Polychromatic Colorings on the Integers

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Abstract

We show that for any set $S \subseteq \mathbb{Z}$, $|S| = 4$ there exists a 3-coloring of $\mathbb{Z}$ in which every translate of S receives all three colors. This implies that S has a codensity of at most $1/3$, proving a conjecture of Newman [D. J. Newman, Complements of finite sets of integers, *Michigan Math. J.* 14 (1967) 481–486]. We also consider related questions in $\mathbb{Z}^d$, $d \geq 2$.

1 Introduction

Throughout the paper, let G denote an arbitrary abelian group. Given $S, T \subseteq G$, $n \in G$, define $S+T = \{s+t : s \in S, t \in T\}$ and $n+S = \{n\}+S$. Any set of the form $n + S$ is called a *translate* of S. Given a subset S of G, a coloring of the elements of G is S*-polychromatic* if every translate of S contains an element of each color. Define the *polychromatic number* of S, denoted $p_G(S)$, to be the largest number of colors allowing an S-polychromatic coloring of the elements of G. We just write $p(S)$ when the choice of G is clear from context.

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We are primarily concerned with the setting where $G = \mathbb{Z}$ and S is finite. If S has cardinality 1 or 2, $p(S) = |S|$. For $|S| = 3$, $p(S)$ can be 2 or 3. For example, if $S = \{0, 1, 5\}$ then every translate of S contains three elements which are each in different congruence classes $\pmod 3$. Thus a 3-coloring of the integers where each congruence class (mod 3) is colored a different color is S-polychromatic, and $p(\{0, 1, 5\}) = 3$. However $p(\{0, 1, 3\}) = 2$. To see that $p(\{0, 1, 3\}) \neq 3$, let χ be a 3-coloring of $\mathbb{Z}$ with $\chi(0)$, $\chi(1)$, and $\chi(3)$ all different. Some element $s \in \{0, 1, 3\}$ has $\chi(s) = \chi(2)$, and there is a translate of $\{0, 1, 3\}$ that contains both s and 2, so the coloring is not polychromatic. Our main result concerns the polychromatic numbers of sets with cardinality 4.

Theorem 1 *If $S \subseteq \mathbb{Z}$ and $|S| = 4$, then $p(S) \geq 3$.*

The proof of Theorem 1 is given in Section 2. For larger sets S, Alon, Kříž, and Nešetřil [2] proved that $p(S) \geq \frac{(1+o(1))|S|}{3\ln|S|}$, while there exists some set S where $p(S) \leq \frac{(1+o(1))|S|}{\ln|S|}$. Subsequently, Harris and Srinivasan [6] established a tight asymptotic lower bound on polychromatic numbers.

Theorem 2 ([2], [6]) *For a finite set $S \subseteq \mathbb{Z}$, $p(S) \geq \frac{(1+o(1))|S|}{\ln|S|}$. Moreover, there exists some set S where $p(S) \leq \frac{(1+o(1))|S|}{\ln|S|}$.*

One motivation for studying polychromatic numbers is that they provide bounds for Turán type problems (see for example [1], [10], [11]). Call $T \subseteq G$ a *blocking set* for S if $G \setminus T$ contains no translate of S, i.e. if for all $n \in G$, $n + S \nsubseteq G \setminus T$. A Turán type problem asks for the smallest blocking set for a given set S. In the case where S is finite and $G = \mathbb{Z}$, any blocking set is countably infinite, so we ask how small the density of a blocking set can be. Following the notation of Newman [8], (he worked in the setting of the natural numbers, but the definitions are equivalent), define for any set $T \subseteq \mathbb{Z}$ its *upper density* $\overline{d}(T)$ and *lower density* $\underline{d}(T)$ as

$$\overline{d}(T) = \limsup_{n\to\infty} \frac{|T \cap [-n, n]|}{2n + 1} \qquad \text{and} \qquad \underline{d}(T) = \liminf_{n\to\infty} \frac{|T \cap [-n, n]|}{2n + 1}.$$

If $\overline{d}(T) = \underline{d}(T)$, we call this quantity the *density* of T and denote it by $d(T)$. Define $\alpha(S)$ to be a measure of how small the density of a blocking set for S can be. Let

$$\alpha(S) = \inf\{d(T) : \ T \text{ is a blocking set for } S \text{ and } d(T) \text{ exists}\}.$$

In Section 3, we describe the relationship between polychromatic colorings and blocking sets, and prove Lemma 3.

Lemma 3 *For any finite set $S \subseteq \mathbb{Z}$, $\alpha(S) \leq 1/p(S)$.*

One of the main consequences of Theorem 1 concerns covering densities of sets of integers. Given a set $S \subseteq G$, we say $T \subseteq G$ is a *complement set* for S if $S + T = G$. We say S *tiles G by translation* if it has a complement set T such that if $s_1, s_2 \in S$,

$t_1, t_2 \in T$, then $s_1 + t_1 = s_2 + t_2$ implies $s_1 = s_2$ and $t_1 = t_2$. In this paper we only consider tilings by translation, so if S tiles G by translation with complement set T we will simply say S tiles G and write $G = S \oplus T$.

Again, our primary interest will be the case where $G = \mathbb{Z}$ and S is finite. For example, if $S = \{0, 1, 5\}$, then S tiles $\mathbb{Z}$ with complement set $T = \{3n : n \in \mathbb{Z}\}$. However $S = \{0, 1, 3\}$ does not tile $\mathbb{Z}$. Newman [9] proved necessary and sufficient conditions for a finite set S to tile $\mathbb{Z}$ if $|S|$ is a power of a prime.

Theorem 4 (Newman [9]) *Let $S = \{s_1, \ldots, s_k\}$ be distinct integers with $|S| = p^\alpha$ where p is prime and α is a positive integer. For $1 \le i < j \le k$ let $p^{e_{ij}}$ be the highest power of p that divides $s_i - s_j$. Then S tiles $\mathbb{Z}$ if and only if $|\{e_{ij} : 1 \le i < j \le k\}| \le \alpha$.*

Later Coven and Meyerowitz [5] gave necessary and suffucient conditions for S to tile $\mathbb{Z}$ when $|S| = p_1^{\alpha_1} p_2^{\alpha_2}$, where p_1 and p_2 are primes. The general question is still open. Kolountzakis and Matolcsi [7] and Amiot [3] have published recent work motivated by what are called rhythmic tilings in music.

If a finite set S tiles $\mathbb{Z}$, it has a complement set of density $1/|S|$. Following Newman [8], we define the *codensity* of a set S, denoted $c(S)$, as a measure of the how small the density of a complement set can be. Let

$$c(S) = \inf\{d(T) : S + T = \mathbb{Z} \text{ and } d(T) \text{ exists}\}.$$

We are interested in the largest codensities for sets of a given cardinality. Define

$$c_k = \sup_{\{S : |S| = k\}} c(S).$$

An example of a complement set for $\{0, 1, 3\}$ is $\{t \in \mathbb{Z} : t \equiv 0 \text{ or } 1 \pmod 5\}$, so $c(\{0, 1, 3\}) \le 2/5$. The following theorem and conjecture on c_4 are due to Newman.

Theorem 5 (Newman [8])

- $c(\{0, 1, 3\}) = 2/5$.
- $c_3 = 2/5$.
- $c(\{0, 1, 2, 4\}) = 1/3$.

Conjecture 6 (Newman [8]) $c_4 = 1/3$.

Weinstein [16] showed that $c_4 < .339934$. Based on a computer search, Bollobás, Janson, and Riordan [4] confirmed Newman's conjecture for sets with diameter at most 22, where the *diameter* of a nonempty finite set of integers is defined to be the difference between the largest and smallest elements in the set. They also conjectured that $c_5 = 3/11$ and $c_6 = 1/4$ (See Remark 5.6 and Question 5.7 in [4]. Note they use different notation).

In Section 3 we prove the following lemma relating blocking sets and complement sets.

Lemma 7 *For any finite set $S \subseteq \mathbb{Z}$, $c(S) = \alpha(S)$.*

Theorem 1, along with Lemmas 3 and 7, suffice to resolve Conjecture 6.

Theorem 8 $c_4 = 1/3$.

Proof: Theorem 5 implies $c(\{0,1,2,4\}) = 1/3$, so it remains to show that for any other set S with cardinality four, $c(S) \leq 1/3$. Let $S \subseteq \mathbb{Z}$ have four elements. Then Theorem 1 implies that $p(S) \geq 3$, and by Lemmas 3 and 7,

$$c(S) = \alpha(S) \leq 1/p(S) \leq 1/3.$$

■

In Subection 3.1, we consider the relationship between polychromatic colorings and tilings. The main result is Theorem 11, which states that a set S tiles an abelian group G by translation if and only if $p(S) = |S|$.

Finally, in Section 4 we turn our attention to polychromatic numbers and tilings for finite sets in $\mathbb{Z}^d$. We begin by proving in Theorem 17 that the bound of Theorem 2 applies to subsets of $\mathbb{Z}^d$. We then show that if a set of points in $\mathbb{Z}^d$ is collinear, determining its polychromatic number is equivalent to determining the polychromatic number of a specific projection of this set into $\mathbb{Z}$. Theorem 11 implies that a set S tiles $\mathbb{Z}^d$ if and only if $p_{\mathbb{Z}_d}(S) = |S|$, so we use this to restate some well-known results on tilings of $\mathbb{Z}^d$ by finite sets in the language of polychromatic colorings. We conclude by applying these results to determine polychromatic numbers of sets with cardinality 3 and 4 in $\mathbb{Z}^d$.

2 Sets of Cardinality Four

In this section we prove that every set of four integers has polychromatic number at least 3. We begin by stating a lemma that reduces the problem of finding a polychromatic coloring of $\mathbb{Z}$ to finding a polychromatic coloring of $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ for a specific choice of m.

Lemma 9 *Let $a, b, c, k, q \in \mathbb{Z}$ with $0 < a < b < c$, $\gcd(a,b,c) = 1$, $k, q \geq 1$, and $m = c - a + b$. Let $S = \{0, ka, kb, kc\}$, $S_1 = \{0, a, b, c\}$, $S_2 = \{0, b-a, b, 2b-a\}$. Then*

(i) $p_{\mathbb{Z}}(S) = p_{\mathbb{Z}}(S_1)$.

(ii) $p_{\mathbb{Z}}(S_1) \geq p_{\mathbb{Z}_m}(S_1)$.

(iii) $p_{\mathbb{Z}_m}(S_1) = p_{\mathbb{Z}_m}(S_2)$.

(iv) If $\gcd(k,q)=1$*, then* $p_{\mathbb{Z}_q}(S)=p_{\mathbb{Z}_q}(S_1)$.

Proof:

(i) Suppose χ is an S-polychromatic coloring. If χ_1 is the coloring defined by $\chi_1(n)=\chi(kn)$, then χ_1 is S_1-polychromatic. Conversely, if χ_1 is an S_1-polychromatic coloring, then the coloring χ defined by $\chi(n)=\chi_1(\lfloor n/k \rfloor)$ is S-polychromatic.

(ii) Suppose χ_m is an S_1-polychromatic coloring on $\mathbb{Z}_m$. Then the coloring χ_1 defined by $\chi_1(n)=\chi_m(r)$ where $0 \le r < m$ is the remainder when n is divided by m is an S_1-polychromatic coloring on $\mathbb{Z}$.

(iii) In $\mathbb{Z}_m$, with addition $\pmod m$, $S_2 = S_1 + (b-a)$. Thus in $\mathbb{Z}_m$, S_1 and S_2 are translates of each other.

(iv) Since $\gcd(k,q)=1$, we can write $\mathbb{Z}_q = \{ik \pmod m : 0 \le i \le m-1\}$. Let χ and χ_1 be colorings of $\mathbb{Z}_q$ such that $\chi(i)=\chi_1(ik \pmod m)$. Then χ is S-polychromatic if and only if χ_1 is S_1-polychromatic.

■

Proof of Theorem 1: Using a computer search, we verified that that for every S with diameter at most 288 there exists an S-polychromatic 3-coloring of $\mathbb{Z}_m$ for some m depending on S. The code for this search has been included as an ancillary file with the preprint of this paper on arXiv.org. By Lemma 9, Part (ii), this gives a periodic S-polychromatic 3-coloring of $\mathbb{Z}$. Hence we suppose that $c \ge 289$.

By Lemma 9, Part (i), it suffices to prove the theorem in the case that $S=\{0,a,b,c\}$ with $0<a<b<c$ and $\gcd(a,b,c)=1$. For the remainder of the proof, let $m=c-a+b$. By Lemma 9, Parts (ii) and (iii), it suffices to show that we can 3-color $\mathbb{Z}_m=\{0,1,\ldots,m-1\}$ so that the translates of $\{0,b-a,b,2b-a\}$ are polychromatic. So for the remainder of the proof we assume $S=\{0,b-a,b,2b-a\}$ and seek an S-polychromatic 3-coloring of $\mathbb{Z}_m$. The key observation regarding S is that it contains two repeated differences: $b-a$ and b.

Define $d_1=\gcd(b,m)$ and $d_2=gcd(b-a,m)$. Since $1=gcd(a,b,c)=\gcd(b-a,b,c-a+b)=\gcd(b-a,b,m)$, we know $gcd(d_1,d_2)=1$. We distinguish two main cases. In the first case, which we call "single cycle," we assume $\min\{d_1,d_2\}=1$ and give a coloring of $\mathbb{Z}_m$. In the second case, which we call "multiple cycle," we assume $\min\{d_1,d_2\}>1$ and partition $\mathbb{Z}_m$ into multiple cycles of length m/d_i for one of the choices of i. We then give a rule for coloring each cycle.

Main case 1 (Single cycle): Suppose $\min\{d_1,d_2\}=1$. Without loss of generality, assume $d_1=1$ (if not, then simply switch all occurences of b and $b-a$ in the argument below). Let $2 \le g \le m-2$ satisfy $gb \equiv b-a \pmod m$, so that $S=\{0,bg,b,b(g+1)\}$.

Applying Lemma 9, Part (iv), with $q = m$ and $k = b$, we can instead work with $S = \{0, g, 1, g+1\} = \{0, 1, g, g+1\}$.

We may assume that $g \le m/2$, as otherwise we could work with the equivalent set $\{0, 1, m-g, m-g+1\}$. Let s be the smallest multiple of 3 such that $g > \lceil m/s \rceil$. We consider four subcases: The first two are (1a) $g = 2$, 3, or 4 and (1b) $5 \le g < 2\lfloor m/s \rfloor$. In the remaining subcases (1c) and (1d), $2\lfloor m/s \rfloor \le g \le \lceil m/(s-3) \rceil$. For $m > 8$, if $2\lfloor m/s \rfloor \le g \le m/2$ then $s > 3$, and for $m > 44$, if $2\lfloor m/s \rfloor \le g \le \lceil m/(s-3) \rceil$ then $s < 9$. Since $m > c \ge 289 > 44$, we can assume $s = 6$, so $2\lfloor m/6 \rfloor \le g \le \lceil m/3 \rceil$. This implies $m = 3g + k$ where $-2 \le k \le 5$ and there are two further subcases to consider, depending on the residue class of m modulo 6: (1c) $m = 3g-2$, $3g-1$, $3g+1$, $3g+2$, $3g+4$, or $3g+5$, and (1d) $m = 3g$ or $3g+3$.

Subcase (1a): Suppose $g = 2$, 3, or 4. Then $S = \{0, 1, 2, 3\}$, $\{0, 1, 3, 4\}$, or $\{0, 1, 4, 5\}$, respectively. In Subcase (1c) we will construct S-polychromatic 3-colorings of $\mathbb{Z}_m$ for each of these sets.

Subcase (1b): Suppose $5 \le g < 2\lfloor m/s \rfloor$. Then split $\mathbb{Z}_m$ into s intervals as equally as possible (i.e. of lengths $\lfloor m/s \rfloor$ and $\lceil m/s \rceil$) and color these intervals 010101…, followed by 121212…, then 202020…, repeating $s/3$ times. Since $\lceil m/s \rceil < g < 2\lfloor m/s \rfloor$, any translate of S' where the pairs $\{0, 1\}$ and $\{g, g+1\}$ lie in different intervals gets all three colors. If one of the pairs $\{0, 1\}$ or $\{g, g+1\}$ straddles two consecutive intervals, this pair may get only the single color common to these two intervals, but then the other pair lies fully inside a third interval which is colored with the remaining two colors.

Subcase (1c): Suppose $m = 3g-2$, $3g-1$, $3g+1$, $3g+2$, $3g+4$, or $3g+5$. In this case we know that $m \not\equiv 0 \pmod 3$ so we can apply Lemma 9, Part (iv), with $q = m$ and $k = 3$, and instead work with one of the sets in $\mathcal{S} = \{\{0, 2, 3, 5\}, \{0, 1, 3, 4\}, \{0, 1, 2, 3\}, \{0, 3, 4, 7\}, \{0, 3, 5, 8\}\}$. For example, if $m = 3g-2$, then multiplying by 3, S is transformed into $\{0, 3, 3g, 3g+3\} \equiv \{0, 2, 3, 5\}$, while if $m = 3g+4$, then multiplying by 3, S is transformed into $\{0, 3, 3g, 3g+3\} \equiv \{0, 3, -4, -1\}$, which is a translate of $\{0, 3, 4, 7\}$.

Thus we have reduced the problem to finding an S-polychromatic 3-coloring of $\mathbb{Z}_m$ for each of the sets $S \in \mathcal{S}$. For each $S \in \mathcal{S}$ we write one interval of a periodic S-polychromatic 3-coloring on $\mathbb{Z}$ in Table 1, and also include one for $\{0, 1, 4, 5\}$ to cover Subcase (1a). Each of these periodic colorings also has the following property: If the coloring has period r, then the periodic 3-coloring with period $r+1$ obtained by adding a prefix of 0 to each interval is also S-polychromatic. In each case this means that for any $h, k \ge 0$ we can create a period $hr + k(r+1)$ S-polychromatic 3-coloring by concatenating a suitable number of the two blocks.

To obtain a coloring of $\mathbb{Z}_m$, we simply need to check that for $r = 3, 6, 7, 9$ we can always express m as a positive integer combination of r and $r+1$. This is the (2 coin) Frobenius problem and can always be done for any integer greater than

S	r	period r	period $r+1$
$\{0,2,3,5\}$	6	001122	0001122
$\{0,1,3,4\}$	6	001212	0001212
$\{0,1,2,3\}$	3	012	0012
$\{0,3,4,7\}$	9	000111222	0000111222
$\{0,3,5,8\}$	9	000111222	0000111222
$\{0,1,4,5\}$	7	0001212	00001212

Table 1: One interval of a periodic coloring for sets in Subcases (1a) and (1c).

$r^2 - r - 1 \leq 71 < 289$.

Subcase (1d): Suppose $m = 3g$ or $3g+3$. If $g \not\equiv 0 \pmod 3$ then simply color $\mathbb{Z}_m$ with the pattern $0120120\ldots012$. If $g \equiv 0 \pmod 3$ and $m = 3g$, color $\mathbb{Z}_m$ in 3 equal intervals, each of length g: $012012\ldots012$ followed by $120120\ldots120$ followed by $201201\ldots201$. Finally, if $g \equiv 0 \pmod 3$ and $m = 3g+3$ we color $\mathbb{Z}_m$ in 3 equal intervals, each of length $g+1$: $012012\ldots0120$ followed by $201201\ldots2012$ followed by $120120\ldots1201$.

Main case 2 (Multiple cycles): Suppose $\min\{d_1, d_2\} > 1$. Since d_1 and d_2 are relatively prime, at most one of them can be a multiple of 3. Choose the smallest of these numbers that is not a multiple of 3, and as in the single cycle case, without loss of generality assume it is d_1.

Let $e_1 = m/d_1$ and $e_2 = m/d_2$. For $0 \leq i < d_1$, let

$$C_i = \{(b-a)i + bj \pmod m : 0 \leq j < e_1\}.$$

Since

$$\mathbb{Z}_m = \{(b-a)i + bj \pmod m : 0 \leq i < d_1, 0 \leq j < e_1\},$$

the C_i's form a partition of $\mathbb{Z}_m$ into d_1 cycles, each with e_1 elements.

Let $c_{i,j}$ denote the jth element of C_i, i.e. $c_{i,j} = i(b-a) + jb \pmod m$. Note that any translate of S contains two consecutive elements of two consecutive cycles, i.e. any translate of S has the form $\{c_{i,j}, c_{i,j+1}, c_{i+1,j}, c_{i+1,j+1}\}$, where the first entry in the subscript is taken $\bmod d_1$ and the second entry is taken $\bmod e_1$. We describe an S-polychromatic 3-coloring for each of four subcases: (2a) e_1 is even, (2b) d_1 is even and e_1 is odd, (2c) d_1 and e_1 are both odd, with $e_1 \leq 17$, and (2d) d_1 and e_1 are both odd, with $e_1 \geq 19$.

Subcase (2a): Suppose e_1 is even. For $i = 0, \ldots, \lfloor d_1/2 \rfloor - 1$, color each C_{2i} by $01010\ldots01$ and each C_{2i+1} by $02020\ldots02$. Finally, if d_1 is odd, color C_{d_1-1} by $1212\ldots12$.

Subcase (2b): Suppose d_1 is even and e_1 is odd. For $i = 0, \ldots, d_1/2 - 1$, color each C_{2i} by $01010\ldots011$ and each C_{2i+1} by $22020\ldots02$.

Subcase (2c): Suppose d_1 and e_1 are both odd, with $e_1 \leq 17$. Since $e_1 e_2 \geq m > c \geq 289$, one of e_1 and e_2 is larger than 17, so $e_2 > e_1$ and hence $d_1 > d_2$. Since d_1 is the smaller of d_1 and d_2 that is not a multiple of 3, d_2 must be a multiple of 3, and thus so is e_1.

We color each C_i with one of three patterns: $012012\ldots012$, $120120\ldots120$, or $201201\ldots201$. Such a coloring is S-polychromatic so long as for all i, C_i and C_{i+1} are colored with different patterns. For $0 \leq i \leq (d_1-3)/2$, color C_{2i} with the first pattern and color C_{2i+1} with the second pattern. Finally, color C_{d_1-1} with the third pattern.

Subcase (2d): Suppose d_1 and e_1 are both odd, with $e_1 \geq 19$. Since d_1 is not divisible by 3 and $\min\{d_1, d_2\} > 1$, $d_1 \geq 5$. Let $e_1 = u+v+w$ be a sum of odd integers u, v, w with $u \geq v \geq w \geq u-2$. Color C_0 in intervals of size u, v, w, using the patterns $0101\ldots010$ then $1212\ldots121$ and then $2020\ldots202$. For each $i \geq 1$, color C_i by taking a "counterclockwise rotation" of length r_i of the coloring of C_{i-1}, so that the color of $c_{i,j+r}$ is the same as the color of $c_{i-1,j}$. For $1 \leq i \leq d_1 - 1$, if $u \leq r_i \leq v+w = e_1 - u$, then each translate of S meeting C_{i-1} and C_i receives all 3 colors.

It remains to show that there are choices of $r_1, \ldots, r_{d_1-1}$ with $u \leq r_i \leq v+w = e_1 - u$ so that of the translates of S meeting C_{d_1-1} and C_0 receive all three colors. The coloring of C_0 is a "clockwise rotation" of length $R = -r_1 - r_2 - \cdots - r_{d_1-1}$ of the coloring of C_{d_1-1}, i.e. the color of $c_{0,j-R}$ is the same as the color of $c_{d_1-1,j}$. Since for each i, $u \leq r_i \leq v+w = e_1 - u$, it suffices to show that there is a multiple of e_1 in the interval $[d_1 u, d_1(e_1-u)]$, ensuring there are choices for the r_i's such that R is congruent to a number between u and $e_1 - u \pmod{e_1}$. This certainly holds if $d_1(e_1 - 2u) \geq e_1 - 1$ which, since $d_1 \geq 5$, holds if $4e_1 \geq 10u - 1$. This inequality is true for $e_1 \geq 19$.

This completes the multiple cycles case and the proof. ∎

3 Colorings, Blocking Sets, Coverings, and Tilings

In this section we prove the results necessary to resolve Newman's conjecture. The key insight in proving Lemma 3 is that the elements of a given color in an S-polychromatic coloring form a blocking set for S. While it is possible for $\alpha(S)$ to be equal to $1/p(S)$ (e.g. if $|S| = 2$ then $\alpha(S) = 1/2 = 1/p(S)$), in general these two quantities are not equal. For example, $p(\{0,1,3\}) = 2$, but by Lemma 7 and Theorem 5, $\alpha(\{0,1,3\}) = 2/5 < 1/2$.

Proof of Lemma 3: Let χ be an S-polychromatic coloring of $\mathbb{Z}$ with $p(S)$ colors. Suppose $d \in \mathbb{Z}$ is greater than the diameter of S and let $I_j = \{n \in \mathbb{Z} : jd \leq n < (j+1)d\}$. By the pigeonhole principle, for some $0 \leq j_1 < j_2 \leq (p(S))^d$ the coloring of the intervals I_{j_1} and I_{j_2} are identical, i.e. for $0 \leq k < d$, $\chi(j_1 d + k) = \chi(j_2 d + k)$.

Let $m = (j_2 - j_1)d$. For any $n \in \mathbb{Z}$, denote by r the remainder when n is divided by m, so $0 \le r < m$. Let χ' be the coloring of $\mathbb{Z}$ where $\chi'(n) = \chi(j_1 d + r)$. Note that χ' uses $p(S)$ colors and is periodic with period m, i.e. for all $n \in \mathbb{Z}$, $\chi(n) = \chi(n+m)$. Furthermore, the coloring under χ' of any d consecutive integers is identical to the coloring under χ of some d consecutive integers, so χ' is S-polychromatic. Let $T_i = \{n \in \mathbb{Z} : \chi'(n) = i\}$. Since any periodic set has a defined density, $d(T_i)$ is defined for each i, and $\sum_{i=1}^{p(S)} d(T_i) = 1$. Since χ' is S-polychromatic, for each i, each translate of S contains an element of T_i, i.e. T_i is also a blocking set for S. Thus for some i, T_i a blocking set for S with density at most $1/p(S)$, which implies that $\alpha(S) \le 1/p(S)$. ■

For any subset T of an abelian group G, let $-T$ denote the set $\{-t : t \in T\}$. Lemma 10 is well-known (see e.g [14]) but for completeness we present a proof.

Lemma 10 *Let G be an abelian group, and $S \subseteq G$. Then $T \subseteq G$ is a complement set for S if and only if $-T$ is a blocking set for S.*

Proof: Suppose T is a complement set for S. For any $n \in G$, $-n \in S + T$, so there must be some $t \in T, s \in S$ such that $t + s = -n$. This implies $t = -n - s$, so $-n - s \in T$, and $n + s \in -T$. Thus for every n, some element of $n + S$ is in $-T$, and $-T$ is a blocking set for S.

Conversely, suppose $-T$ is a blocking set for S. For the sake of contradiction, assume T is not a complement set for S, i.e. there is some $-n \in G$ such that $-n \notin S + T$. This implies that for all $s \in S$, $-n - s \notin T$, which means for all $s \in S$, $n + s \notin -T$. Thus $n + S \subseteq G \setminus -T$, and so $-T$ is not a blocking set for S, a contradiction. ■

Proof of Lemma 7: Lemma 10 implies that T is a complement set for S if and only if $-T$ is a blocking set for S. If they exist, the densities of T and $-T$ are the same. ■

3.1 Polychromatic Colorings and Tilings

We now describe some relationships between polychromatic colorings and tilings.

Theorem 11 *Let G be any abelian group. A finite set $S \subseteq G$ tiles G by translation if and only if $p(S) = |S|$. Moreover, if χ is an S-polychromatic coloring of G with $|S|$ colors and T is the set of elements of G colored by χ with any given color, then $S \oplus T = G$.*

Proof: ($\Rightarrow$): Let $S = \{s_1, s_2, \ldots, s_k\}$, and suppose S tiles G with complement set $T \subseteq G$. For each $n \in G$, define a coloring χ on G so that $\chi(n) = i$ if $n = s_i + t$ for some $t \in T$. By the definition of tiling, this coloring is well-defined. For the sake of contradiction, assume χ is not S-polychromatic. Then for some l where $1 \le l \le k$, there exists $n \in G$ and $s_i, s_j \in S$ with $i \ne j$ such that $\chi(n + s_i) = \chi(n + s_j) = l$. Then there exist $t_1, t_2 \in T$, $t_1 \ne t_2$, such that $n + s_i = t_1 + s_l$ and $n + s_j = t_2 + s_l$. Subtracting these equations, we find that $s_i - s_j = t_1 - t_2$. Thus $t_2 + s_i = t_1 + s_j$, which is a contradiction.

$(\Leftarrow)$: Let $S = \{s_1, s_2, \ldots, s_k\}$, suppose $p(S) = |S|$, and let χ be an S-polychromatic coloring of G with $|S|$ colors. Then for all $n \in G$, if $i \neq j$ then $\chi(n+s_i) \neq \chi(n+s_j)$. Let $T \subseteq G$ be the set of elements colored with a given color. We show that $S \oplus T = G$. First assume for the sake of contradiction that two translates of S share an element, i.e. there exist $s_i, s_j \in S$, $i \neq j$, $t_1, t_2 \in T$, $t_1 \neq t_2$, such that $s_i + t_1 = s_j + t_2$. Let $n = t_1 - s_j = t_2 - s_i$, so $t_1 = n + s_j$ and $t_2 = n + s_i$. Since $\chi(t_1) = \chi(t_2)$ we get $\chi(n + s_j) = \chi(n + s_i)$, so two elements of $n + S$ are colored identically, which is a contradiction.

It remains to show that $S+T = G$. Suppose there is some $n \in G$ such that $n \notin S+T$. Then for all i, $n - s_i \notin T$, which implies that the $|S|$ elements of $n-S$ are colored with at most $|S|-1$ colors, i.e. two are colored identically. Suppose $\chi(n-s_i) = \chi(n-s_j)$, where $i \neq j$. Let $m = n - s_j - s_i$. Then $m + S$ contains both $m + s_i = n - s_j$ and $m + s_j = n - s_i$. Since these integers are colored identically, $m + S$ is a translate of S that does not contain all colors, which is a contradiction. ■

Sets of integers with cardinality $n = 3$ or 4 always have polychromatic number n or $n - 1$, and a corollary of Theorem 11 is that they have polychromatic number $n - 1$ if and only if they do not tile $\mathbb{Z}$. According to Remark 5.6 in [4], $c(\{0, 1, 3, 4, 8\}) = 3/11 > 1/4$. Thus by Lemma 3, $\{0, 1, 3, 4, 8\}$ is an example of a set with cardinality 5 and polychromatic number 3. The results of [2] and [6] imply that for sets S with large cardinality n the cardinality and polychromatic number of S can differ by a factor of $1/\ln n$.

We now state some other corollaries of Theorem 11.

Corollary 12 *If a finite set S tiles an abelian group G by translation, then any S-polychromatic coloring of G with $|S|$ colors is also a $(-S)$-polychromatic coloring.*

Proof: Suppose S tiles G. By Theorem 11, there exists an S-polychromatic coloring χ of G with $|S|$ colors. Let $T \subseteq G$ be the set of all elements of a given color. Again by Theorem 11, $S + T = G$. Therefore by Lemma 10, $-T$ is a blocking set for S, i.e. for all $n \in G$, $n + S \nsubseteq G \setminus (-T)$. This implies that for all $n \in G$, $-n - S \nsubseteq G \setminus T$, i.e. T is a blocking set for $-S$. Since T is a blocking set for $-S$ for every color choice, every translate of $-S$ contains every color, i.e. the coloring χ is $(-S)$-polychromatic. ■

Define $t(S)$ to be the cardinality of the largest subset of S that tiles G.

Corollary 13 *For any finite subset S of an abelian group G, $p(S) \geq t(S)$.*

If $S \subseteq \mathbb{Z}$, $|S| \leq 3$, then $p(S) = t(S)$. But these parameters can be different for sets of integers with at least four elements. For example, $S = \{0, 1, 3, 7\}$ is an example of a set where $t(S) = 2$, but $p(S) = 3$.

Question 14 *For sets S of a given cardinality, how large can the gap between $t(S)$ and $p(S)$ be?*

4 Polychromatic Colorings in $\mathbb{Z}^d$

In this section we consider polychromatic numbers in the case where $G = \mathbb{Z}^d$, $d \geq 2$. We will frequently "project" a set $S \subseteq \mathbb{Z}^d$ to another set $S' \subseteq \mathbb{Z}^{d-1}$ as follows. Let $d \geq 2$, and $\boldsymbol{w} = (w_1, \ldots, w_{d-1}, 1) \in \mathbb{Z}^d$. Define $f_{\boldsymbol{w}} : \mathbb{Z}^d \to \mathbb{Z}^{d-1}$ so that if $\boldsymbol{s} = (v_1, \ldots, v_d) \in \mathbb{Z}^d$,

$$f_{\boldsymbol{w}}(\boldsymbol{s}) = (v_1, \ldots, v_{d-1}) - v_d(w_1, \ldots, w_{d-1}).$$

We call $f_{\boldsymbol{w}}(\boldsymbol{s})$ a *projection* of $\boldsymbol{s}$ along $\boldsymbol{w}$. Given a set $S = \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_k\} \subseteq \mathbb{Z}^d$, we call a set $S' = \{\boldsymbol{s}'_1, \ldots, \boldsymbol{s}'_k\} \subseteq \mathbb{Z}^{d-1}$ a *projection* of S along $\boldsymbol{w}$ if for $1 \leq i \leq k$, $\boldsymbol{s}'_i$ is a projection of $\boldsymbol{s}_i$ along $\boldsymbol{w}$.

For example, if $\boldsymbol{s} = (2, 7, 4)$ and $\boldsymbol{w} = (3, 1, 1)$, the projection of $\boldsymbol{s}$ along $\boldsymbol{w}$ is $f_{\boldsymbol{w}}(\boldsymbol{s}) = (2, 7) - 4(3, 1) = (-10, 3)$. As another example, note that if $\boldsymbol{s} = (v_1, \ldots, v_d) \in \mathbb{Z}^d$, the vector $\boldsymbol{s'} = (v_1, \ldots, v_{d-1}) \in \mathbb{Z}^{d-1}$ is a projection of $\boldsymbol{s}$ along $\boldsymbol{w} = (0, \ldots, 0, 1)$.

Lemma 15 *Let $d \geq 2$, and $\boldsymbol{w} = (w_1, \ldots, w_{d-1}, 1) \in \mathbb{Z}^d$. Let $S = \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_k\} \subseteq \mathbb{Z}^d$, and suppose $S' = \{\boldsymbol{s}'_1, \ldots, \boldsymbol{s}'_k\} \subseteq \mathbb{Z}^{d-1}$ is a projection of S along $\boldsymbol{w}$. Then $p_{\mathbb{Z}^d}(S) \geq p_{\mathbb{Z}^{d-1}}(S')$.*

Proof: We show that if χ_1 is an S'-polychromatic r-coloring of $\mathbb{Z}^{d-1}$, we can define an S-polychromatic r-coloring χ_2 on $\mathbb{Z}^d$. For all $\boldsymbol{n} \in \mathbb{Z}^d$, let $\chi_2(\boldsymbol{n}) = \chi_1(f_{\boldsymbol{w}}(\boldsymbol{n}))$. If $\boldsymbol{n} \in \mathbb{Z}^d$ and $\boldsymbol{n'} = f_{\boldsymbol{w}}(\boldsymbol{n}) \in \mathbb{Z}^{d-1}$, then for all i, $\chi_2(\boldsymbol{n} + \boldsymbol{s}_i) = \chi_1(f_{\boldsymbol{w}}(\boldsymbol{n} + \boldsymbol{s}_i)) = \chi_1(\boldsymbol{n'} + \boldsymbol{s}'_i)$. Since $\boldsymbol{n'} + S'$ is polychromatic under χ_1, $\boldsymbol{n} + S$ is polychromatic under χ_2, and $p_{\mathbb{Z}^d}(S) \geq p_{\mathbb{Z}^{d-1}}(S')$. ■

Proposition 16 *Let $d \geq 2$. For any $S \subseteq \mathbb{Z}^d$, there is a projection $S' \subseteq \mathbb{Z}^{d-1}$ where $|S| = |S'|$.*

Proof: Let $S = \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_k\} \subseteq \mathbb{Z}^d$ and suppose $\boldsymbol{w} = (w_1, \ldots, w_{d-1}, 1) \in \mathbb{Z}^d$. For $1 \leq i \leq k$ let $\boldsymbol{s}'_i = f_{\boldsymbol{w}}(\boldsymbol{s}_i)$. For $1 \leq i \leq k$, let s_{id} denote the last coordinate of $\boldsymbol{s}_i$ and note that if $i \neq j$, $\boldsymbol{s}'_i = \boldsymbol{s}'_j$ if and only if

$$\boldsymbol{w} = \frac{1}{s_{id} - s_{jd}}(\boldsymbol{s}_i - \boldsymbol{s}_j).$$

In other words $\boldsymbol{s}'_i = \boldsymbol{s}'_j$ if and only if $\boldsymbol{w}$ is parallel to $\boldsymbol{s}_i - \boldsymbol{s}_j$. Since the number of differences $\boldsymbol{s}_i - \boldsymbol{s}_j$ is finite, we can choose $\boldsymbol{w}$ so that it is not parallel to any of these. For this choice of $\boldsymbol{w}$, for all $1 \leq i \neq j \leq k$, $\boldsymbol{s}'_i \neq \boldsymbol{s}'_j$ Then $S' = \{\boldsymbol{s}'_1, \ldots, \boldsymbol{s}'_k\}$, is a projection S with $|S'| = |S|$. ■

Theorem 17 *Fix $d \geq 2$. For a finite set $S \subseteq \mathbb{Z}^d$, $p(S) \geq \frac{(1+o(1))|S|}{\ln |S|}$.*

Proof: Given $S \subseteq \mathbb{Z}^d$, Proposition 16 implies we can project $d-1$ times to ultimately obtain a set $S' \subseteq \mathbb{Z}$, with $|S'| = |S|$. Theorem 2, along with repeated application of Lemma 15 implies $p_{\mathbb{Z}^d}(S) \geq p_{\mathbb{Z}}(S') \geq \frac{(1+o(1))|S'|}{\ln |S'|} = \frac{(1+o(1))|S|}{\ln |S|}$. ■

Theorem 18 *Let $d \geq 2$. Let $S = \{\boldsymbol{s}_0, \boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_k\}$ be a set of $k+1$ collinear points in $\mathbb{Z}^d$ where for each i, $\boldsymbol{s}_i = (l_i a_1, l_i a_2, \ldots, l_i a_d)$, where $0 = l_0 < l_1 < l_2 < \cdots < l_k$, $a_i \in \mathbb{Z}$, $a_1 > 0$, and $\gcd(a_1, a_2, \ldots, a_d) = 1$. Let $S' = \{0, l_1, l_2, \ldots, l_k\} \subseteq \mathbb{Z}$. Then $p_{\mathbb{Z}^d}(S) = p_{\mathbb{Z}}(S')$.*

Proof: Let $S'' = \{0, l_1 a_1, l_2 a_1, \ldots, l_k a_1\} \subseteq \mathbb{Z}$. By an argument identical to Lemma 9, Part (i), since S'' is a dilation of S', $p_{\mathbb{Z}}(S') = p_{\mathbb{Z}}(S'')$. Since S'' can be obtained from S by a sequence of $d-1$ projections, Lemma 15 implies $p_{\mathbb{Z}^d}(S) \geq p_{\mathbb{Z}}(S'') = p_{\mathbb{Z}}(S')$. For the other direction, suppose χ_2 is an S-polychromatic r-coloring of $\mathbb{Z}^d$. Let χ_1 be the r-coloring of $\mathbb{Z}$ where for all $n \in \mathbb{Z}$, $\chi_1(n) = \chi_2(n(a_1, a_2, \ldots, a_d))$. Let $n \in \mathbb{Z}$ and $\boldsymbol{n'} = n(a_1, \ldots, a_d) \in \mathbb{Z}^d$. Then for all i, $\chi_1(n + l_i) = \chi_2(\boldsymbol{n'} + \boldsymbol{s}_i)$, and since χ_2 is S-polychromatic, χ_1 is S'-polychromatic and $p_{\mathbb{Z}^d}(S) \leq p_{\mathbb{Z}}(S')$. ■

Now we return to the subject to tilings. Lemma 19 and Theorems 20, 21, and 22 are well-known in the field of discrete geometry (see e.g. Section III of [14]) as simple examples of "splitting" groups. We restate them here using the language of polychromatic colorings.

Lemma 19 *If a set $S \subseteq G$ tiles a nontrivial subgroup H of G, then S tiles G.*

Proof: Suppose $S \oplus T = H$. Let V be a set containing of one element from each coset of H. Then by properties of cosets, $H \oplus V = G$. In other words, for any $n \in G$, there is a unique $h \in H$, $v \in V$ such that $n = h + v$. Further, there is a unique $s \in S$, $t \in T$ such that $h = s + t$. Thus $n = (s+t)+v = s+(t+v)$ and $S + (T+V) = G$. To show uniqueness, suppose $n = s' + (t' + v')$ where $s \neq s'$. Then $(s+t)+v = (s'+t')+v'$ which implies $h + v = h' + v'$ for some $h, h' \in H$. Since $H \oplus V = G$, this implies $v = v'$, and so $s + t = s' + t'$. Since $S \oplus T = H$, $s = s'$ and $t = t'$. Thus $S \oplus (T+V) = G$. ■

For any $d \geq 1$, let $\mathbf{0}$ denote the element $(0, 0, \ldots, 0) \in \mathbb{Z}^d$ and let $\boldsymbol{e}_i$ denote the element $(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^d$ with all 0's except for a 1 in the ith position. For $\boldsymbol{s} = (v_1, \ldots, v_d) \in \mathbb{Z}^d$, let $-\boldsymbol{s} = (-v_1, \ldots, -v_d)$. Define the d-semicross $SC_d = \{\mathbf{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\}$ and the d-cross $C_d = \{\mathbf{0}, \boldsymbol{e}_1, -\boldsymbol{e}_1, \boldsymbol{e}_2, -\boldsymbol{e}_2, \ldots, \boldsymbol{e}_d, -\boldsymbol{e}_d\}$. Theorem 11 implies that any finite set $S \subseteq G$ with $p(S) = |S|$ tiles G, and we use this insight to show that these sets tile $\mathbb{Z}^d$.

Theorem 20 *For all $d \geq 1$, the d-semicross $SC_d = \{\mathbf{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\}$ tiles $\mathbb{Z}^d$.*

Proof: Consider the coloring $\chi : \mathbb{Z}^d \to [d+1]$ where $\chi(v_1, \ldots, v_d) = v_1 + 2v_2 + 3v_3 + \cdots + dv_d \pmod{d+1}$. On any translate $\boldsymbol{n} + SC_d \in \mathbb{Z}^d$, the colors $\chi(\boldsymbol{n} + \mathbf{0}), \chi(\boldsymbol{n} + \boldsymbol{e}_1), \chi(\boldsymbol{n} + \boldsymbol{e}_2), \ldots, \chi(\boldsymbol{n} + \boldsymbol{e}_d)$ are $\chi(\boldsymbol{n}), \chi(\boldsymbol{n}) + 1, \chi(\boldsymbol{n}) + 2, \ldots, \chi(\boldsymbol{n}) + d \pmod{d+1}$. They are all different, so χ is SC_d-polychromatic with $|SC_d| = d + 1$ colors. By Theorem 11, SC_d tiles $\mathbb{Z}^d$. ■

Theorem 21 *For all $d \geq 1$, the d-cross $C_d = \{\mathbf{0}, \boldsymbol{e}_1, -\boldsymbol{e}_1, \boldsymbol{e}_2, -\boldsymbol{e}_2, \ldots, \boldsymbol{e}_d, -\boldsymbol{e}_d\}$ tiles $\mathbb{Z}^d$.*

Proof: The $(2d+1)$-coloring $\chi : \mathbb{Z}^d \to [2d+1]$ where $\chi(v_1, \ldots, v_d) = v_1 + 2v_2 + 3v_3 + \cdots + dv_d \pmod{2d+1}$ is C_d-polychromatic: On any translate $\boldsymbol{n} + C_d \in \mathbb{Z}^d$, the colors $\chi(\boldsymbol{n}+\mathbf{0}), \chi(\boldsymbol{n}+\boldsymbol{e}_1), \chi(\boldsymbol{n}-\boldsymbol{e}_1), \chi(\boldsymbol{n}+\boldsymbol{e}_2), \chi(\boldsymbol{n}-\boldsymbol{e}_2), \dots, \chi(\boldsymbol{n}+\boldsymbol{e}_d), \chi(\boldsymbol{n}-\boldsymbol{e}_d)$ are $\chi(\boldsymbol{n}), \chi(\boldsymbol{n})+1, \chi(\boldsymbol{n})-1, \chi(\boldsymbol{n})+2, \chi(\boldsymbol{n})-2, \dots, \chi(\boldsymbol{n})+d, \chi(\boldsymbol{n})-d \pmod{2d+1}$. ■

Theorem 22 *Let $d \geq 2$. Let $S \subseteq \mathbb{Z}^d$ be a set that contains $\mathbf{0}$ and $j \leq d$ other elements $\boldsymbol{s}_1, \dots, \boldsymbol{s}_j$, where no nontrivial integer linear combination of $\{\boldsymbol{s}_1, \dots, \boldsymbol{s}_j\}$ is $\mathbf{0}$. Then S tiles $\mathbb{Z}^d$.*

Proof: Let $H \subseteq \mathbb{Z}^d$ be the set of all integer linear combinations of $\{\boldsymbol{s}_1, \dots, \boldsymbol{s}_j\}$. By Theorem 20, there is a set $T \subseteq \mathbb{Z}^j$ such that $\{\mathbf{0}, \boldsymbol{e}_1, \dots, \boldsymbol{e}_j\} \oplus T = \mathbb{Z}^j$. Let $M : \mathbb{Z}^j \to \mathbb{Z}^d$ be the unique linear transformation which maps $\boldsymbol{e}_i$ to $\boldsymbol{s}_i$ for each $i \leq j$. Then $\{\mathbf{0}, \boldsymbol{s}_1, \dots, \boldsymbol{s}_j\}$ tiles H with complement set $\{M(t) : t \in T\}$. Since H is a subgroup of $\mathbb{Z}^d$, by Proposition 19, S tiles $\mathbb{Z}^d$. ■

We can now determine the polychromatic number of any set of cardinality 3 or 4 in $\mathbb{Z}^d$, $d \geq 2$.

Theorem 23 *Let $d \geq 2$ and suppose $S \subseteq \mathbb{Z}^d$ has cardinality 3. Then $p_{\mathbb{Z}^d}(S) = 3$ if the three points are in general position or if they are collinear and some projection $S' \subseteq \mathbb{Z}$ of S has $p_{\mathbb{Z}}(S') = 3$. Otherwise $p_{\mathbb{Z}^d}(S) = 2$.*

Proof: Theorem 22 implies that if $d \geq 2$ and $S \subseteq \mathbb{Z}^d$ consists of three points in general position, then S tiles $\mathbb{Z}^d$, and thus $p(S) = 3$. If $S \subseteq \mathbb{Z}^d$ has three collinear points, then Theorem 18 implies the problem is equivalent to finding the polychromatic number of a set of three integers, which is either 2 or 3 and can be determined using Theorem 4. ■

Theorem 24 *Let $d \geq 2$ and suppose $S \subseteq \mathbb{Z}^d$ has cardinality 4. Then*

- *If all points of S are collinear, $p_{\mathbb{Z}^d}(S)$ is 3 or 4.*
- *If exactly three points of S are collinear, $p_{\mathbb{Z}^d}(S) = 4$.*
- *If $d \geq 3$ and S has four points in general position, $p_{\mathbb{Z}^d}(S) = 4$.*
- *If $d = 2$ and S has four points in general position, $p_{\mathbb{Z}^2}(S)$ is 3 or 4.*

Proof: For $d \geq 2$ and a set $S \subseteq \mathbb{Z}^d$ with $|S| = 4$, Proposition 16 implies that there is a set $S' \subseteq \mathbb{Z}$ where $|S'| = 4$ and S' is a projection of S. Thus Theorem 1 and Lemma 15 imply that $p(S) \geq 3$. Determining whether $p(S)$ is 3 or 4 is equivalent to determining whether S tiles $\mathbb{Z}^d$. As with the $|S| = 3$ case, we can examine cases depending on how many points of S are collinear.

If the four points of S are in general position, then if none is a nontrivial integer linear combination of the others, $p(S) = 4$ by Theorem 22. Otherwise, we can assume $S \subseteq \mathbb{Z}^2$. In this case, $p(S)$ can be 3, for example if $S = \{(0,0), (1,0), (0,1), (1,2)\} \subseteq \mathbb{Z}^2$. It can also be 4, for example if $S = \{(0,0), (1,0), (0,1), (1,1)\} \subseteq \mathbb{Z}^2$. Szegedy [15] gave an algorithm to determine if a set of cardinality 4 tiles $\mathbb{Z}^2$.

If the four points of S are all collinear, then $p(S)$ is determined by applying Theorems 18 and 4.

If exactly three of the four points are collinear, then without loss of generality, assume that the three collinear points are $\{\mathbf{0}, (a, 0, \ldots, 0), (b, 0, \ldots 0)\}$, where a and b do not have the same parity. Since the fourth point $\boldsymbol{s}$ can be projected anywhere onto the line, by Proposition 15 it suffices to show that there exists $c \in \mathbb{Z}$ such that $p_{\mathbb{Z}}(\{0, a, b, c\}) = 4$. By Theorem 4, the value $c = a + b$ is has this property. ∎

The fact that $p_{\mathbb{Z}^d}(S) = 4$ if S contains exactly three collinear points implies that for any set S of three integers, there is a 4-coloring of $\mathbb{Z}$ so that every translate of S gets three different colors. Here is an explicit example of one such coloring. Without loss of generality we need only consider sets of the following form: Let $S = \{0, a, b\} \subseteq \mathbb{Z}$ where a and b are positive with a even and b odd (note that we do not specify which is larger). Define the *alternating block 4-coloring relative to* S as follows: Given any $m \in \mathbb{Z}$, let q_m and r_m be the unique integers such that $m = 2aq_m + r_m$, where $-a \leq r_m < a$. Let $X(m) = 0$ if $r_m \geq 0$, $X(m) = 1$ otherwise. Let $Y(m) = 0$ if m is even, $Y(m) = 1$ otherwise. Define χ, the alternating block 4-coloring relative to S, so that $\chi(m) = (X(m), Y(m))$.

Theorem 25 *Let $S = \{0, a, b\} \subseteq \mathbb{Z}$ with $a, b > 0$, a even, and b odd. If the integers are colored with the alternating block 4-coloring relative to S then every translate of S has elements of three different colors.*

Proof: For any translate $n + S = \{n, n + a, n + b\}$ of S, $X(n) \neq X(n + a)$, while $Y(n) = Y(n + a) \neq Y(n + b)$. Thus χ has the property that any translate of S contains elements with three different colors. ∎

Given a set of three integers, the alternating block 4-coloring shows that there is a 4-coloring of the integers so that every translate gets three different colors. If $S \subset \mathbb{Z}$, $|S| = 4$, is there a 5-coloring of $\mathbb{Z}$ so that every translate of S has 4 colors? More generally, we ask the following question.

Question 26 *Let $d \geq 1$. Given $k, n \in \mathbb{Z}$ with $k \leq n$, let $p(n, k)$ denote the minimum r so that any $S \subseteq \mathbb{Z}$ with $|S| = n$ has an r-coloring where every translate of S gets at least k colors. What is an asymptotic upper bound on $p(n, k(n))$ for natural choices of $k(n)$?*

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