Q_2-free families in the Boolean lattice

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Abstract

For a family $\mathcal{F}$ of subsets of $[n] = \{1, 2, \ldots, n\}$ ordered by inclusion, and a partially ordered set P, we say that $\mathcal{F}$ is P-free if it does not contain a subposet isomorphic to P. Let $ex(n, P)$ be the largest size of a P-free family of subsets of $[n]$. Let Q_2 be the poset with distinct elements a, b, c, d, $a < b, c < d$; i.e., the 2-dimensional Boolean lattice. We show that $2N - o(N) \leq ex(n, Q_2) \leq 2.283261N + o(N)$, where $N = \binom{n}{\lfloor n/2 \rfloor}$. We also prove that the largest Q_2-free family of subsets of $[n]$ having at most three different sizes has at most $2.20711N$ members.

1 Introduction

Let Q_n be the n-dimensional Boolean lattice corresponding to subsets of an n-element set ordered by inclusion. A poset $P = (X, \leq)$ is a subposet of $Q = (Y, \leq')$ if there is an injective map $f : X \to Y$ such that for $x_1, x_2 \in X$, $x_1 \leq x_2$ implies $f(x_1) \leq' f(x_2)$. For a poset P, we say that a set of elements $\mathcal{F} \subseteq 2^{[n]}$ is P-free if $(\mathcal{F}, \subseteq)$ does not contain P as a subposet. Let $ex(n, P)$ be the size of the largest P-free family of subsets of $[n]$. We say that the set of all i-element subsets of $[n]$, $\binom{[n]}{i}$, is the ith layer of Q_n. Finally, let $N(n) = N = \binom{n}{\lfloor n/2 \rfloor}$; i.e., N is the size of a middle layer of a Boolean lattice.

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The classical theorem of Sperner [12] says that $ex(n, Q_1) = N$. Most asymptotic bounds for $ex(n, P)$ are expressed in terms of N. Many largest P-free families are simply unions of largest layers in Q_n. For example, Erdős generalized Sperner's result in [5], showing that the size of the largest subposet of Q_n which does not contain a chain with k elements, C_k, is equal to the number of elements in the $k-1$ largest layers of Q_n; i.e., for a fixed k, $ex(n, C_k) = (k-1)N + o(N)$. De Bonis, Katona and Swanepoel showed in [4] that $ex(n, \bowtie) = 2N + o(N)$, where $\bowtie$ is a subposet of Q_n consisting of distinct sets a, b, c, d such that $a, b \subset c, d$. De Bonis and Katona, as well as Thanh showed in [3], [13] that $ex(n, V_{r+1}) = N + o(N)$, where V_{r+1} is a subposet of Q_n with distinct elements f, g_i, $i = 1, \ldots, r$, $f \subset g_i$ for $i = 1, \ldots, r$. More generally, for a poset $K_{s,t}$, with distinct elements $f_1, \ldots, f_s \subset g_1, \ldots, g_t$, and a poset $P_k(s)$, with distinct elements $f_1 \subset \cdots \subset f_k \subset g_1, g_2, \ldots, g_s$, Katona and Tarjan [9] and later De Bonis and Katona [3] proved that $ex(n, K_{s,t}) = 2N + o(N)$ and $ex(n, P_k(s)) = kN + o(N)$, respectively. Griggs and Katona proved in [7] that $ex(n, \mathsf{N}) = N + o(N)$, for a poset N with distinct elements a, b, c, d, such that $a \subset c, d$, and $b \subset c$. Griggs and Lu [8] proved that $ex(n, P_k(s,t)) = (k-1)N + o(N)$, where $P_k(s,t)$ is a poset with distinct elements $f_1, f_2 \ldots, f_s \subset g_2 \subset g_3 \subset \cdots \subset g_{k-1} \subset h_1, \ldots, h_t$, $k \geq 3$. They also showed that $ex(n, O_{4k}) = N + o(N)$, $ex(n, O_{4k-2}) \leq (1 + \sqrt{2}/2)N + o(N)$, where O_i is a poset of height two which is a cycle of length i as an undirected graph. More generally, they proved that if $G = (V, E)$ is a graph and P is a poset with elements $V \cup E$, with $v < e$ if $v \in V$, $e \in E$ and v incident to e, then $ex(n, P) \leq (1 - \sqrt{1 - 1/(\chi(G) - 1)})N + o(N)$. Bukh [2] proved that $ex(n, T) = kN + o(N)$, where T is a poset whose diagram is a tree and k is an integer which is one less than the height of T. As a general reference in poset theory, see [14].

The smallest poset, P, for which $ex(n, P)$ is not known to be an integer multiple of N, is $P = Q_2$. This manuscript is devoted to this little poset for which we still do not know whether $ex(n, Q_2) = kN + o(N)$ for an integer k. We show that $2N - o(N) \leq ex(n, Q_2) \leq 2.283261N + o(N)$. We believe that $ex(n, Q_2) = 2N + o(N)$. Next, are our main results.

Theorem 1 *If $\mathcal{F} \subset Q_n$ is Q_2-free, then $2N - o(N) \leq |\mathcal{F}| \leq 2.283261N + o(N)$.*

Theorem 2 *Let $\mathcal{F} \subset Q_n$ be a Q_2-free family, $\mathcal{F} = \mathcal{S} \cup \mathcal{T} \cup \mathcal{U}$, where $\mathcal{S}$ is a subset of minimal elements of $\mathcal{F}$, $\mathcal{U}$ is a subset of maximal elements of $\mathcal{F}$ and $\mathcal{T} = \mathcal{F} \setminus (\mathcal{S} \cup \mathcal{U})$ such that for any $T \in \mathcal{T}$, $S \in \mathcal{S}$, $U \in \mathcal{U}$, $|T| = k$, $|U| > k$, $|S| < k$. Then $|\mathcal{F}| \leq N(3 + \sqrt{2})/2 + o(N) \leq 2.20711N + o(N)$. In particular, if $\mathcal{F}$ is a Q_2-free subset of three layers of Q_n, then $|\mathcal{F}| \leq 2.20711N + o(N)$.*

We prove the main theorems in Sections 2 and 3, prove supporting lemmas in Section 4.

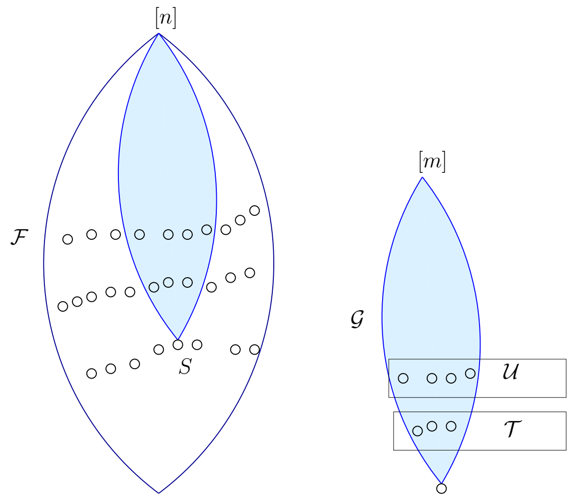

Figure 1: Local argument

2 Proof of Theorem 1

Sketch of the proof. We consider a Q_2-free family, $\mathcal{F}$, of subsets of $[n]$. Using a standard argument, we assume that all members of $\mathcal{F}$ have size between $n/2 - n^{2/3}$ and $n/2 + n^{2/3}$. We bound $\mathcal{F}$ in terms of the number of full chains containing exactly 3 sets or exactly 1 set of $\mathcal{F}$. In doing this, we introduce an auxiliary graph corresponding to 2-element subsets in local sub-lattices, express the number of chains in terms of the size of that graph, and optimize the resulting expression. This produces the upper bound in the statement of the theorem. The lower bound is achieved by $\mathcal{F} = \binom{[n]}{\lfloor n/2 \rfloor} \cup \binom{[n]}{\lfloor n/2 \rfloor + 1}$.

Let us now begin the proof in full. Let $\mathcal{F}$ be a Q_2-free family of subsets of $[n]$, let $\mathcal{S}$ be the set of minimal elements of $\mathcal{F}$.

Lemma 1 $\sum_{|k-n/2| \geq n^{2/3}} \binom{n}{k} \leq 2^{n-\Omega(n^{1/3})} = 2^{-\Omega(n^{1/3})} N$.

Proof of Lemma 1. We note that the expression $2^{-n} \sum_{|k-n/2| \geq n^{2/3}} \binom{n}{k}$ computes the probability that a $B(n, 1/2)$ binomial random variable, X, takes on values outside of the interval $\left(n/2 - n^{2/3}, n/2 + n^{2/3}\right)$. Using a standard Chernoff bound, $\Pr\left(|X - n/2| \geq \delta(n/2)\right) \leq 2 \exp\left\{-(n/2)\delta^2/2\right\}$. Observing that the left-hand side sums $\binom{n}{k}$ over all k for which $|k - n/2| \geq \delta(n/2)$ and setting $\delta = 2n^{-1/3}$, we can conclude:

$$\sum_{|k-n/2|\geq n^{2/3}} \binom{n}{k} \leq 2^{n+1} e^{-n^{1/3}}.$$

Since $\binom{n}{n/2} = \Omega(n^{-1/2})2^n$, we may conclude that $\sum_{|k-n/2|\geq n^{2/3}} \binom{n}{k} \leq 2^{-\Omega(n^{1/3})}\binom{n}{n/2}$. Note that, for every C there exists a c such that $\sum_{|k-n/2|\geq cn^{1/2}\ln n} \binom{n}{k} \leq n^{-C}\binom{n}{n/2}$. So, we could in fact have chosen a more precise $\Omega(n^{1/2}\ln n)$ as our error term, rather than the more convenient $n^{2/3}$. □

As a result of Lemma 1, we can assume that all elements in $\mathcal{F}$ are close to the middle layer. A full chain in Q_n is a chain containing $n+1$ sets. For $i = 1, 2, 3$ and a set $F \in \mathcal{F}$, let $\Upsilon_n^i(F, \mathcal{F})$ denote the set of full chains in Q_n that contain F and exactly $i-1$ other members of $\mathcal{F}$. Let $\Upsilon_n^i(\mathcal{F})$ be the set of full chains in Q_n that contain exactly i members of $\mathcal{F}$.

Lemma 2 $|\mathcal{F}| \leq \left(2 + \frac{1}{n!}\left(|\Upsilon_n^3(\mathcal{F})| - |\Upsilon_n^1(\mathcal{F})|\right)\right) N \leq \left(2 + \frac{1}{n!}\sum_{S\in\mathcal{S}}\left(|\Upsilon_n^3(S,\mathcal{F})| - |\Upsilon_n^1(S,\mathcal{F})|\right)\right) N.$

Proof of Lemma 2. Let Υ be the set of all full chains in Q_n. Let $\mathcal{X} = \{(F,\sigma) : F \in \mathcal{F}, \sigma \in \Upsilon, F \in \sigma\}$. Since each $\sigma \in \Upsilon$ contains at most 3 sets from $\mathcal{F}$, we have that

$$|\mathcal{X}| = 3|\Upsilon_n^3(\mathcal{F})| + 2|\Upsilon_n^2(\mathcal{F})| + |\Upsilon_n^1(\mathcal{F})|.$$

On the other hand, any $F \in \mathcal{F}$ is contained in $|F|!(n-|F|)! \geq \lfloor n/2\rfloor!\lceil n/2\rceil!$ full chains from Υ. Thus $|\mathcal{F}|\lfloor n/2\rfloor!\lceil n/2\rceil! \leq |\mathcal{X}| = 3|\Upsilon_n^3(\mathcal{F})| + 2|\Upsilon_n^2(\mathcal{F})| + |\Upsilon_n^1(\mathcal{F})|$.

Since the terms $|\Upsilon_n^i(\mathcal{F})|$ sum to $n!$, $|\mathcal{X}| = 2n! + |\Upsilon_n^3(\mathcal{F})| - |\Upsilon_n^1(\mathcal{F})|$. Thus,

$$\begin{aligned}
\lfloor n/2\rfloor!\lceil n/2\rceil!|\mathcal{F}| &\leq 2n! + |\Upsilon_n^3(\mathcal{F})| - |\Upsilon_n^1(\mathcal{F})|,\\
|\mathcal{F}| &\leq 2N + \frac{1}{\lfloor n/2\rfloor!\lceil n/2\rceil!}(|\Upsilon_n^3(\mathcal{F})| - |\Upsilon_n^1(\mathcal{F})|) = \left(2 + \frac{1}{n!}\left(|\Upsilon_n^3(\mathcal{F})| - |\Upsilon_n^1(\mathcal{F})|\right)\right) N.
\end{aligned}$$

The second inequality in the lemma follows from the fact that every member of $\Upsilon_n^3(\mathcal{F})$ contains a member of $\mathcal{S}$. □

Fix $S \in \mathcal{S}$. We shall bound $\left(|\Upsilon_n^3(S,\mathcal{F})| - |\Upsilon_n^1(S,\mathcal{F})|\right)$. Let $\mathcal{G} = \mathcal{G}(S) = \{F \setminus S : F \in \mathcal{F}, S \subseteq F\}$. We see that $\mathcal{G}$ is a system of subsets of an m-element set, where $m = n - |S|$, see Figure 1. Moreover, $\emptyset \in \mathcal{G}$, and since $\mathcal{F}$ is Q_2-free, for any $X \in \mathcal{G}$, there is at most one set $Y \in \mathcal{G} \setminus \emptyset$, such that $Y \subseteq X$. We see also that $|\Upsilon_n^i(S,\mathcal{F})| = |S|!|\Upsilon_m^i(\emptyset,\mathcal{G})|$.

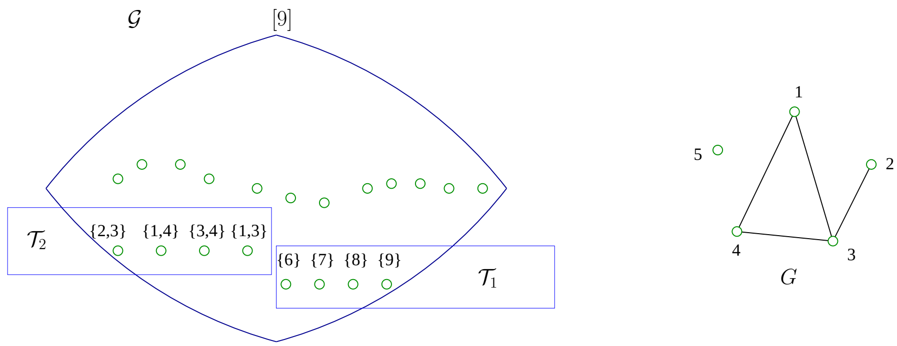

Figure 2: Family $\mathcal{G}$ and graph G, $m = 9$, $\eta = 5$.

Let $\mathcal{T}$ be the set of minimal elements in $\mathcal{G} - \{\emptyset\}$ and $\mathcal{U} = \mathcal{G} - (\mathcal{T} \cup \{\emptyset\})$. Let $\mathcal{T}_i = \{T \in \mathcal{T} : |T| = i\}$, $i = 1, 2, 3, \ldots$. Without loss of generality let $\mathcal{T}_1 = \{\{\eta+1\}, \{\eta+2\}, \ldots, \{m\}\}$, as a result $\mathcal{T}_2$ is a set of some two-element subsets of $[\eta]$. We create an auxiliary graph G corresponding to $\mathcal{G}$ by letting the vertex set be $[\eta]$ and the edge set be $\mathcal{T}_2$. See Figure 2 for illustration. Let $e, \overline{e}$ be the number of edges and nonedges in G, respectively. Let $\Upsilon_i = \Upsilon^i_m(\emptyset, \mathcal{G})$, $i = 1, 3$.

We shall then express the bounds on $|\Upsilon_1|$ and $|\Upsilon_3|$ in terms of proportions $a = \eta/m$ and $b = \frac{e}{\binom{\eta}{2}}$. Note that $0 \le a, b \le 1$. Finally, set

$$\mu = \begin{cases} 1, & a < 1/2 \\ \frac{1-a}{a}, & a \ge 1/2. \end{cases}$$

Next, we state the technical lemmas which are proved in Section 4.

Lemma 3 $|\Upsilon_1| \ge m! \left[b(a^3 - a^2)\mu + (a^2 - a^3)\mu + O(m^{-1}) \right]$.

Lemma 4 $|\Upsilon_3| \le m! \left[b^2 \left(a^4/2 - a^3 \right) + b \left(a^3 - 3a^4/4 \right) + \left(a^4/4 - a^2 + a \right) + O(m^{-1}) \right]$.

With Lemmas 3 and 4,

$$|\Upsilon_3| - |\Upsilon_1| \le m! \left[b^2 \left(a^4/2 - a^3 \right) + b \left(a^3 - 3a^4/4 - a^3\mu + a^2\mu \right) + \left(a^4/4 - a^2 + a - a^2\mu + a^3\mu \right) + O(m^{-1}) \right].$$

Lemma 5 *With $0 \le a, b \le 1$ and $\mu = \mu(a)$ as defined above,*

$$b^2 \left(a^4/2 - a^3 \right) + b \left(a^3 - 3a^4/4 - a^3\mu + a^2\mu \right) + \left(a^4/4 - a^2 + a - a^2\mu + a^3\mu \right) \le 0.283261.$$

Using Lemma 5,

$$|\Upsilon^3_m(\emptyset,\mathcal{G})| - |\Upsilon^1_m(\emptyset,\mathcal{G})| \le \left[0.283261 + O(m^{-1})\right] m!. \tag{1}$$

For a final calculation, we need the so-called LYM inequality, proven by Yamamoto [15], Bollobás [1], Lubell [10], and Meshalkin [11].

Lemma 6 (LYM inequality) *If $\mathcal{A}$ is an antichain in Q_n, then* $\sum_{A\in\mathcal{A}} \binom{n}{|A|}^{-1} \le 1$.

Returning to Lemma 2, we have

$$|\mathcal{F}| \le N\left(2 + \frac{1}{n!}\sum_{S\in\mathcal{S}}\left(|\Upsilon^3_n(S,\mathcal{F})| - |\Upsilon^1_n(S,\mathcal{F})|\right)\right).$$

Using inequality (1), we have

$$|\mathcal{F}| \le N\left(2 + \sum_{S\in\mathcal{S}}\frac{1}{n!}|S|!\left[0.283261 + O((n-|S|)^{-1}))\right](n-|S|)!\right).$$

LYM and the fact that that $(n-|S|)^{-1} \le (n/2 - n^{2/3})^{-1}$ give

$$|\mathcal{F}| \le N\left(2.283261 + O((n/2-n^{2/3})^{-1})\right) = 2.283261N + o(N).$$

This concludes the proof of the main theorem.

3 Proof of Theorem 2

For ease of notation, in this proof let $N' = \binom{n}{k}$. Suppose $\mathcal{F}$ is a Q_2-free family from 3 layers, L_1, L_2, L_3, of the Boolean lattice Q_n, where $L_1 = \binom{[n]}{k-1}$, $L_2 = \binom{[n]}{k}$, and $L_3 = \binom{[n]}{k+1}$. Let $\mathcal{S} = \mathcal{F}\cap L_1$, $\mathcal{T} = \mathcal{F}\cap L_2$, and $\mathcal{U} = \mathcal{F}\cap L_3$. We may assume that $|k-n/2| < n^{2/3}$ as a result of Lemma 1. Furthermore, it will be useful to assume that $|\mathcal{S}|, |\mathcal{U}| \le N'$; otherwise we could delete at most $\binom{n}{k-1} - N' = O(n^{-1/3})N' = o(N')$ members of $\mathcal{S}$ and $O(n^{-1/3})N' = o(N')$ members of $\mathcal{U}$ to ensure that the resulting sets are at most N'. Let Υ be the set of 3-element chains contained in $L_1\cup L_2\cup L_3$, and $\Upsilon_i = \{\sigma\in\Upsilon : |\sigma\cap\mathcal{F}| = i\}$, $i = 0,1,2,3$.

We count ordered pairs, one element is a member of $\mathcal{F}$ and the other is a chain from Υ. That is, $\mathcal{X} := \{(F,\sigma) : F\in\mathcal{F}, \sigma\in\Upsilon, F\in\sigma\}$. Then

$$|\mathcal{X}| = 3|\Upsilon_3| + 2|\Upsilon_2| + |\Upsilon_1| = 2|\Upsilon| + |\Upsilon_3| - |\Upsilon_1| - 2|\Upsilon_0| \le 2|\Upsilon| + |\Upsilon_3| - |\Upsilon_1|.$$

On the other hand,

$$|\mathcal{X}| = (k+1)k|\mathcal{U}| + k(n-k)|\mathcal{T}| + (n-k+1)(n-k)|\mathcal{S}|.$$

Putting together these expressions for $|\mathcal{X}|$ and using the fact that $|\Upsilon| = N'k(n-k)$, we have

$$(k+1)k|\mathcal{U}| + k(n-k)|\mathcal{T}| + (n-k+1)(n-k)|\mathcal{S}| \leq 2N'k(n-k) + |\Upsilon_3| - |\Upsilon_1|. \tag{2}$$

For $X \in L_1$, $Y \in L_2$, and $Z \in L_3$, define

$$f(X) = |\{T \in \mathcal{T} : X \subset T\}|; \quad g(Z) = |\{T \in \mathcal{T} : Z \supset T\}|;$$
$$\breve{f}(Y) = |\{S \in \mathcal{S} : S \subset Y\}|; \quad \breve{g}(Y) = |\{U \in \mathcal{U} : U \supset Y\}|.$$

Note that $\sum_{X\in\mathcal{S}} f(X) = \sum_{Y\in\mathcal{T}} \breve{f}(Y)$ and $\sum_{Z\in\mathcal{U}} g(Z) = \sum_{Y\in\mathcal{T}} \breve{g}(Y)$.

We shall bound $|\Upsilon_3| - |\Upsilon_1|$ by counting the chains that contain an element of $\mathcal{T}$, $\mathcal{S}$ and $\mathcal{U}$, then chains containing an element of $\mathcal{T}$, $L_1 \setminus \mathcal{S}$, $L_3 \setminus \mathcal{U}$.

$$\begin{aligned} |\Upsilon_3| - |\Upsilon_1| &\leq \sum_{Y\in\mathcal{T}} \left[\breve{f}(Y)\breve{g}(Y) - \left(k - \breve{f}(Y)\right)(n-k-\breve{g}(Y))\right] \\ &= (n-k)\sum_{X\in\mathcal{S}} f(X) + k\sum_{Z\in\mathcal{U}} g(Z) - |\mathcal{T}|k(n-k). \end{aligned} \tag{3}$$

Now, we shall find a bound on $\sum f$ and $\sum g$ in terms of $|\mathcal{S}|$ and $|\mathcal{U}|$. Recall that we were able to assume that $|\mathcal{S}|, |\mathcal{U}| \leq N'$.

Lemma 7 *If $N' = \binom{n}{k}$ and $|k - n/2| = O(n^{2/3})$, then*

$$\sum_{X\in\mathcal{S}} f(X) \leq (k+1)\sqrt{|\mathcal{S}|\,(N' - |\mathcal{U}|)} + O(n^{5/6})N',$$
$$\sum_{Z\in\mathcal{U}} g(Z) \leq (n-k+1)\sqrt{|\mathcal{U}|\,(N' - |\mathcal{S}|)} + O(n^{5/6})N'.$$

Proof of Lemma 7. Consider any $X \in L_1$. One can associate members of L_2 lying above X with the elements of $[n] - X$ that they contain. Furthermore, one can associate members of L_3 lying above X with the pairs of elements of $[n] - X$ that they contain. So, for any $X \in L_1$, then $|\{U \in \mathcal{U} : U \supset X\}| \leq \binom{n-k+1}{2}$. But if $X \in \mathcal{S}$, there are $\binom{f(X)}{2}$ members of L_3 that cannot be above X. Hence, for each $X \in \mathcal{S}$,

$$|\{U \in \mathcal{U} : U \supset X\}| \leq \binom{n-k+1}{2} - \binom{f(X)}{2}.$$

Symmetrically, for any $Z \in L_3$, $|\{S \in \mathcal{S} : S \subset Z\}| \leq \binom{k+1}{2}$ but if $Z \in \mathcal{U}$, then

$$|\{S \in \mathcal{S} : S \subset Z\}| \leq \binom{k+1}{2} - \binom{g(Z)}{2}.$$

Now we double-count the pairs (X, U) such that $X \in L_1$, $U \in \mathcal{U}$ and $X \subset U$:

$$\binom{k+1}{2}|\mathcal{U}| = \sum_{X \in L_1} |\{U \in \mathcal{U} : U \supset X\}|.$$

We can partition the members of $X \in L_1$ according to whether or not $X \in \mathcal{S}$ and use the estimates above. To wit,

$$\begin{aligned} |\mathcal{U}| &\leq \binom{k+1}{2}^{-1} \left(\sum_{X \in \mathcal{S}} \left(\binom{n-k+1}{2} - \binom{f(X)}{2} \right) + \sum_{X \in L_1 - \mathcal{S}} \binom{n-k+1}{2} \right) \\ &= \binom{k+1}{2}^{-1} \left(|L_1| \binom{n-k+1}{2} - \sum_{X \in \mathcal{S}} \binom{f(X)}{2} \right). \end{aligned}$$

Since $|L_1| = \binom{n}{k-1}$, then the first term simplifies to $\binom{n}{k+1}$. Hence,

$$|\mathcal{U}| \leq \binom{n}{k+1} - \frac{1}{(k+1)_2} \sum_{X \in \mathcal{S}} (f(X))_2.$$

Jensen's inequality allows us to bound $\sum_{X \in \mathcal{S}} f^2(X) \geq \frac{1}{|\mathcal{S}|} \left(\sum_{X \in \mathcal{S}} f(X) \right)^2$. Furthermore, since $f(X) \leq n - k + 1$ and $|S| \leq \binom{n}{k-1}$, $\sum_{X \in \mathcal{S}} f(X) \leq \binom{n}{k-1}(n - k + 1)$.

$$\begin{aligned} |\mathcal{U}| &\leq \binom{n}{k+1} - \frac{1}{(k+1)_2|\mathcal{S}|} \left(\sum_{X \in \mathcal{S}} f(X) \right)^2 + \binom{n}{k-1} \frac{n-k+1}{(k+1)_2} \\ &= N' \frac{n-k+1}{k+1} - \frac{1}{(k+1)_2|\mathcal{S}|} \left(\sum_{X \in \mathcal{S}} f(X) \right)^2. \end{aligned}$$

Rearranging the terms gives

$$\left(\sum_{X \in \mathcal{S}} f(X) \right)^2 \leq (k+1)_2 |\mathcal{S}| \left(N' \frac{n-k+1}{k+1} - |\mathcal{U}| \right).$$

Now, we solve for the summation and make some easy estimates:

$$\begin{aligned}\sum_{X\in\mathcal{S}} f(X) &\leq (k+1)\sqrt{|\mathcal{S}|\,(N'-|\mathcal{U}|)+|\mathcal{S}|\frac{n-2k}{k+1}N'}\\ &\leq (k+1)\sqrt{|\mathcal{S}|\,(N'-|\mathcal{U}|)}+(k+1)\sqrt{|\mathcal{S}|\frac{|n-2k|}{k+1}N'}\\ &\leq (k+1)\sqrt{|\mathcal{S}|\,(N'-|\mathcal{U}|)}+O(n^{5/6})N'\end{aligned}$$

Symmetrically, $\sum_{Z\in\mathcal{U}} g(Z) \leq (n-k+1)\sqrt{|\mathcal{U}|\,(N'-|\mathcal{S}|)}+O(n^{5/6})N'$, and this concludes the proof of Lemma 7. □

Returning to (2) and using (3) we have:

$$\begin{aligned}&(k+1)k|\mathcal{U}|+k(n-k)|\mathcal{T}|+(n-k+1)(n-k)|\mathcal{S}|\\ &\qquad\leq 2\binom{n}{k}k(n-k)+|\Upsilon_3|-|\Upsilon_1|\\ &\qquad\leq 2N'k(n-k)+(n-k)\sum_{X\in\mathcal{S}} f(X)+k\sum_{Z\in\mathcal{U}} g(Z)-|\mathcal{T}|k(n-k).\end{aligned}\tag{4}$$

As $|k-n/2|=O(n^{2/3})$, we can utilize the estimates in Lemma 7 to bound $\sum_{X\in\mathcal{S}} f(X)$ and $\sum_{Z\in\mathcal{U}} g(Z)$ and divide (4) by $k(n-k)$ to get

$$\frac{k+1}{n-k}|\mathcal{U}|+|\mathcal{T}|+\frac{n-k+1}{k}|\mathcal{S}|\leq 2N'+\frac{k+1}{k}\sqrt{|\mathcal{S}|\,(N'-|\mathcal{U}|)}+\frac{n-k+1}{n-k}\sqrt{|\mathcal{U}|\,(N'-|\mathcal{S}|)}+O(n^{-1/6})N'-|\mathcal{T}|.$$

The goal is to get $2|\mathcal{F}|$ on the left-hand side of the inequality. What this enables us to do is to eliminate $|\mathcal{T}|$ from the right-hand side. We may disregard all small-order terms because they are of magnitude at most $O(n^{-1/6})N'$:

$$\begin{aligned}2|\mathcal{U}|+2|\mathcal{T}|+2|\mathcal{S}| &\leq 2N'+|\mathcal{U}|+|\mathcal{S}|+\sqrt{|\mathcal{S}|\,(N'-|\mathcal{U}|)}+\sqrt{|\mathcal{U}|\,(N'-|\mathcal{S}|)}+O(n^{-1/6})N'\\ &\leq \frac{3+\sqrt{2}}{2}N'+O(n^{-1/6})N'.\end{aligned}$$

Here the last inequality is obtained by maximizing function $f(u,s)=2+\sqrt{s(1-u)}+\sqrt{u(1-s)}+u+s$, $0\leq u,s\leq 1$. The maximum occurs when $s=u=(2+\sqrt{2})/4$.

Therefore,

$$|\mathcal{F}|\leq\frac{3+\sqrt{2}}{2}N'+o(N')\leq 2.20711N'+o(N').\tag{5}$$

Consider now a more general setting. Recall that $N = \binom{n}{n/2}$. Let $\mathcal{F}$ be a Q_2-free family of sets in Q_n. Let $\mathcal{F} = \mathcal{S} \cup \mathcal{T} \cup \mathcal{U}$, where $\mathcal{S} \subset \binom{[n]}{k_{\mathcal{S}}}$, $\mathcal{T} \subset \binom{[n]}{k}$ and $\mathcal{U} \subset \binom{[n]}{k_{\mathcal{U}}}$, where $k_{\mathcal{S}} < k < k_{\mathcal{U}}$. We may assume that $n/2 - n^{2/3} < k_{\mathcal{S}} < k < k_{\mathcal{U}} < n/2 + n^{2/3}$. Otherwise, by Lemma 1, at least one of $\mathcal{S}$, $\mathcal{T}$ or $\mathcal{U}$ has size $o(N)$ and so $|\mathcal{F}| \leq (2 + o(1))N$.

Consider a Symmetric Chain Decomposition of Q_n (see Greene and Kleitman [6] for the existence of such a decomposition) and, in particular, the N' disjoint chains that contain elements of $\binom{[n]}{k}$, call them $P_1, \ldots, P_{N'}$. We can create a new family $\mathcal{F}' = \mathcal{U}' \cup \mathcal{T} \cup \mathcal{S}'$ such that we shift $\mathcal{S}$ and $\mathcal{U}$ to the layers directly below and above $\mathcal{T}$, respectively, along each chain P_i. Formally, let

$$\begin{aligned} \mathcal{S}' &= \left\{ P_i \cap \binom{[n]}{k-1} : \text{ there is } S \in \mathcal{S} \cap P_i, i = 1, \ldots, q \right\}, \\ \mathcal{U}' &= \left\{ P_i \cap \binom{[n]}{k+1} : \text{ there is } U \in \mathcal{U} \cap P_i, i = 1, \ldots, q \right\}. \end{aligned}$$

Note that $\mathcal{F}'$ is Q_2-free and consists of three consecutive layers. Thus, the inequality (5) gives that $|\mathcal{F}'| \leq \frac{3+\sqrt{2}}{2} N' + o(N')$.

There might be unshifted elements, but not too many. In fact, both $|\mathcal{S}| - |\mathcal{S}'|$ and $|\mathcal{U}| - |\mathcal{U}'|$ are at most $N - N'$. So,

$$\begin{aligned} |\mathcal{F}| &= |\mathcal{F}'| + (|\mathcal{S}| - |\mathcal{S}'|) + (|\mathcal{U}| - |\mathcal{U}'|) \\ &\leq \frac{3+\sqrt{2}}{2} N' + o(N') + 2(N - N') \\ &= \left(\frac{\sqrt{2}-1}{2} \right) N' + 2N + o(N) \\ &\leq \left(\frac{\sqrt{2}-1}{2} \right) N + 2N + o(N) \\ &\leq \frac{3+\sqrt{2}}{2} N + o(N) \approx 2.20711N + o(N). \end{aligned}$$

□

4 Proofs of Lemmas

4.1 Proof of Lemma 3

In order to find the lower bound on $|\Upsilon_1|$, we shall consider $\Upsilon_1' = \{\sigma \in \Upsilon_1 : \emptyset \in \sigma\}$; i.e., the set of full chains in Q_m containing only $\emptyset$ and no other sets from $\mathcal{G}$.

Recall that $\mathcal{G} = \{\emptyset\} \cup \mathcal{T} \cup \mathcal{U}$, where $\mathcal{T}$ are the minimal elements of $\mathcal{G} - \{\emptyset\}$, and $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \cdots$, where $\mathcal{T}_i$ is the family of sets from $\mathcal{T}$ of size i. Without loss of generality, the one-element members of $\mathcal{T}$ are $\{\eta+1\}, \{\eta+2\}, \ldots, \{m\}$, which correspond to 1-element subsets of $[m]-[\eta]$. A graph G is defined on vertex set $[\eta]$ with edges corresponding to sets from $\mathcal{T}_2$, $e = |E(G)|, \overline{e} = |E(\overline{G})|$, $d(v), \overline{d}(v)$ is the degree of v in G and $\overline{G}$, respectively.

Let $x \in [\eta], t \in [m]-[\eta]$.

If $\{x,t\} \in \mathcal{G}$ then denote $\mathcal{C}_1(x,t)$ to be the set of full chains of the form $\emptyset, \{x\}, \{x,y\}, \{x,y,t\}, \ldots$, where $y \in [\eta]$, $\{x,y\} \notin \mathcal{G}$. We have that $\mathcal{C}_1(x,t) \subseteq \Upsilon_1'$ and $|\mathcal{C}_1(x,t)| \geq (m-3)!\overline{d}(x)$.

If $\{x,t\} \notin \mathcal{G}$ then denote $\mathcal{C}_2(x,t)$ to be the set of full chains of the form $\emptyset, \{x\}, \{x,t\}, \ldots$ unless such a chain passes through $A \cup \{t\}$ where $A \cup \{t\} \in \mathcal{G}$. We have that $\mathcal{C}_2(x,t) \subseteq \Upsilon_1'$ and $|\mathcal{C}_2(x,t)| \geq (m-2)! - (m-3)!\overline{d}(x)$.

Observe also for any $t, t' \in [m]-[\eta]$, $\mathcal{C}_1(x,t) \cap \mathcal{C}_1(x,t') = \emptyset$ and $\mathcal{C}_2(x,t) \cap \mathcal{C}_2(x,t') = \emptyset$. Thus for each $x \in [\eta]$, the number of chains in Υ_1' passing through x is at least

$$\sum_{\substack{t \in [m]-[\eta] \\ \{x,t\} \in \mathcal{G}}} |\mathcal{C}_1(x,t)| + \sum_{\substack{t \in [m]-[\eta] \\ \{x,t\} \notin \mathcal{G}}} |\mathcal{C}_2(x,t)| \geq \sum_{t \in [m]-[\eta]} \min\{(m-3)!\overline{d}(x), (m-2)! - (m-3)!\overline{d}(x)\}.$$

Thus,

$$\begin{aligned} |\Upsilon_1| \geq |\Upsilon_1'| &\geq \sum_{x \in [\eta]} \sum_{t \in [m]-[\eta]} \min\{(m-3)!\overline{d}(x), (m-2)! - (m-3)!\overline{d}(x)\} \\ &= (m-\eta)(m-3)! \left(2\overline{e} - \sum_{x \in [\eta]} \max\left\{0, 2\overline{d}(x) - m + 2\right\} \right). \end{aligned} \tag{6}$$

Consider the set D of all sequences of η nonnegative real numbers which are at most $\eta - 1$, and which add up to $2\overline{e}$. Note that the degree sequence of $\overline{G}$ is in D. Thus,

$$\begin{aligned}\sum_{x\in[\eta]} \max\{0, 2\overline{d}(x) - m + 2\} &\leq \max_{(d_1,\ldots,d_n)\in D} \sum_{i=1}^{\eta} \max\{0, 2d_i - m + 2\} \\ &\leq \frac{2\overline{e}}{\eta - 1}(2\eta - 2 - m + 2).\end{aligned}$$

Returning to (6), and recalling that $a = \eta/m$, and

$$\mu = \begin{cases} 1, & a < 1/2, \\ \frac{1-a}{a}, & a \geq 1/2, \end{cases}$$

we have

$$\begin{aligned}(m-\eta)(m-3)!\left(2\overline{e} - \sum_{x\in[\eta]} \max\left\{0, 2\overline{d}(x) - m + 2\right\}\right) &\geq \begin{cases} (m-\eta)(m-3)!2\overline{e}, & 2\eta \leq m-2; \\ (m-\eta)(m-3)!2\overline{e}\left[\frac{m-\eta-1}{\eta-1}\right], & 2\eta > m-2 \end{cases} \\ &\geq (m-\eta)(m-3)!2\overline{e}(\mu - O(m^{-1})).\end{aligned}$$

Therefore, since $b = e/\binom{\eta}{2}$,

$$\begin{aligned}|\Upsilon_1| &\geq (m-\eta)(m-3)!2\overline{e}(\mu - O(m^{-1})) \\ &= m!\frac{m-\eta}{m(m-1)(m-2)}\eta^2(1-b)(\mu - O(m^{-1})) \\ &= m!\left[b(a^3 - a^2)\mu + (a^2 - a^3)\mu - O(m^{-1})\right].\end{aligned}$$

□

4.2 Proof of Lemma 4

For each $T \in \mathcal{T}$, let $\mathcal{U}_T = \{U \in \mathcal{U} : U \supset T\}$ and let

$$\mathcal{U}'_T = \{V \supset T : \ |V| = |T| + 1, \ \ \nexists T_0 \in \mathcal{T} - \{T\}, \ T_0 \subset V\}.$$

We say that $\mathcal{G}' = \emptyset \cup \mathcal{T} \cup \bigcup_{T\in\mathcal{T}} \mathcal{U}'_T$ is a **compressed family**.

We have that $|\Upsilon^3_m(\emptyset, \mathcal{G})| \leq |\Upsilon^3_m(\emptyset, \mathcal{G}')|$. Indeed, if a chain contains both $T \in \mathcal{T}$ and $U \in \mathcal{U}$, then there is some $U' \in \mathcal{U}'_T$ that this chain contains also.

Let $\Upsilon'_3 = \Upsilon^3_m(\emptyset, \mathcal{G}')$. Recall that $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \cdots$, where $\mathcal{T}_i$ is the family of sets from $\mathcal{T}$ of size i. To bound $|\Upsilon'_3|$, we count first the number of full chains from Υ'_3 containing sets from $\mathcal{T}_1$, then those containing sets from $\mathcal{T}_2$, and finally those containing sets from $\mathcal{T}_i$, $i \geq 3$.

Recall that the graph G is defined on vertex set $[\eta]$ with edges corresponding to sets from $\mathcal{T}_2$. Let α_1 be the number of triples from $[\eta]$ which induce exactly one edge in G. Let B_0 be the set of 4-element sets from $[\eta]$ which do not induce an edge in G, and let $\beta_0 = |B_0|$.

There are at most $(m-\eta)\eta(m-2)!$ chains from Υ'_3 containing sets from $\mathcal{T}_1$. There are $2\alpha_1(m-3)!$ such chains containing sets from $\mathcal{T}_2$. We need to do some more work to bound the number of chains from Υ'_3 containing sets from $\mathcal{T}_i$, $i \geq 3$. Call the set of such chains Y.

Recall that if $T, U \in \mathcal{G}'$, $T \subseteq U$, $T \neq \emptyset$, then the number of full chains through T and U is $|T|!(|U-|T|)!(m-|U|)!$. Since $\mathcal{G}'$ is a compressed family, we have that $|T| = |U| - 1$. Moreover, if T, U belong to a chain in Y, we have that $|U| \geq 4$. Let

$$\mathcal{U}^* = \{U \in \mathcal{G}' \setminus \mathcal{T} : U \in C \in Y\}.$$

Since for each $U \in \mathcal{G}'$ there is at most one $T \in \mathcal{G}'$, $T \neq \emptyset$ such that $T \subseteq U$, we have

$$|Y| = \sum_{U \in \mathcal{U}^*} \sum_{T \subseteq U, T \in \mathcal{T}} |T|!(m-|U|)! \leq \sum_{U \in \mathcal{U}^*} (|U|-1)!(m-|U|)! = \sum_{U \in \mathcal{U}^*} \frac{1}{|U|}|U|!(m-|U|)! \leq \frac{1}{4} \sum_{U \in \mathcal{U}^*} |U|!(m-|U|)!.$$

The last summation counts the number of full chains containing a set from $\mathcal{U}^*$. Since for each $U \in \mathcal{U}^*$, there is $B \in B_0$, $B \subseteq U$, we have that the number of full chains passing through sets in $\mathcal{U}^*$ is at most the number of full chains passing through B_0 sets. Thus

$$|Y| \leq \frac{1}{4} \sum_{B \in B_0} |B|!(m-|B|)! \leq \frac{1}{4} \sum_{B \in B_0} |4|!(m-|4|)! = \frac{1}{4} 4!(m-4)!|B_0| \leq 3!(m-4)!\beta_0.$$

So, we have that

$$|\Upsilon_3| \leq |\Upsilon'_3| \leq (m-\eta)\eta(m-2)! + 2(m-3)!\alpha_1 + 3!(m-4)!\beta_0. \tag{7}$$

To bound the last two terms, we use the following lemma.

Lemma 8 *With α_1 and β_0 defined as above for G, an η-vertex graph, and $a = \eta/m$,*

$$\alpha_1 + \frac{3}{m-3}\beta_0 \leq \frac{\eta^3}{8} + \frac{e^2}{\eta}(a-2) + \frac{1}{4}e\eta(4-3a) - \frac{1}{8}(1-a)\eta^3 + O(m^2).$$

Proof. Let $\beta_i = \beta_i(G)$ be the number of 4-element subsets of the vertex set of G spanning exactly i edges. Moreover, let

$$\beta_2(G) = \beta_{\|}(G) \cup \beta_{\wedge}(G), \quad \beta_3(G) = \beta_{\triangle}(G) \cup \beta_{\dashv}(G) \cup \beta_{\sqcup}(G),$$

where $\beta_{\|}$, and $\beta_{\wedge}$ count such subsets inducing two disjoint edges, and two adjacent edges, respectively; $\beta_{\triangle}$, $\beta_{\dashv}$, and $\beta_{\sqcup}$ count the number of such subsets inducing triangle, a star of three edges, and a path with three edges, respectively. We also denote $\overline{\beta}_i(G) = \beta_i(\overline{G})$, $\overline{\beta}_{\|}(G) = \beta_{\|}(\overline{G})$, $\overline{\beta}_{\wedge}(G) = \beta_{\wedge}(\overline{G})$, $\overline{\beta}_{\triangle}(G) = \beta_{\triangle}(\overline{G})$, $\overline{\beta}_{\dashv}(G) = \beta_{\dashv}(\overline{G})$, $\overline{\beta}_{\sqcup}(G) = \beta_{\sqcup}(\overline{G})$. Note that $\overline{\beta}_{\triangle}(G) = \beta_{\dashv}(G)$, $\overline{\beta}_{\sqcup}(G) = \beta_{\sqcup}(G)$. Let $d(v)$ and $\overline{d}(v)$ be the degree of v in G and in $\overline{G}$, respectively. Then,

$$\begin{aligned}
\sum_{v\in V} d(v)\binom{\overline{d}(v)}{2} &= 2\beta_1 + 4\beta_{\|} + 2\beta_{\wedge} + 2\beta_{\sqcup} + 3\beta_{\dashv} + \overline{\beta}_{\wedge}, \\
\sum_{v\in V} \binom{d(v)}{3} &= \beta_{\dashv} + \overline{\beta}_{\wedge} + 2\overline{\beta}_1 + 4\overline{\beta}_0, \\
\sum_{v\in V} \overline{d}(v)\binom{d(v)}{2} &= 2\overline{\beta}_1 + 4\overline{\beta}_{\|} + 2\overline{\beta}_{\wedge} + 2\overline{\beta}_{\sqcup} + 3\overline{\beta}_{\dashv} + \beta_{\wedge}, \\
\sum_{v\in V} \binom{\overline{d}(v)}{3} &= \overline{\beta}_{\dashv} + \beta_{\wedge} + 2\beta_1 + 4\beta_0.
\end{aligned}$$

Observe that the last two equations are complementary of the first two. In order to bound $\alpha_1 + \frac{3}{m-3}\beta_0$, we shall express everything in terms of βs, and then in terms of e.

Since

$$\alpha_1(\eta - 3) = 2\beta_1 + 4\beta_{\|} + 2\beta_{\wedge} + 3\beta_{\triangle} + 2\beta_{\sqcup} + \overline{\beta}_{\wedge},$$

and

$$\binom{\eta}{4} = \beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 = \beta_0 + \beta_1 + (\beta_{\|} + \beta_{\wedge}) + (\beta_{\triangle} + \beta_{\sqcup} + \beta_{\dashv}) + (\overline{\beta}_{\|} + \overline{\beta}_{\wedge}) + \overline{\beta}_1 + \overline{\beta}_0,$$

we have, by recalling that $a = \eta/m > (\eta - 3)/(m - 3)$,

$$\begin{aligned}
Q &:= (\eta - 3)\left[\alpha_1 + \frac{3}{m-3}\beta_0\right] \\
&= 3\frac{\eta - 3}{(m-3)}\beta_0 + 2\beta_1 + 4\beta_{\|} + 2\beta_{\wedge} + 3\beta_{\triangle} + 2\beta_{\sqcup} + \overline{\beta}_{\wedge} \\
&< 3\binom{\eta}{4} + (3a - 3)\beta_0 + (-1)\beta_1 + 1\beta_{\|} + (-1)\beta_{\wedge} + 0\beta_{\triangle} + (-1)\beta_{\sqcup} + (-3)\beta_{\dashv} \\
&\quad + (-2)\overline{\beta}_{\wedge} + (-3)\overline{\beta}_{\|} + (-3)\overline{\beta}_1 + (-3)\overline{\beta}_0.
\end{aligned}$$

At the price of slightly increasing the right hand side, we collect the terms in order to utilize the various formulas that sum the degrees:

$$\begin{aligned} Q &\le 3\binom{\eta}{4} + \frac{1}{4}\left[2\beta_1 + 4\beta_{\|} + 2\beta_{\wedge} + 2\beta_{\sqcup} + (1-3)\overline{\beta}_{\wedge} - 6\overline{\beta}_1 - 12\overline{\beta}_0\right] \\ &\quad + \frac{(1-a)}{4}\left[2\overline{\beta}_1 + 4\overline{\beta}_{\|} + 2\overline{\beta}_{\wedge} + 2\beta_{\sqcup} + (1-3)\beta_{\wedge} - 6\beta_1 - 12\beta_0\right] \\ &= 3\binom{\eta}{4} + \frac{1}{4}\left[\sum_v d(v)\binom{\overline{d}(v)}{2} - 3\sum_v \binom{d(v)}{3}\right] + \frac{1-a}{4}\left[\sum_v \overline{d}(v)\binom{d(v)}{2} - 3\sum_v \binom{\overline{d}(v)}{3}\right] \\ &= 3\binom{\eta}{4} + \frac{\eta-3}{8}\left[\sum_v d(v)(\eta - 2d(v))\right] + (1-a)\frac{(\eta-3)}{8}\left[\sum_v \overline{d}(v)(\eta - 2\overline{d}(v))\right]. \end{aligned}$$

We can use the fact that $\overline{d}(v) = \eta - d(v) - 1$ and collect terms

$$Q \le 3\binom{\eta}{4} + \frac{\eta-3}{8}(-4+2a)\sum_v d^2(v) + \frac{\eta-3}{8}(4\eta - 4 - 3a\eta + 4a)\sum_v d(v) + \frac{\eta-3}{8}(a-1)\eta(\eta-1)(\eta-2).$$

Using the fact that $\sum_v d(v) = 2e$ and $\sum_v d^2(v) \ge 4e^2/\eta$,

$$Q \le \frac{\eta^4}{8} + e^2(a-2) + \frac{1}{4}e\eta^2(4-3a) - \frac{1}{8}(1-a)\eta^4 + O(\eta^3).$$

Dividing Q by $\eta - 3$ and observing that $\eta \le m$, this concludes the proof of Lemma 8. □

Now, we return to the upper bound (7) on $|\Upsilon_3|$, recalling that $a = \eta/m$ and $e = b\binom{\eta}{2}$,

$$|\Upsilon_3| \le (m-\eta)\eta(m-2)! + 2(m-3)!\alpha_1 + 3!(m-4)!\beta_0.$$

Because of Lemma 8,

$$\begin{aligned} |\Upsilon_3| &\le m!\left[a(1-a) + \frac{2}{m^3}\left(\frac{\eta^3}{8} + \frac{e^2}{\eta}(a-2) + \frac{1}{4}e\eta(4-3a) - \frac{1}{8}(1-a)\eta^3 + O(m^{-1})\right)\right] \\ &= m!\left[b^2\left(a^4/2 - a^3\right) + b\left(a^3 - 3a^4/4\right) + \left(a^4/4 - a^2 + a\right) + O(m^{-1})\right]. \end{aligned}$$

This concludes the proof of Lemma 4. □

4.3 Proof of Lemma 5

The estimations here can be checked by a symbolic manipulation program.

Set

$$Q' := b^2\left(a^4/2 - a^3\right) + b\left(a^3 - 3a^4/4 - a^3\mu + a^2\mu\right) + \left(a^4/4 - a^2 + a - a^2\mu + a^3\mu\right).$$

If $0 \le a < 1/2$ we have that $\mu = 1$ and

$$Q' = b^2\left(a^4/2 - a^3\right) + b\left(-3a^4/4 + a^2\right) + \left(a^4/4 + a - 2a^2 + a^3\right) \le 0.25,$$

which is achieved when $b = 1$ and $a = 1/2$.

If $1/2 \le a \le 1$ we have that $\mu = (1-a)/a$ and

$$Q' = b^2\left(a^4/2 - a^3\right) + b\left(2a^3 - 3a^4/4 - 2a^2 + a\right) + \left(a^4/4 + a^2 - a^3\right)\right) < 0.283261.$$

The maximum is achieved when $a \approx 0.935$ and $b \approx 0.285$. □

5 Conclusions

The method we use is local, it allows us to count the number of full chains with three or one element in $\mathcal{F}$. Using this method, one could not get a bound better than $2.25N$ for $ex(n, Q_2)$. To see this, consider a set system with elements from $[m]$, where m is even and $[m] = M_1 \cup M_2$, $M_1 = [m/2], M_2 = \{m/2+1, m/2+2, \ldots, m\}$. $\mathcal{G} = \{\emptyset\} \cup \mathcal{T} \cup \mathcal{U}$, where

$$\mathcal{T} = \binom{M_1}{2} \cup \binom{M_2}{2},$$

$$\mathcal{U} = \{\{a, b, c\} : a, b \in M_2, c \in M_1\} \cup \{\{a, b, c\} : a, b \in M_1, c \in M_2\}.$$

We have that the number of full chains in Q_m containing three elements of $\mathcal{G}$ is at $4\binom{m/2}{2}m/2(m-3)!$. On the other hand, each full chain contains at least one nonempty set from $\mathcal{F}$. Thus $|\Upsilon_3| \ge m!/4$.

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