**, thus X_0 is nice. Consider X_1 and X_3. By Lemma 7 (4) we have that X_1 and X_3 are almost nice, as both are defined as $X_0^+(w_0)$ with root w_0. By construction, out-$\deg_{X_3}(w_0) = 3$ and in-$\deg_{X_3}(w_0) = 0$, which proves (v). For X_1 note that, if $\deg(w_0) = 4$, then **(D4)** holds for w_0 in X_1, and if $\deg(w_0) = 5$, then **(D5)** holds for w_0 since $X_1 \subseteq X_0 \subseteq \mathcal{P}$. Thus X_1 is nice.

(iii), (iv): Next consider $X_2 = X_0^+(v) \cup X_0^+(w_0) \cup \{P\}$ with root v and path P as defined above, see the middle part of Figure 5. Each of $X_0^+(v)$, $X_0^+(w_0)$ is almost nice by Lemma 7 (4) and the niceness of X_0. We have neither $w_0 \to_{X_0} v$ (by assumption) nor $v \to_{X_0} w_0$ (as in-$\deg_{X_0}(w_0) = 0$). So $X_0^+(v) \cup X_0^+(w_0)$ satisfies **(D1)**–**(D5)**, except perhaps for w_0 and v. As X_2 additionally contains the path P from v to w_0, we have that **(D4)** is satisfied for w_0 and thus X_2 is nice when $\deg(v) = 2$ and almost nice when $\deg(v) = 4$. Because X_0 is nice, i.e., satisfies **(D2)**, we have out-$\deg_{X_0}(v) = 0$ when $\deg(v) = 2$. As $X_2 - P \subseteq X_0$ and P is outgoing at v, we have out-$\deg_{X_2}(v) = 1$.

(vi): The system $X_4 = X_0^+(w_0) \cup \{P'\}$ is almost nice, because P' and $X_0^+(w_0)$ share only vertex w_0, $X_0^+(w_0)$ is almost nice by Lemma 7 (4), and P' is incoming at w_0.

(vii): These properties are corollaries of the almost-niceness of X and the considerations for the root in the previous items. □

Coloring reducible configurations based on X_1, X_2, X_3, X_4. Recall that a coloring is good if each monochromatic component is a path of length at most 14. A *reducible configuration* is a non-empty subgraph H of G, such that any good L-coloring of $G - V(H)$ (which exists by the minimality of G) can be extended to a good L-coloring of G in which every edge between a vertex in H and a vertex outside of H is colored properly. Showing that G has a reducible configuration will conclude the proof of Theorem 1. For convenience we say that X_i is reducible if the subgraph H of G induced by the vertices in X_i is a reducible configuration, $i = 1, 2, 3, 4$.

Lemma 9. *Each of X_1, X_2, X_3, X_4 is reducible, whenever it is defined.*

Proof. Consider $j \in \{1, 2, 3, 4\}$. Let r be the root of X_j, H be the subgraph of G consisting of all vertices and undirected edges in the path system X_j. Let V_1 be the set of vertices of H and H' be the subgraph of G induced by V_1, i.e., $H \subseteq H'$. Let $W \subseteq V_1$ be the set of endvertices of X_j. Recall, that by Lemma 8(vii), if $w \in W - \{r\}$, then $\deg(w) \in \{2, 4\}$, $\deg(r) \in \{2, 3, 4, 5\}$, and if $u \in V_1 - W$, then $\deg(u) = 3$. In addition, the niceness or almost niceness of X_j and the degree conditions for r given in Lemma 8 any vertex from V_1 has at most one neighbor not in V_1 and each vertex in $V_1 - \{r\}$ has at most one incident edge from $E(G) - E(H)$. In particular, $E(H') - E(H)$ is a matching, unless $j = 4$, in which case $E(H') - E(H)$ might contain two edges incident to r. In case $j \neq 3$, let $E_1 = E(H') - E(H)$. Otherwise (when $j = 3$) let $e^* = ru^*$ denote the unique edge in

$E(H') - E(H)$ incident to the root r and let $E_1 = E(H') - (E(H) \cup e^*)$. In Figure 5 on the right we have $r = w_0$ and $u^* = w_1$. Note that if $j \in \{1, 2\}$, then there are no edges from $E(H') - E(H)$ incident to r.

We shall be coloring different sets of vertices of G one after another.

- First we make a good L-coloring c' of $G - V_1$, which exists by the minimality of G. Note that $G - V_1$ might be empty.

We shall color V_1 so that no vertex in V_1 has the same color as its neighbor (if exists) in $V(G) - V_1$ and such that each monochromatic path with vertices in V_1 is contained in the union of two paths from X_j.

- Consider $A \subseteq V_1$, the set of vertices that have a neighbor in $V(G) - V_1$. As X_j satisfies **(D4)** no vertex of degree 4 in H' is in A, except for possibly the root r. We color each vertex $v \in A$ such that its color is from $L(v)$ and differs from the color of its neighbor in $V(G) - V_1$.

Now, no matter how we color $V_1 - A$, each monochromatic path has all its vertices completely in V_1 or completely in $V(G) - V_1$. Since the coloring of $V(G) - V_1$ is good, each monochromatic path there has length at most 14. So, we only need to color $V_1 - A$ so that each monochromatic path with vertices in V_1 has length at most 14.

- Consider the vertices of E_1. First assume $r \in V(E_1)$, this could be only if $j = 2$ or $j = 4$. If $r \in A$, then r is already colored and if $r \notin A$, we give r any color from its list. Next, we color every neighbor of r in E_1 with a color from its respective list different from the color of r. Finally, we color the remaining vertices of E_1 from their lists such that each edge of E_1 has endpoints of different colors. If $r \notin V(E_1)$, i.e., E_1 is a matching, color $V(E_1)$ such that each edge is colored properly.

This ensures that eventually every monochromatic component of H' is a subgraph of H or $H \cup e^*$ in case $j = 3$.

- Consider the set B of vertices from $V_1 - A$ not incident to E_1 and of degree 3. Note that $r \notin B$ because it is either of degree different from 3 or is incident to E_1 in case when $j = 3$. Hence B consists only of inner vertices of X_j, i.e., $B = V_1 - (A \cup V(E_1) \cup W)$. For any $u \in B$ all three edges incident to u are in H, so u lies on at least two paths in X_j. We consider the paths in X_j in any order and when we process a path $P \in X_j$, we color the vertices in $B \cap V(P)$. For the current path P and the current vertex $u \in B \cap V(P)$, consider the neighbor u' of u not in P. If u' is not colored, color u arbitrarily from its list. Otherwise, color u with a color different from the color of u'.

This ensures that every monochromatic component of $H' - W$ is completely contained in some path in X_j. It remains to color the vertices in $W - A$ and in $e^* = ru^*$ (if it exists) in such a way that e^* is not monochromatic and at most two monochromatic components of $H' - W$ are part of the same monochromatic component of H'.

- Consider the vertices in $W - A$ and the vertex u^* (if it exists). Recall that u^* is an inner vertex of some path in X_j and hence $u^* \notin W$. For each $u \in W - A$, consider the paths in X with in-endvertex u and let $S(u)$ be the set of immediate neighbors of u on those paths, i.e., $v \in S(u)$ if $uv \in E(H)$ and u is the in-endvertex of the path in X_j containing uv. In particular, $S(r) = \emptyset$, and for $u \neq r$ we have $|S(u)| = 1$ if $\deg(u) = 4$ and $|S(u)| = 2$ for $\deg(u) = 2$. Additionally let $S(u^*) = \{r\}$ when considering X_3. We apply the following rules to still uncolored vertices (initially the set $(W - A) \cup \{u^*\}$) as long as any of these is applicable:

  Rule 1: If for some uncolored vertex u three of its neighbors have the same color a, we color u with a color in $L(u)$ different from a.

  Rule 2: If Rule 1 does not apply, but for some uncolored vertex u some $u' \in S(u)$ is already colored, we color u with a color from its list different from the color of u'.

  Rule 3: If neither Rule 1 nor Rule 2 applies, and the root r is uncolored, consider the set of colors appearing on $N(r)$ and a color a that is repeated the most in $N(r)$. Let $b \in L(r) - \{a\}$. Then b is repeated at most twice in $N(r)$ since $|N(r)| \leq 5$. Moreover, b is repeated at most once in $N(r)$ if $|N(r)| = 2$ or 3. Assign color b to r.

We claim that if none of the three rules applies, then all vertices are colored. Indeed, if neither Rule 1 nor Rule 2 applies and some vertex u_1 is uncolored, we have that $u_1 \neq r$ and $S(u_1)$ is uncolored, which implies $S(u_1) \subseteq (W - A) \cup \{u^*, r\}$. Let u_2 be any vertex in $S(u_1)$. So, $u_1, u_2 \in W$ and thus u_2u_1 is a path of length 1 in X with in-endvertex u_1 and out-endvertex u_2. As u_2 is uncolored and Rule 2 does not apply we have that $S(u_2)$ is uncolored. Continuing this way we obtain a sequence $u_1, u_2, \ldots$ of uncolored vertices such that for each $i = 1, 2, \ldots$, $u_{i+1} \in S(u_i)$ and $u_{i+1}u_i$ is a path of length 1 in X with in-endvertex u_i and out-endvertex u_{i+1}. As G is finite, we have $u_i = u_k$ for some $i < k$, which contradicts Lemma 8(i), stating that X_j is acyclic. This shows that if none of Rule 1, Rule 2, Rule 3 applies, then all vertices in H' are colored. So, applying Rule 1–Rule 3 as long as possible colors all the remaining vertices of G.

Next we shall show that the produced coloring is good, or more specifically that each monochromatic components of H' is a subpath of the union of two paths from X_j. Rule 1 and Rule 3 ensure that every vertex $v \in W - A$ has at most two neighbors in the same color as v. If u^* exists, then $\deg(u^*) = 3$, and hence Rule 2 ensures that $e^* = ru^*$ is colored properly. Moreover, for every vertex $u \in W$ let $X(u)$ be the set of paths P in X_j containing u, for which the neighbor of u in P has the same color as u. Then Rule 1 and Rule 2 ensure that $X(u) = \emptyset$, or $X(u)$ consists of exactly one path with in-endvertex u, or $X(u)$ consists of at most two paths, both with out-endvertex u.

Recall that we colored the vertices in A so that no vertex in X_j has a neighbor outside of X_j in the same color. Moreover, we colored $V(E_1) \cup B \cup \{u^*\}$ in such a way that every monochromatic component of $X_j - W$ is completely contained in a path of X_j. Finally, we colored the vertices in W so that every monochromatic component of X_j is the union of at most two monochromatic components of $X_j - W$. Together this implies that every monochromatic component is contained in the union of at most two paths in X_j. To summarize, we see that our coloring is good on $V(G) - V_1$. Now, each path of X_j is facial, i.e., has at most 8 edges by Lemma 5 and each monochromatic component in V_1 is a path contained in the union of some two paths from X_j. This monochromatic path has length at most 14, because it is induced and hence contains at most 7 edges from each of the two paths. So, our coloring is good on V_1. Finally, since no vertex of V_1 has the same color as its neighbor (if exists) in $V(G) - V_1$, the vertices of each monochromatic component are completely contained in V_1 or in $V(G) - V_1$. Thus the coloring is good. This concludes the proof of Lemma 9 saying that X_j, $j = 1, 2, 3, 4$, is reducible. □

To conclude the proof of Theorem 1, we see that Lemma 9 shows that G has a reducible configuration, contradicting the fact that G is a minimal counterexample.

4 Proof of Theorem 2

For every integer $t \geq 2$ we define two planar graphs of girth 4, denoted by A_t and B_t, respectively. The graph A_t consists of a path P_t on t vertices and two special vertices u and w, such that the vertices along P_t are joined by an edge alternatingly to u and w. For example, A_2 is a path on 4 vertices and the left of Figure 6 shows A_5. The graph B_t consists of A_t with special vertices u and w, and for every neighbor v of u there is another copy of A_t, with special vertices being identified with v and w, respectively. See the middle of Figure 6.

Note that for every $t \geq 2$ the graph B_t has girth 4 and the two special vertices u and w are at distance 3 (counted by the number of edges) in B_t.

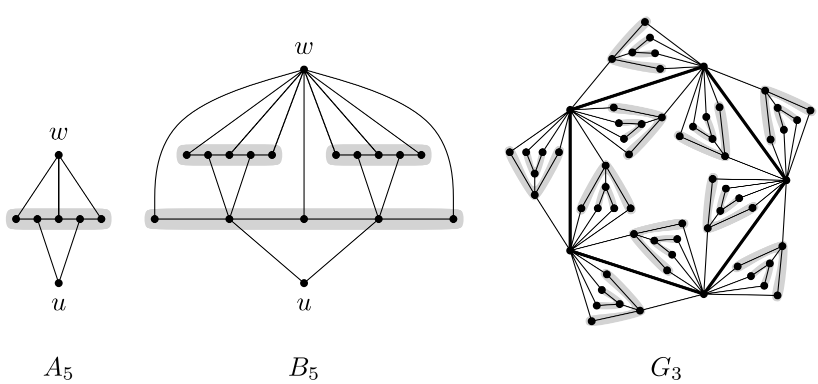


Figure 6: The graph A_5, B_5 and G_3.

We construct G_t inductively. For $t = 2$, we define G_t to be the 5-cycle. Clearly, in any 2-vertex coloring of G_2 there is a monochromatic P_2.

For $t \geq 3$, let G be a copy of G_{t-1}. We obtain G_t from G by considering every edge xy in G, taking two copies B, B' of B_t with special vertices u, w and u', w', respectively, and identifying x, u' and w, as well as y, u and w'. Note that G_t has girth 4 and is indeed planar: We can embed B and B' on different "sides" of the edge xy, as in the right of Figure 6.

Now, fix any 2-vertex coloring of G_t and consider the inherited coloring of G_{t-1}. By induction hypothesis there is a monochromatic copy Q of P_{t-1} in G_{t-1}, say in color 1. Let x be an endpoint of Q and y be the neighbor of x in Q. Consider the copy B of H_t where x is identified with w and y is identified with u and the copy A of A_t in B_t with special vertices $x = w$ and $u = y$.

If the copy P of P_t in A is not monochromatic, then at least one vertex v of P has color 1. If v is a neighbor of $x = w$, we have can extend Q by v into monochromatic P_t in color 1. Otherwise, v is a neighbor of $u = y$ and we consider the copy A' of A_t with special vertices $u' = v$ and $w' = x$. Again, if the copy P' of P_t in A' is not monochromatic, then at least one vertex v' of P' has color 1. If v' is a neighbor of x, then $Q \cup v'$ is a monochromatic P_t in color 1. Otherwise v' is a neighbor of v, and $Q \cup \{v, v'\}$ forms a monochromatic P_t in color 1. $\square$

5 Conclusions and open questions

In this paper, we proved that for any planar graph of girth 6 and any assignment of lists of 2 colors to each vertex, there is a coloring from these lists such that monochromatic components are paths of lengths at most 14. This extends a corresponding recent result of Borodin *et al.* [9] for planar graphs of girth 7. Our result can be interpreted as a statement about linear arboricity with short paths.

The proof uses discharging and reducible configurations. Compared to most of the previous discharging proofs, where the reducible configurations are small, here, the reducible configuration can be arbitrarily large. A similar approach was used by Havet and Sereni [21], who argued that every graph of maximum average degree less than 3 (which includes planar graphs of girth at least 6) has a 2-defective 2-list-coloring. The main difference between this proof and the proof of Theorem 1 is that Havet and Sereni can assume that in a minimal counterexample every edge is incident to a vertex of degree at least 4. Indeed, if there are two adjacent vertices u, v of degree at most 3 each, then a 2-defective 2-list-coloring of $G - \{u, v\}$ can easily be extended to a 2-defective 2-list-coloring of G. Such a reduction does not work in our case, since we can not bound the length of a longest monochromatic path. The reducible configurations of Havet and Sereni are not only simpler (they contain no vertices of degree 3), with their coloring of such a configuration one can get arbitrarily long monochromatic paths. Thus, our Lemma 9 requires less and proves more then the corresponding statement of Havet and Sereni [21, Lemma 2].

According to Table 1 the remaining open questions concern 2-colorings of planar graphs of girth 5 or 6. Figure 7 shows a planar graph of girth 5 that contains a monochromatic P_3 in every 2-coloring, i.e., $k_d(5, 2) \geq 2$, $k_f(5, 2) \geq 3$, and $k_p(5, 2) \geq 4$. Indeed, we may assume without loss of generality, that in a given 2-coloring vertices u and v both have color 1. Then u_i and v_i, for $i = 1, 2, 3$, have color 2 or there is a monochromatic P_3 in color 1. Similarly, w_1, w_2, w_3 have color 1 or there is a monochromatic P_3 in color 2. But then these three vertices form a monochromatic P_3 in color 1.

This also follows from Montassier and Ochem [25] who provided an example of a planar graph of girth 5 such that in any red/blue-coloring of its vertices there is a red P_3 or a vertex of degree at least 4 in the subgraph induced by blue vertices.

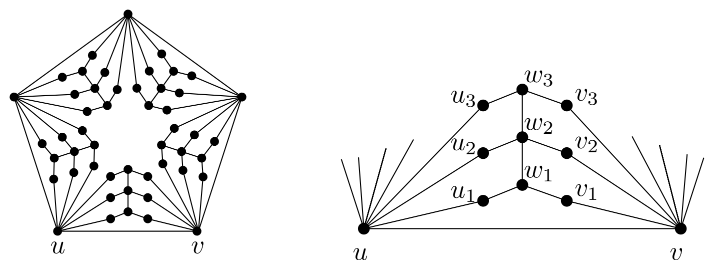


Figure 7: Every 2-coloring contains a monochromatic path on 3 vertices.

However, to the best of our knowledge, it is open whether $k_f(5, 2)$ and $k_p(5, 2)$ are finite. On the other hand, it is still possible that every planar graph of girth 5 and 6 admits a 2-coloring where every monochromatic component is a subgraph of P_3 and P_2, respectively.

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