Conditions on Ramsey non-equivalence

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Abstract

Given a graph H, a graph G is called a *Ramsey graph of H* if there is a monochromatic copy of H in every coloring of the edges of G with two colors. Two graphs G, H are called *Ramsey equivalent* if they have the same set of Ramsey graphs. Fox *et al.* [*J. Combin. Theory Ser. B* **109** (2014), 120–133] asked whether there are two non-isomorphic connected graphs that are Ramsey equivalent. They proved that a clique is not Ramsey equivalent to any other connected graph. Results of Nešetřil *et al.* showed that any two graphs with different clique number [*Combinatorica* **1**(2) (1981), 199–202] or different odd girth [*Comment. Math. Univ. Carolin.* **20**(3) (1979), 565–582] are not Ramsey equivalent. These are the only structural graph parameters we know that "distinguish" two graphs in the above sense. This paper provides further supportive evidence for a negative answer to the question of Fox *et al.* by claiming that for wide classes of graphs, chromatic number is a distinguishing parameter. In addition, it is shown here that two connected graphs are not Ramsey equivalent if they have at most 5 vertices, if they belong to a special class of trees, or to classes of graphs with clique-reduction properties. An infinite class of graphs is given such that any graph in this class is not Ramsey equivalent to any other connected graph.

Keywords: Ramsey, Ramsey equivalence, chromatic number, Ramsey classes

1 Introduction

Given a graph H, a graph G is called a *Ramsey graph of H* if there is a monochromatic copy of H in every coloring of the edges of G with two colors. If G is a Ramsey graph of H, we write $G \to H$ and say that G arrows H. We denote by $\mathcal{R}(H)$ the set of all graphs that arrow H, and call it the *Ramsey class of H*. So the Ramsey number $R(H)$ is the smallest integer n such that $K_n \in \mathcal{R}(H)$, where K_n denotes the complete graph on n vertices. Two graphs G, H are called *Ramsey equivalent* if they have the same Ramsey class. We write $G \overset{R}{\sim} H$ if G is Ramsey equivalent to H, and write $G \overset{R}{\nsim} H$ otherwise. The study of Ramsey classes was initiated by the fundamental work of Nešetřil and Rödl [20] and Burr, Erdős, and Lovász [7]. However, the notion of Ramsey equivalence of graphs was raised only recently by Szabó *et al.* [26]. It was shown in [26], that there are two non-isomorphic graphs that are Ramsey equivalent, for example $G \overset{R}{\sim} H$, where $G = K_t$ and H is a vertex disjoint union of K_t and K_2 for $t \geq 4$. In this example H is a disconnected graph. Fox *et al.* formulated a question for connected graphs:

Question 1 ([10])**.** *Are there two non-isomorphic connected graphs G and H with $G \overset{R}{\sim} H$?*

Note that $G \overset{R}{\not\sim} H$ if and only if there exists a graph Γ such that $\Gamma \to H$ and $\Gamma \not\to G$ or $\Gamma \to G$ and $\Gamma \not\to H$. In this case we call Γ a graph, *distinguishing* G and H. So in order to prove that $G \overset{R}{\not\sim} H$, it is sufficient to explicitly construct a distinguishing graph. Another approach is to identify a graph parameter s, such that $s(G) \neq s(H)$ implies that $G \overset{R}{\not\sim} H$. In this case, we say that s is a *Ramsey distinguishing parameter.* The only structural graph parameters that we know to be Ramsey distinguishing are the clique number ω and the odd girth g_o, where ω is the largest number of vertices in a clique of the graph, and g_o is the length of its shortest odd cycle. Specifically, it is shown in [22, 21] that if $\omega(H) = \omega$ and $g_o(H) = g_o$ then there are Ramsey graphs $G, G' \in \mathcal{R}(H)$ such that $\omega(G) = \omega$ and $g_o(G') = g_o$. Note that if Question 1 has a negative answer, then any graph parameter is a Ramsey distinguishing parameter for the class of connected graphs.

In this paper, we provide a supporting evidence for a 'No'-answer to Question 1 by the following theorems, focusing on another graph parameter, the chromatic number, χ.

Observation 2. *If G and H are graphs, $\chi(G) = 2$ and $\chi(H) > 2$ then $G \overset{R}{\not\sim} H$.*

Indeed, a sufficiently large complete bipartite graph arrows any fixed bipartite graph [5]. But it does not contain, and thus does not arrow any non-bipartite graph. Here, we prove that for several large classes of connected graphs, the chromatic number is a Ramsey distinguishing parameter. A graph is called *clique-splittable* if its vertex set can be partitioned into two subsets, each inducing a subgraph of smaller clique number. Note that any graph G with $\chi(G) \leq 2\omega(G) - 2$ is clique-splittable. In particular all cliques and all planar graphs containing a triangle are clique-splittable. The triangle-free clique-splittable graphs are precisely the bipartite graphs.

Theorem 3. *If G, H are graphs, G is clique-splittable and $\chi(G) < \chi(H)$, then $G \overset{R}{\not\sim} H$.*

Corollary 4. *If G, H are graphs, $\chi(G) \leq 2\omega(G) - 2$ and $\chi(G) \neq \chi(H)$, then $G \overset{R}{\not\sim} H$.*

Theorem 3 distinguishes pairs of graphs of distinct chromatic number under some splittability condition. The following theorem requires stronger assumptions but also applies to graphs of the same chromatic number. If H is a subgraph (proper subgraph) of G, we write $H \subseteq G$ ($H \subsetneq G$).

Theorem 5. *Let a connected graph G satisfy the following two properties:*

1) *There is an independent set $S \subset V(G)$ such that $\omega(G - S) < \omega(G)$.*

2) *There is a proper $\chi(G)$-vertex-coloring of G in which some two color classes induce a subgraph of a matching.*

Let H be a connected graph, not isomorphic to G, such that either $H \subsetneq G$ or $\chi(H) \geq \chi(G)$. Then $G \overset{R}{\not\sim} H$.

In Theorems 3 and 5 we distinguish pairs of graphs under certain properties. Call a graph G *Ramsey isolated* if $G \overset{R}{\not\sim} H$ for any connected graph H not isomorphic to G. Note that Question 1 asks whether every connected graph is Ramsey isolated or not. We apply the previous results to identify large families of Ramsey isolated graphs. The k-wheel is the graph on $k+1$ vertices obtained from a cycle of length k by adding a vertex adjacent to all vertices of the cycle.

Theorem 6.

1. *If G is connected, $\chi(G) = \omega(G)$ and there is a proper $\chi(G)$-vertex-coloring of G in which some two color classes induce a matching in G, then G is Ramsey isolated.*
2. *Any path and any star are Ramsey isolated.*
3. *Every connected graph G on at most 5 vertices and not isomorphic to the complete bipartite graph $K_{2,3}$ or the 4-wheel is Ramsey isolated.*

Theorem 7. *Any two connected non-isomorphic graphs on at most 5 vertices are not Ramsey equivalent.*

Remark 8. *If F distinguishes G and H then F has at least $\min\{R(G), R(H)\}$ vertices. The distinguishing graphs used in the proof of Theorem 6 are rather large, except for stars. However, for most pairs G, H of distinct connected graphs on at most 5 vertices there is a distinguishing graph on $\min\{R(G), R(H)\}$ vertices.*

A tree T on k vertices is called *balanced* if deleting some edge splits T into components of order at most $\lceil \frac{k+1}{2} \rceil$ each. The extremal function $\text{ex}(n, H)$ is the largest number of edges in an n-vertex graph with no copy of H. The Erdős-Sós-Conjecture states that $\text{ex}(n, T) \leq \frac{k-2}{2} n$ for any tree T on k vertices. We remark that recently, Ajtai, Komlós, Simonovits and Szemerédi announced a proof of the conjecture for large k [1, 2, 3]. We state here a much weaker conjecture:

Conjecture 9. *There is a positive ϵ and an integer n_ϵ such that $\text{ex}(n, T) \leq \frac{k-1-\epsilon}{2} n$ for any tree on k vertices and $n > n_\epsilon$.*

Theorem 10. *If Conjecture 9 is true then any two trees of different order are not Ramsey equivalent. If T_k is a balanced tree on k vertices and T_ℓ is any tree on $\ell \geq k+1$ vertices, then $T_k \overset{R}{\not\sim} T_\ell$.*

The next theorem makes use of multicolor Ramsey numbers. Here $R(G_1, G_2, G_3)$ is the minimum integer n such that any coloring of the edges of K_n in red, blue, and green has a red G_1, a blue G_2, or a green G_3. We write $G \overset{R}{\not\sim}_k H$ if G and H are not Ramsey equivalent in k colors, i.e., there is a graph Γ such that any k-coloring of its edges contains a monochromatic G (we write $\Gamma \to_k G$) and there is such a coloring avoiding monochromatic H, or vice versa.

Theorem 11. *If G and H are graphs then $G \overset{R}{\not\sim} H$ if one of the following conditions holds:*

- *There is a graph F such that $R(G, G, F) < R(H, H, F)$.*
- *$G \subseteq H$ and there is $k \geq 2$ with $G \overset{R}{\not\sim}_k H$.*

The paper is structured as follows. In Section 2 we provide a more detailed summary of known results on Ramsey equivalence, as well as some observations. In Section 3 we give known and our preliminary results that will be used in proving the main theorems.

Section 4 contains the proofs of the main results, the Appendix provides lemmas used to prove Remark 8. The proof of Theorem 6.3 uses Theorem 7, which in turn uses only Theorem 6.1 and Theorem 6.2. However for the sake of better structure, we present the proofs of Theorems 6.1 - 6.3 and Theorem 7 in the order of their numbering. Finally, Section 5 contains conclusions and open questions. We refer the reader to [27] for all standard notations in graph theory and to Figure 7 for notations used for small graphs. We omit floors and ceilings as long as the meaning is clear from context. We also assume that the graphs under consideration have at least one edge.

2 Overview and Observations

As mentioned in the introduction, two connected non-isomorphic graphs G and H are not Ramsey equivalent if one of them is a clique [10], or if $\omega(G) \neq \omega(H)$ [22] or when $g_o(G) \neq g_o(H)$ [21]. Note that it is an open question whether $\max\{\text{girth}(F) : F \to G\} = \text{girth}(G)$ for all graphs G [19], where girth(G) is the length of a shortest cycle in G. There are several other results about Ramsey classes that are useful in checking whether some two given graphs are Ramsey equivalent or not. Note that two graphs are Ramsey equivalent if and only if they have the same set of minimal Ramsey graphs. A graph Γ is *minimal Ramsey* for H if $\Gamma \to H$, but $\Gamma' \not\to H$ for any proper subgraph Γ' of Γ. We need to define a few graphs to state the known results: P_n, C_n is the path, cycle on n vertices, respectively, $H_{t,d}$ is a graph on $t+1$ vertices such that one vertex has degree d and the other vertices induce K_t, $K_{a,b}$ is the complete bipartite graph with parts of sizes a and b, respectively. It was shown in [23] that if H does not contain $K_{a,b}$, $a, b \geq 3$ or $1 = a \leq b \leq 2$, then there is $\Gamma \in \mathcal{R}(H)$ such that Γ does not contain $K_{a,b}$. In general, for an integral graph parameter s, we define

$$\mathcal{R}_s = \min\{s(\Gamma) : \Gamma \text{ is minimal Ramsey for } H\}.$$

If $\mathcal{R}_s(H) \neq \mathcal{R}_s(G)$ then clearly $G \overset{R}{\not\sim} H$. When $s = \delta$, the minimum degree, a number of results have been obtained: $\mathcal{R}_\delta(K_t) = (t-1)^2$, [7, 11], $\mathcal{R}_\delta(H_{t,1}) = t-1$, [10], $\mathcal{R}_\delta(H_{t,d}) = d^2$, for $2 \leq d \leq t$, [13], $\mathcal{R}_\delta(K_{a,b}) = 2\min\{a,b\} - 1$, [11], $\mathcal{R}_\delta(H) = 1$ if H is a tree, [26], or when H is $K_{t,t}$ plus a pending edge, [10], $\mathcal{R}_\delta(C_n) = 3$ for even $n \geq 4$, [26]. Burr, Erdős, Lovász [7] conjectured that for every integer χ there is H with $\chi(H) = \chi$ and $R_\chi(H) = (\chi-1)^2 + 1$. This conjecture has been proven by Zhu [28]. Burr *et al.* proved that $R_\chi(G) \leq R(G)$ [7]. Hence $G \overset{R}{\not\sim} H$ if G is an n-vertex graph and $\chi(H) > 4^n$ because $R_\chi(H) \geq \chi(H) > 4^n \geq R(G) \geq R_\chi(G)$. In [16] the authors asked whether $\mathcal{R}_\Delta(H)$ is bounded by a function depending on maximum degree $\Delta(H)$ only. They prove that if H is a tree, then $2\Delta(H) - 1 \leq \mathcal{R}_\Delta(H) \leq 4(\Delta(H) - 1)$. We observe that for a non-bipartite H, $\mathcal{R}_\Delta(H) \geq 2\Delta(H)$, see Lemma 18. In [24] a threshold t_H is given, such that almost all graphs with density larger than t_H are Ramsey for H and almost all with smaller density are not. The introduction of [6] covers several results on graphs with infinitely many minimal Ramsey graphs.

It remains unclear whether a connected graph could or could not be Ramsey equivalent to its subgraph. Note that an edge-transitive graph H is not Ramsey equivalent to any of its subgraphs because coloring the edges of $\Gamma - e$ for a Ramsey minimal graph Γ of H in two colors, gives a monochromatic copy of $H - e'$ for some edge e'. Since $H - e'$ is isomorphic to $H - e''$ for any two edges, $H - e'$ contains any proper subgraph H' of H. We see that $\Gamma - e \to H'$, but $\Gamma - e \not\to H$.

3 Preliminary Lemmas

The following lemma is an easy generalization of the Focusing Lemma in [10].

Lemma 12 (Focusing Lemma, [10]). *Let $(A \cup B, E)$ be a bipartite graph with a 2-edge-coloring. Then there is a subset $B' \subseteq B$, $|B'| \geq |B|/2^{|A|}$, such that for each $a \in A$ all edges from a to B' are of the same color.*

Lemma 13 ([22]). *For any graph G there is a graph $F \in \mathcal{R}(G)$ with $\omega(G) = \omega(F)$.*

We write $V(\mathcal{H})$ and $E(\mathcal{H})$ for the vertex set, respectively the edge set, of a graph or hypergraph $\mathcal{H}$. A hypergraph is k-uniform if every hyperedge has size k. The girth of a hypergraph is the smallest number $k \geq 2$ of distinct vertices $v_0, \ldots, v_{k-1}$ and distinct hyperedges $E_0, \ldots, E_{k-1}$ such that for each i, $0 \leq i \leq k-1$, $\{v_i, v_{i+1}\} \subseteq E_i$ (indices taken modulo k). The independence number of a hypergraph is the size of a largest set of vertices which does not contain a hyperedge completely. The chromatic number of a hypergraph is the smallest number of colors in a proper vertex coloring, i.e., a coloring without monochromatic hyperedges.

Lemma 14 ([9]). *For any integers $k, g \geq 2$ and any $\epsilon > 0$ there is an integer n and a k-uniform hypergraph on n vertices with girth at least g and independence number less than ϵn.*

From this lemma one easily derives the following well-known result.

Lemma 15 ([9]). *For any integers $k, g, \chi \geq 2$ there is a k-uniform hypergraph with girth at least g and chromatic number at least χ.*

Proof. Let $\epsilon < \frac{1}{\chi}$ and let $\mathcal{H}$ denote a k-uniform hypergraph with girth at least g and independence number at most $\epsilon|V(\mathcal{H})|$, which exists by Lemma 14. In any χ-coloring of $V(\mathcal{H})$ there is a set of at least $\frac{1}{\chi}|V(\mathcal{H})| > \epsilon|V(\mathcal{H})|$ vertices of the same color. Hence this color class induces an edge of $\mathcal{H}$. Thus the coloring is not proper and $\chi(\mathcal{H}) > \chi$. □

For graphs F, G and for $\epsilon > 0$ we write $F \xrightarrow{\epsilon} G$ if for any set $S \subseteq V(F)$ with $|S| \geq \epsilon|V(F)|$, we have $F[S] \rightarrow G$. Here $F[S]$ denotes the subgraph of F induced by the vertices in S.

Lemma 16 ([10]).
For any $\epsilon > 0$ and any graph H, there is a graph F with $\omega(F) = \omega(H)$ and $F \xrightarrow{\epsilon} H$.

Proof. Let F' be a graph such that $F' \rightarrow H$ and $\omega(F') = \omega(H)$. Such a graph exists by Lemma 13. Further let $\mathcal{H}$ denote a $|V(F')|$-uniform hypergraph of girth at least 4 and no independent set of size $\epsilon|V(\mathcal{H})|$, which exists by Lemma 14. Construct a graph F by placing a copy of F' on the vertices of each hyperedge of $\mathcal{H}$. Then F is a graph on $|V(\mathcal{H})|$ vertices with $\omega(F) = \omega(F') = \omega(H)$. Each vertex set of size at least $\epsilon|V(\mathcal{H})|$ induces a hyperedge in $\mathcal{H}$ and thus a copy of F' in F which arrows H. □

We have the following corollary, since any graph which arrows H contains H.

Lemma 17. *For any $\epsilon > 0$ and any graph H, there is a graph F with $\omega(F) = \omega(H)$ and each set of $\epsilon|V(F)|$ vertices in F containing a copy of H.*

Lemma 18. *If a graph H is not bipartite then $\mathcal{R}_\Delta(H) \geq 2\Delta(H)$. The lower bound is tight.*

Proof. Let $\Delta = \Delta(H)$ and suppose F is a graph with $\Delta(F) \leq 2\Delta - 1$. It is sufficient to prove that $F \not\to H$. Consider a partition $V_1 \dot\cup V_2$ of $V(F)$ with the maximum number of edges between V_1 and V_2. If there is a vertex $v \in V_1$ with at least Δ neighbors in V_1, then v has at most $\Delta - 1$ neighbors in V_2. Thus the partition $(V_1 \setminus \{v\}) \dot\cup (V_2 \cup \{v\})$ has at least one more edge between the parts than the original partition, a contradiction. Hence both $F[V_1]$ and $F[V_2]$ have maximum degree at most $\Delta - 1$. Color all edges between V_1 and V_2 red and all other edges blue. Then the red subgraph is bipartite and the blue subgraph has maximum degree at most $\Delta - 1$. Thus $F \not\to H$.
The lower bound is tight since $K_{2\Delta+1} \to H$, $\Delta \geq 3$, where H is the graph of maximum degree Δ obtained from $K_{1,\Delta}$ by adding an edge between two leaves. □

Lemma 19. *Let G and H be graphs.*
If $\mathrm{ex}(n, G) < \mathrm{ex}(n, H)/2$ or if H is connected and $\mathrm{ex}(n, G) < \sqrt{n}\,\mathrm{ex}(\sqrt{n}, H)$, then $G \overset{R}{\not\sim} H$.
In particular, if G is a forest and H contains a cycle, then $G \overset{R}{\not\sim} H$.

Proof. Assume first that $\mathrm{ex}(n, G) < \mathrm{ex}(n, H)/2$. Let F be a graph on n vertices with $\mathrm{ex}(n, H) \geq 2\,\mathrm{ex}(n, G) + 1$ edges without a copy of H. In any 2-coloring of the edges of F one of the color classes contains at least $\mathrm{ex}(n, G) + 1$ edges, and thus a copy of G. Hence $F \to G$, but $F \not\to H$.
Assume now that H is connected and $\mathrm{ex}(n^2, G) < n\,\mathrm{ex}(n, H)$. Let F be a graph on n vertices and $\mathrm{ex}(n, H)$ edges not containing H. Let $F^* = F \times F$ be the Cartesian product of F with itself, that is, $V(F^*) = V(F) \times V(F)$ and $\{(u, v), (x, y)\} \in E(F^*)$ if and only if $u = x$ and $vy \in E(F)$ or $v = y$ and $ux \in E(F)$. Then F^* has n^2 vertices and $2n\,\mathrm{ex}(n, H)$ edges. In any 2-edge-coloring of F^* there is a color class with at least $n\,\mathrm{ex}(n, H)$ edges. This color class contains a copy of G, thus $F^* \to G$. On the other hand, we can color the edges of $E(F^*)$ without creating monochromatic copies of H by coloring an edge $\{(u, v), (x, y)\}$ red if $u = x$ and blue otherwise. Note that each color class is a vertex disjoint union of n copies of F and thus does not contain H, as H is connected. Thus $F^* \not\to H$.

For the second part of the statement let G be any forest and H be any graph with a cycle C. We have $\mathrm{ex}(n, G) \leq |V(G)|n$ and $\mathrm{ex}(n, H) \geq \mathrm{ex}(n, C) \geq \Omega(n^{1+\frac{1}{|V(C)|-1}})$ [18]. Hence for sufficiently large n we have $\mathrm{ex}(n, G) < \mathrm{ex}(n, C)/2$ and thus $G \overset{R}{\not\sim} H$ by the first part of the Lemma. □

4 Proofs of Theorems

4.1 Proof of Theorem 3

If $\chi(G) = 2$ then G is not Ramsey equivalent to any graph H of higher chromatic number by Observation 2. So, assume that $\chi(G) \geq 3$.
Let $\omega = \omega(G)$, $k = \chi(G)$, $\chi(H) > k$. We assume $\omega(G) = \omega(H)$, otherwise $G \overset{R}{\not\sim} H$ by Lemma 13. Let $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, such that $G_1 = G[V_1]$ and $G_2 = G[V_2]$ each have clique number less than ω. Let $G^\star$ be a vertex disjoint union of G_1 and G_2, in particular $\omega(G^\star) < \omega$. We shall construct a graph Γ such that $\Gamma \to G$ and $\Gamma \not\to H$.
The building blocks of Γ are a hypergraph $\mathcal{H}$ and graphs F and F' such that:

- $\mathcal{H}$ is a 3-chromatic, k-uniform hypergraph of girth at least $|V(H)| + 1$. It exists by Lemma 15.

- F is a graph such that $\omega(F) < \omega$ and every set of at least $\epsilon_1|V(F)|$ vertices in F contains a copy of $G^\star$, where $\epsilon_1 = 2^{-|V(\mathcal{H})|}$. Such a graph exists by Lemma 17.
- F' is a graph such that $\omega(F') = \omega(F) < \omega$ and $F' \stackrel{\epsilon}{\to} F$ for $\epsilon = 2^{-|V(\mathcal{H})||V(F)|}$. Such a graph exists by Lemma 16.

Note that $|V(\mathcal{H})|$ depends on $|V(H)|$ and $\chi(G)$; $|V(F)|$ in turn depends on $|V(\mathcal{H})|$, $\omega(G)$, and G, so $|V(F)|$ depends only on H and G. So, ϵ and ϵ_1 are constants depending on H and G.
Construct a graph Γ by replacing the vertices $v_1, \ldots, v_n$ of $\mathcal{H}$ with pairwise vertex disjoint copies of F' on vertex sets $V_1, \ldots, V_n$ and placing a complete bipartite graph between two copies of F' if and only if the corresponding vertices belong to the same hyperedge of $\mathcal{H}$, see Figure 1.

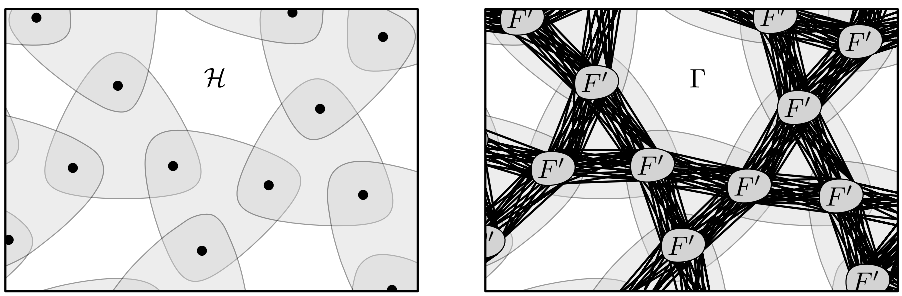

Figure 1: Left: The k-uniform hypergraph $\mathcal{H}$, $k = 3$. Right: The graph Γ.

To show that $\Gamma \not\to H$ color each edge with both endpoints in some V_i red, $i = 1, \ldots, n$, and all other edges blue. The red subgraph is a vertex disjoint union of copies of F', its clique number is strictly less than ω, so it does not contain H, whose clique number is ω. The blue subgraph is a union of complete k-partite graphs induced by V_i, $i = 1, \ldots, n$. To see that the blue subgraph does not contain a copy of H, consider any copy of H in Γ and consider sets $V_{i_1}, \ldots, V_{i_\ell}$ intersecting the vertex set of this copy. Since $\mathcal{H}$ has girth at least $|V(H)|+1$, $v_{i_1}, \ldots, v_{i_\ell}$ do not form a cycle in $\mathcal{H}$, thus the blue graph induced by $V_{i_1}, \ldots, V_{i_\ell}$ is k-partite. However, $\chi(H) > k$, so the blue subgraph does not contain a copy of H.

Next we shall show that $\Gamma \to G$. Consider a 2-edge-coloring of Γ. Recall that $n = |V(\mathcal{H})| = n(k, |V(H)|)$. We write $v_i \sim v_j$ if there is a hyperedge in $\mathcal{H}$ containing both v_i and v_j.

Claim 1. *For any m, $1 \le m \le n$, any i, $1 \le i \le m$, V_i contains a subset V_i' that is the vertex set of a monochromatic copy of F and such that for any $v \in V_i'$ and any j with $v_i \sim v_j$, $i < j \le m$, all edges from v to V_j', are of the same color.*

We prove Claim 1 by induction on m using the Focusing Lemma (Lemma 12). When $m = 1$, we see that $\Gamma[V_1]$ is isomorphic to F' and $F' \stackrel{\epsilon}{\to} F$. So in particular $F' \to F$ and there is a monochromatic copy of F on some vertex set V_1'. Assume that $V_1', V_2', \ldots, V_m'$ form vertex sets of monochromatic copies of F satisfying the conditions of Claim 1. Apply the Focusing Lemma to the bipartite graph with parts $U_m = V_1' \cup \cdots \cup V_m'$ and V_{m+1}. It gives a subset $V_{m+1}^* \subseteq V_{m+1}$ such that for any $v \in U_m$, all edges between v and V_{m+1}^*, if any, are of the

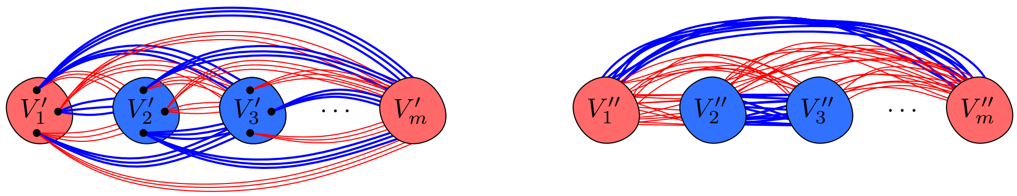

Figure 2: Illustrations of Claim 1 (left) and Claim 2 (right). Thin edges are red and thick edges blue.

same color and such that $|V^*_{m+1}| \geq 2^{-|V_1' \cup \cdots \cup V_m'|}|V_{m+1}| = 2^{-m|V(F)|}|V_{m+1}| \geq \epsilon |V_{m+1}|$. Thus $\Gamma[V^*_{m+1}]$ contains a monochromatic copy of F, because $\Gamma[V_{m+1}]$ is isomorphic to F' and $F' \overset{\epsilon}{\to} F$. Call the vertex set of this copy V'_{m+1}.

Claim 2. *For any m, $1 \leq m \leq n$, and i, $m \leq i \leq n$, each V_i contains a subset $V_i'' \subseteq V_i'$ that is the vertex set of a monochromatic copy of $G^\star$ and such that for each j with $v_i \sim v_j$, $i < j \leq n$, V_i'', V_j'' are partite sets of a monochromatic complete bipartite graph.*

We prove Claim 2 by induction on $n-m$ using the pigeonhole principle. When $m = n$, we see that V_n' forms the vertex set of a monochromatic F, that in turn contains a monochromatic $G^\star$. Denote the vertex set of this $G^\star$ as V_n''. Assume that $V_m'', V_{m+1}'', \ldots, V_n''$ form vertex sets of monochromatic copies of $G^\star$ satisfying the conditions of Claim 2. Consider V'_{m-1} and recall from Claim 1 that each vertex in V'_{m-1} sends only red or only blue edges to each V_i'' with $v_{m-1} \sim v_i$, $i = m, \ldots, n$. If $v_{m-1} \sim v_n$ then at least half of the vertices in V'_{m-1} send monochromatic stars of the same color to V_n''. If $v_{m-1} \sim v_{n-1}$ then at least half of those send monochromatic stars of the same color to V''_{n-1}, and so on. So at least $2^{-(n-m)}|V'_{m-1}|$ vertices of V'_{m-1} send monochromatic stars of the same color to each V_i'' with $v_{m-1} \sim v_i$ for $i = m, \ldots, n$. We denote the set of these vertices by V^*_{m-1}. Since $\Gamma[V'_{m-1}]$ forms the vertex set of a monochromatic F, and $|V^*_{m-1}| \geq 2^{-(n-m)}|V'_{m-1}| \geq \epsilon_1 |V'_{m-1}|$, the definition of F implies that $\Gamma[V^*_{m-1}]$ contains a monochromatic copy of $G^\star$. We denote the vertex set of this copy by V''_{m-1}.

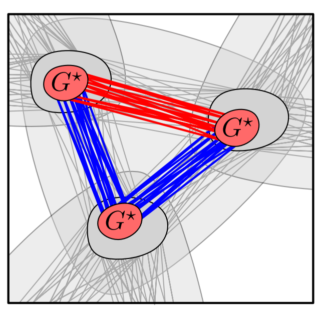

Figure 3: A set of k red copies of $G^\star$ corresponding to the k vertices of a hyperedge of $\mathcal{H}$. Here $k = 3$. The complete bipartite graph between any two of these k copies is also monochromatic.

Applying Claim 2 with $m = 1$, we see that each vertex v_i of $\mathcal{H}$ corresponds to a monochromatic copy of $G^\star$ with vertex set V_i'', such that all edges between any two such copies from a common hyperedge have the same color. Assigning the color of this $G^\star$ to v_i gives a 2-coloring of $V(\mathcal{H})$. Since $\chi(\mathcal{H}) > 2$, there is a monochromatic hyperedge, without loss of generality with red vertices $v_1, \dots, v_k$. Thus in Γ there are k red copies of $G^\star$ on vertex sets $V_1'', \dots, V_k''$, such that V_i'', V_j'' are partite sets of monochromatic complete bipartite graphs, for all i, j, $1 \le i < j \le k$, see Figure 3. If at least one such bipartite graph is red, then there is a red copy of G obtained by taking a red $G_1 \subseteq G^*$ from one part and a red $G_2 \subseteq G^*$ from the other part. So we can assume that all such bipartite graphs are blue, forming a complete k-partite graph with each part of size $|V(G)|$. Since $\chi(G) = k$, there is a blue copy of G. Thus $\Gamma \to G$. Since $\Gamma \not\to H$, we have that $G \overset{R}{\not\sim} H$. This concludes the proof of Theorem 3. □

Proof of Corollary 4. Let G and H be two graphs such that $\chi(G) \neq \chi(H)$ and $\chi(G) \le 2\omega(G) - 2$. Consider an arbitrary proper $\chi(G)$-vertex-coloring of G. Let V_1 denote the union of $\lfloor \frac{\chi(G)}{2} \rfloor$ color classes and $V_2 = V(G) \setminus V_1$. Since $\omega(G) \ge \frac{\chi(G)}{2} + 1$, every maximum clique contains a vertex from both sets V_i, $i = 1, 2$. Thus, G is clique splittable. So, if $\chi(H) > \chi(G)$ then $G \overset{R}{\not\sim} H$ by Theorem 3. If $\chi(H) < \chi(G)$, then $\chi(H) < \chi(G) \le 2\omega(H) - 2$ (where we assume $\omega(H) = \omega(G)$ by Lemma 13). Thus, H is clique-splittable with the same arguments as above. Hence $G \overset{R}{\not\sim} H$ by Theorem 3. □

4.2 Proof of Theorem 5

Our construction is similar to the one from Lemma 3.9 in [13].
Consider a connected graph H. We may assume $\omega(G) = \omega(H)$ by Lemma 13. Note that G is clique-splittable since $\omega(S) = 0$ and $\omega(G - S) < \omega(G)$. Further note that if G is bipartite, the conditions of the theorem imply that G is a union of a matching and a set of independent vertices. However, G is assumed to be connected, and thus it must be a single edge. Since a single edge is Ramsey isolated, we can assume that $\chi(G) \ge 3$.

In the first part of the proof, we assume that $H \not\subseteq G$ and $\chi(H) \ge \chi(G)$. Let $s = |S|$, $k = \chi(G) = \chi(H)$ and let m denote the size of a matching induced by two color classes of some proper k-vertex-coloring of G. Note that $m \ge 1$ since there is at least one edge between any two color classes. Further let $n = |V(G)|$, $\omega = \omega(G)$ and G_S be a vertex disjoint union of $G - S$ and S independent vertices, i.e., G_S is the graph obtained from G by deleting all edges incident to S. Then $\omega(G_S) < \omega$. Let G' be a vertex disjoint union of $m' = (k-2)(s-1) + m$ copies of G_S and G_0' be a vertex disjoint union of m' copies of G. Let $\epsilon = 2^{-|V(G')|k} = 2^{-m'nk}$. Let F be a graph with $F \overset{\epsilon}{\to} G'$ and $\omega(F) = \omega(G') < \omega$, which exists by Lemma 16. We construct a graph Γ by taking the vertex disjoint union of a copy of G_0' and $k-2$ copies of F denoted by $F_1, \dots, F_{k-2}$ and placing a complete bipartite graph between F_i and F_j, $1 \le i < j \le k-2$ and between F_i and G_0', $i = 1, \dots, k-2$, see Figure 4.

We shall show that $\Gamma \to G$, but $\Gamma \not\to H$. Color all edges within each F_i and within G_0' red and all other edges blue. Since $\omega(F) < \omega = \omega(H)$, $H \not\subseteq F$, and thus $H \not\subseteq F_i$, $i = 1, \dots, k-2$. Since $H \not\subseteq G$ and H is connected, we have that $H \not\subseteq G_0'$. Thus there is no red copy of H. On the other hand, the blue subgraph is a complete $(k-1)$-partite graph, but $\chi(H) \ge \chi(G) = k$. Thus there is no blue copy of H.
It remains to show that $\Gamma \to G$. Consider a 2-edge-coloring of Γ. Assume for the sake of contradiction that there is no monochromatic copy of G. We prove the following claim,

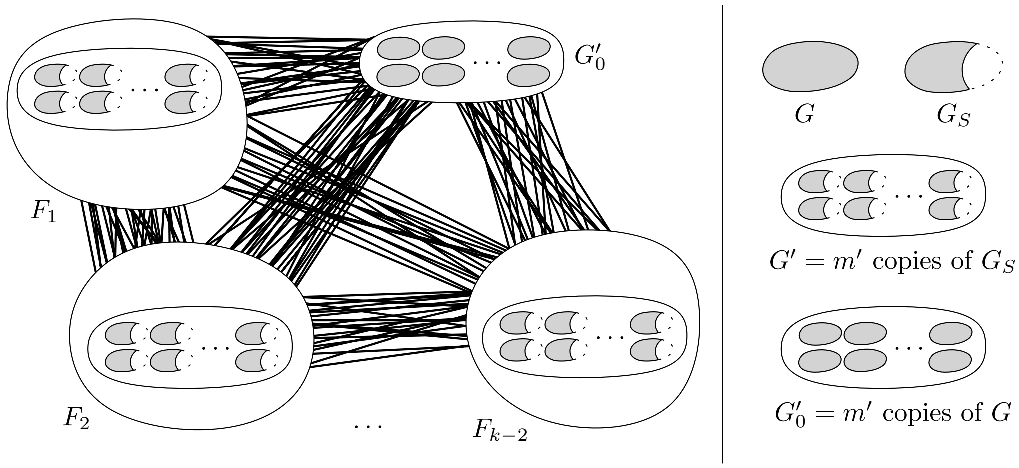

Figure 4: The graph Γ consisting of $k-2$ copies $F_1, \ldots, F_{k-2}$ of F and one copy of G'_0 and all possible edges between distinct copies. We have $F \overset{\epsilon}{\to} G'$, G' consists of m' copies of G_S and G'_0 consists of m' copies of G.

similar to Claim 1 in the proof of Theorem 3, by induction on p (up to renaming colors), see Figure 5 for an illustration.

Claim. *For each p, $1 \leq p \leq k-2$, and each i, $1 \leq i \leq p$, there is a red copy G'_i of G' in F_i. Moreover for each i, $0 \leq i < p$, each vertex v in G'_i and each j, $i < j \leq p$, all edges between v and G'_j are of the same color.*

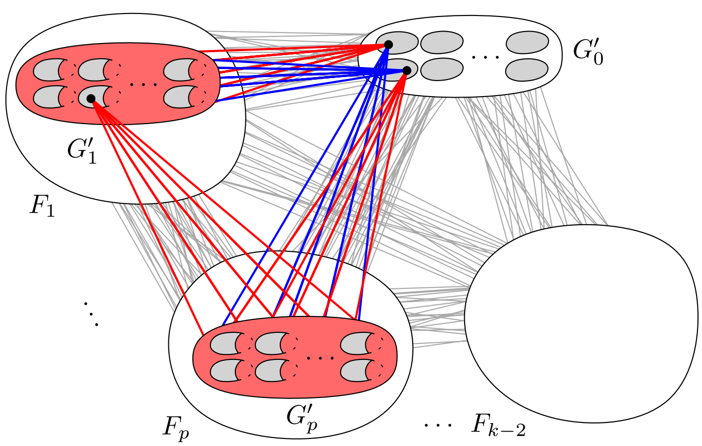

Figure 5: Illustrating the statement of the Claim.

There is a set V_1 of $2^{-m'n}|V(F_1)| \geq \epsilon|V(F)|$ vertices in F_1 such that for each vertex in G'_0 all edges to V_1 are of the same color by the Focusing Lemma (Lemma 12). Since $F \overset{\epsilon}{\to} G'$ there is a monochromatic copy G'_1 of G' in $F_1[V_1]$. Assume without loss of generality that G'_1 is red. This proves the Claim for $p = 1$, and for $k = 3$.

Suppose $k-2 \geq p \geq 2$ and there are red subgraphs $G'_1, \dots, G'_{p-1}$, satisfying the conditions of the Claim. We apply the Focusing Lemma to the complete bipartite graph with one part $V(G'_0) \cup \dots \cup V(G'_{p-1})$ and the other part $V(F_p)$. There is a set $V_p \subseteq V(F_p)$ of size $2^{-|V(G')|p}|V(F_p)| \geq \epsilon|V(F)|$, such that for each vertex v in $G'_0, \dots, G'_{p-1}$ all edges from v to V_p are of the same color. Since $F \stackrel{\epsilon}{\to} G'$ there is a monochromatic copy G'_p of G' in $F_p[V_p]$. It remains to prove that G'_p is red. Assume G'_p is blue. Consider the vertices of G'_1. All of them send monochromatic stars to G'_p. At most $s-1$ of these stars are blue, as otherwise these stars together with a blue subgraph of G'_p isomorphic to G_S form a blue copy of G. Since the number of vertex disjoint copies of G_S in G'_1 is $m' > s-1$, there is a red copy G^* of G_S in G'_1 whose vertices send only red stars to G'_p. Taking G^* and s vertices from G'_p gives a red copy of G, a contradiction. So we may assume that G'_p is red, which completes the proof of the Claim.

Consider the red G'_i, $1 \leq i \leq k-2$, given by the Claim for $p = k-2$. We say that a vertex in $V(G'_i)$, $i = 0, \dots, k-3$ is bad for G'_j if it sends a red star to G'_j, for some $j > i$. Since for each G'_j there are at most $s-1$ bad vertices, there are at most $(k-2)(s-1)$ bad vertices overall. Since G'_0 has $m' = (k-2)(s-1)+m$ vertex disjoint copies of G, there are at least $m \geq 1$ copies $G^0_1, \dots, G^0_m$ of G in G'_0 without bad vertices. Since each G'_i, $i = 1, \dots k-2$, has $m' = (k-2)(s-1)+m$ disjoint copies of G_S, there is at least one copy G''_i of G_S in G'_i without bad vertices, $i = 1, \dots, k-2$. Note that all G''_is are red, $i = 1, \dots, k-2$, all edges between them are blue, and all edges between a G''_i and G^0_j are blue, $i = 1, \dots, k-2$, $j = 1, \dots, m$, see Figure 6.

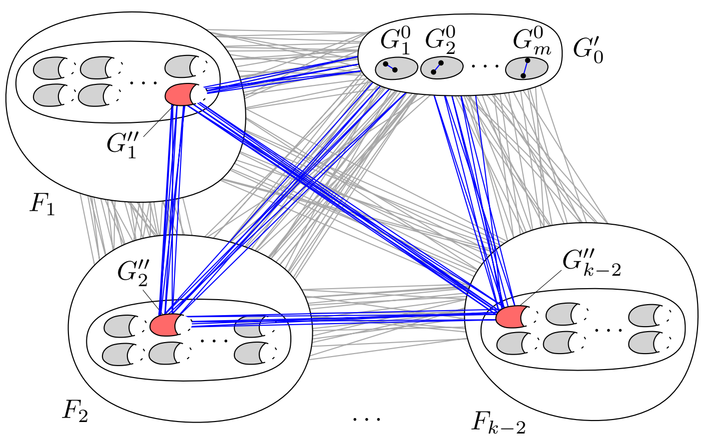

Figure 6: One red copy of G_S in each of $F_1, \dots, F_{k-2}$ and m copies $G^0_1, \dots, G^0_m$ of G in G'_0 where all edges between distinct copies are blue. If each G^0_i, $i = 1, \dots, m$, has a blue edge we find a blue copy of G in here.

By assumption each G^0_j, $j = 1, \dots, m$, has a blue edge, since otherwise there is a red copy of G. But then we can find a blue copy of G by identifying these blue edges with the matching of size m induced by the union of two color classes of G, picking the other vertices of these two color classes from G^0_1 and the vertices of the other $k-2$ color classes of G from G''_i, $i = 1, \dots, k-2$. Since $|V(G''_i)| = |V(G)|$, there is sufficient number of vertices for each color class. Altogether we have a contradiction to our assumption that there are no monochromatic copies of G. Hence $\Gamma \to G$. This concludes the proof in case when $H \not\subseteq G$ and $\chi(H) \geq \chi(G)$.

Now, in the second part of the proof, we assume that $H \subseteq G$. Then $\chi(H) \leq \chi(G)$. Since we assume that $\omega(G) = \omega(H)$, we have $\omega(H - S) < \omega(H)$. Thus, H is clique-splittable. Assume first that $\chi(H) < \chi(G)$. Then we have $G \overset{R}{\not\sim} H$ by Theorem 3, applied with roles of G and H switched. The last case to consider is when $\chi(H) = \chi(G)$ (and $H \subseteq G$). Now any proper $\chi(G)$-vertex-coloring of G with two color classes inducing a subgraph of a matching gives such a coloring of H, too. Thus, the first part of the proof applied with roles of G and H switched shows that $G \overset{R}{\not\sim} H$. □

4.3 Proof of Theorem 6

Proof of 6.1: Assume that $\chi(G) = \omega(G)$ and in some proper $\chi(G)$-vertex-coloring of G two color classes induce a matching. Then G satisfies the requirements of Theorem 5. If $\omega(H) \neq \omega(G)$ then $H \overset{R}{\not\sim} G$ by Lemma 13. So, we can assume that $\omega(H) = \omega(G)$. If $H \subseteq G$ or $\chi(H) \geq \chi(G)$, then $G \overset{R}{\not\sim} H$ by Theorem 5. If $H \not\subseteq G$ and $\chi(H) < \chi(G)$, then $\omega(H) = \omega(G) = \chi(G) > \chi(H)$. Thus $\chi(H) < \omega(H)$, a contradiction. □

Proof of 6.2: To see that a star $S = K_{1,t}$ is not Ramsey equivalent to any other graph, observe that $K_{1,2t-1}$ is a minimal Ramsey graph for S, but $K_{1,2t-1}$ is minimal Ramsey for neither any connected subgraph of S nor any connected graph that is not a subgraph of S. It remains to show that a path is not Ramsey equivalent to any other connected graph. Let $G = P_m$, a path on m vertices, and H be a connected graph not isomorphic to G. If H is a path of different length, then $G \overset{R}{\not\sim} H$ since $R(P_m) = m + \lfloor \frac{m}{2} \rfloor - 1$ [12] and hence $R(G) \neq R(H)$. So assume H is not a path. If H is not a tree, then by Lemma 19 we have $G \overset{R}{\not\sim} H$. Otherwise, H is a tree and $\Delta(H) \geq 3$. Then $R_\Delta(H) \geq 2\Delta(H) - 1 \geq 5$ [16], while an easy argument due to Alon *et al.* [4] shows that $R_\Delta(G) \leq 4$. Indeed, for any 4-regular graph F with girth at least $m+1$ we have $F \to P_m$ as follows. Considering any 2-edge-coloring of F, we see that since F has average degree 4 at least one color class has average degree at least 2, i.e., contains a cycle. Since $\text{girth}(F) \geq m+1$, this monochromatic cycle has length at least $m+1$, and thus contains P_m. □

Proof of 6.3: Figure 7 shows all non-trivial connected graphs on at most 5 vertices. Let $S = \{C_4, P_4 + e, C_4 + e, C_5, K_{2,3}, H_4, W_4\}$. Observe that any connected graph on at most 5 vertices which is not in S satisfies the conditions of Theorem 6.1 or 6.2 and thus is Ramsey isolated.
It remains to prove that each graph in $S' = \{C_4, P_4 + e, C_4 + e, C_5, H_4\} = S \setminus \{K_{2,3}, W_4\}$ is Ramsey isolated. Recall that $K_{2,3}$ and W_4 are excluded by assumption. First of all we consider $G \in \{C_4, P_4 + e, C_4 + e\}$. We have that $R(G) = 6$. Consider a connected graph H which is not isomorphic to G. If $|V(H)| \leq 5$ we have $G \overset{R}{\not\sim} H$ by Theorem 7. We claim that $R(H) > 6$ if $|V(H)| \geq 6$. Indeed, if H is a star then coloring the edges of a C_6 in K_6 red and all other edges blue does not yield a monochromatic H. If H is not a star, then color a copy of $K_{1,5}$ in K_6 red and all other edges blue. Then the red edges form a star and the blue connected subgraph contains only 5 vertices, so the coloring has no monochromatic H. Altogether we have $H \overset{R}{\not\sim} G$. Thus G is Ramsey isolated.

Next consider $G \in \{C_5, H_4\}$ and a connected graph H which is not isomorphic to G. If $|V(H)| \leq 5$ we have $G \overset{B}{\not\to} H$ by Theorem 7. If H is bipartite then $G \overset{B}{\not\to} H$ by Observation 2. We have that $R(G) \leq 10$ and we claim that $R(H) > 10$ if $|V(H)| \geq 6$ and H is not bipartite. Indeed color the edges of K_{10} with two vertex disjoint red copies of K_5 and all other edges blue. Then each connected component of the red subgraph has 5 vertices and the blue subgraph is bipartite. In particular there is no monochromatic copy of H. We conclude that $G \overset{B}{\not\to} H$, so G is Ramsey isolated. □

4.4 Proof of Theorem 7 and Remark 8 (Small Graphs)

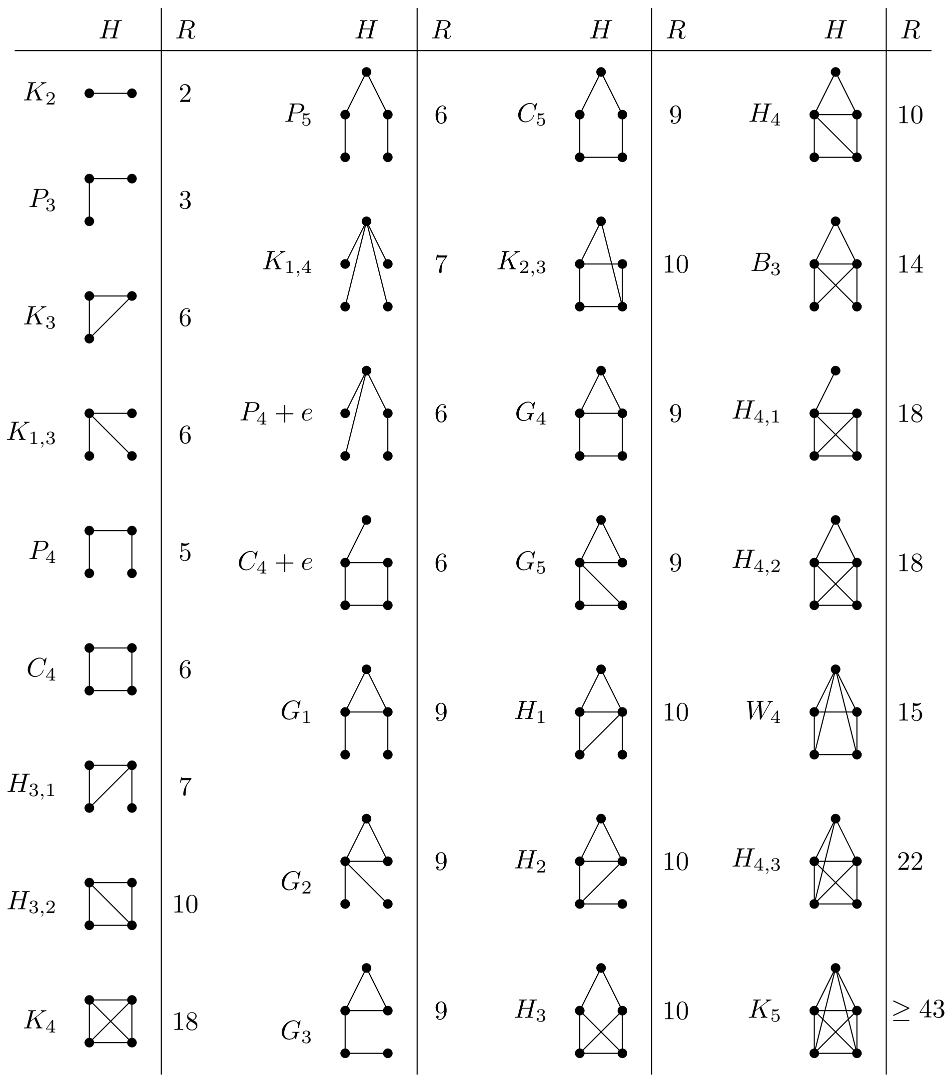

Figure 7: The connected graphs on at most 5 vertices with their Ramsey numbers $R = R(H)$.

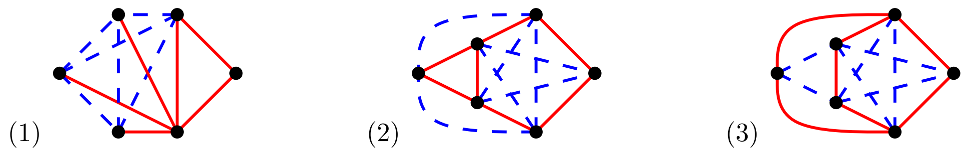

Figure 8: A coloring of $H_{5,2}$ without monochromatic P_5 (1), a coloring of $H_{5,4}$ without monochromatic C_4 (2) and a coloring of $H_{5,4}$ without monochromatic K_3 (3).

Proof of Theorem 7: Figure 7 shows all nontrivial connected graphs on at most 5 vertices. Let $S = \{C_4, P_4 + e, C_4 + e, C_5, K_{2,3}, H_4, W_4\}$. First of all note that Theorem 6.1 and 6.2 imply that each connected graph on at most 5 vertices, that is not in S, is Ramsey isolated. It remains to prove that $G \overset{R}{\not\sim} H$ for any pair of distinct graphs $G, H \in S$. We summarize known Ramsey numbers for all connected graphs on at most 5 vertices in Figure 7 where the values are taken from [8, 15]. We consider the graphs in S grouped according to their Ramsey number. Let $S_1 = \{C_4, P_4 + e, C_4 + e\}$, $S_2 = \{K_{2,3}, H_4\}$, and and $S_3 = \{C_5, W_4\}$. Here S_1 contains the graphs from S of Ramsey number 6, S_3 those of Ramsey number 10, and S_3 those of larger Ramsey number.
Since $G \overset{R}{\not\sim} H$ if $R(G) \neq R(H)$, we have $G \overset{R}{\not\sim} H$ for $G \in S_i$ and $H \in S_j$, $1 \leq i < j \leq 3$. It remains to distinguish the graphs within each set S_i from each other. Since $R(C_5) \neq R(W_4)$ the graphs in S_3 are distinguished by their Ramsey number. The graphs in S_2 are distinguished since $K_{2,3}$ is bipartite but H_4 is not, see Observation 2. It remains to consider S_1. We have $H_{5,4} \not\rightarrow C_4$ and $H_{5,4} \not\rightarrow C_4 + e$ due to the coloring given in Figure 8. However, we claim $H_{5,4} \rightarrow P_4 + e$. Indeed, consider a 2-edge-coloring of $H_{5,4}$ and a vertex u of degree 5. Without loss of generality u is incident to 3 red edges ux, uy and uz. Then there is a red $P_4 + e$ or all edges between $\{x, y, z\}$ and $V(H_{5,4}) \setminus \{u, x, y, z\}$ are blue. But in the latter case the vertices in $H_{5,4}$ other than u give a blue $P_4 + e$. In particular $H_{5,4} \rightarrow P_4 + e$ and thus $P_4 + e \overset{R}{\not\sim} C_4$ and $P_4 + e \overset{R}{\not\sim} C_4 + e$. Finally $\mathcal{R}_\delta(C_4) = 3$ [11] and $\mathcal{R}_\delta(C_4 + e) = 1$ [10] (for the latter see a remark in the conclusion of [10]). Thus $C_4 \overset{R}{\not\sim} C_4 + e$, which concludes the proof. □

Proof of Remark 8. Next we show that all but 11 pairs from the $\binom{31}{2} = 465$ pairs of distinct connected graphs on at most 5 vertices are distinguished by a small graph. For 447 such pairs of such graphs $\{G, H\}$ we give a distinguishing graph on $\min\{R(G), R(H)\}$ vertices, which is clearly best-possible. Among others, we will use graphs Γ and Γ' given in Figure 9 and Figure 10 respectively. The graph Γ' is obtained from K_7 by adding two independent vertices of degree 5 such that these two vertices have exactly 4 common neighbors.
First of all note that two graphs G, H of different Ramsey number are distinguished by K_n where $n = \min\{R(G), R(H)\}$ which is the smallest possible order of a distinguishing graph. This result distinguishes already lots of graphs using small graphs. It remains to distinguish pairs of connected graphs on at most 5 vertices of the same Ramsey number. Hence we need to consider the following sets of graphs corresponding to Ramsey number 6, 9, 10 and 18 respectively;
Ramsey number 6: $\{K_3, K_{1,3}, C_4, P_5, P_4 + e, C_4 + e\}$. We have $K_{1,5} \rightarrow K_{1,3}$ (pigeonhole principle) but $K_{1,5} \not\rightarrow K_3, C_4, P_5, P_4 + e, C_4 + e$ ($K_{1,5}$ does not contain these), $H_{5,2} \rightarrow P_4 + e$ (Lemma 20) but $H_{5,2} \not\rightarrow K_3, C_4, P_5, C_4 + e$ (Figure 8), $H_{5,4} \rightarrow P_5$ (Lemma 21) but $H_{5,4} \not\rightarrow K_3, C_4, C_4 + e$ (Figure 8), $K_{5,5} \rightarrow C_4, C_4 + e$ (Lemma 22) but $K_{5,5} \not\rightarrow K_3$ (since K_3 not

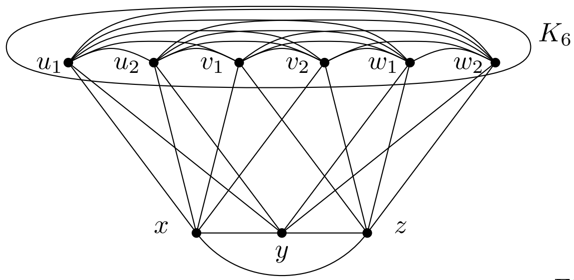

Figure 9: Graph Γ.

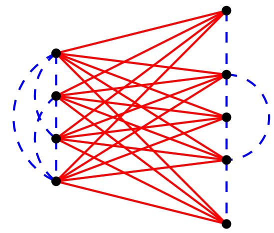

Figure 10: A coloring of Γ' without monochromatic G_3.

bipartite). It remains open to distinguish C_4 and $C_4 + e$ by some small graph.

Ramsey number 9: $\{G_1, G_2, G_3, C_5, G_4, G_5\}$. We have $\Gamma \to G_1, G_3$ (Lemma 24, 26) but $\Gamma \not\to G_2, G_5$ (Lemma 18), $\Gamma' \to G_1$ (Lemma 30) but $\Gamma' \not\to G_3$ (Figure 10), $H_{8,5} \to G_1, G_2, G_3$ (Lemma 28) but $H_{8,5} \not\to G_4, G_5, C_5$ (Figures 11, 12). We conjecture $H_{8,6} \to G_4$ (motivated by Lemma 32) but $H_{8,6} \not\to G_5$ (Figure 12). It remains open to distinguish C_5 from G_4 and G_5 by small graphs.

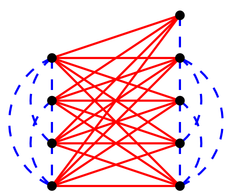

Figure 11: A coloring of $H_{8,5}$ without monochromatic G_4 and C_5.

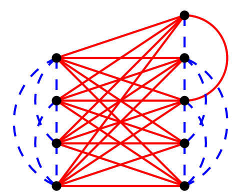

Figure 12: A coloring of $H_{8,6}$ without monochromatic G_5.

Ramsey number 10: $\{H_{3,2}, K_{2,3}, H_1, H_2, H_3, H_4\}$. We have $K_{12,12} \to K_{2,3}$ (Lemma 33) but the other graphs are not bipartite, $H_{9,6} \to H_{3,2}, H_1, H_2$ (Lemma 35) but $H_{9,6} \not\to H_3, H_4$ (Figures 13, 14). In this case it remains to distinguish each pair within the sets $\{H_{3,2}, H_1, H_2\}$ and $\{H_3, H_4\}$ with a small graph.

Ramsey number 18: $K_4, H_{4,1}, H_{4,2}$. We have not found any small distinguishing graph for the pairs in this case. $\square$

4.5 Proof of Theorem 10 (Trees)

Assume first that Conjecture 9 is true. Let T_k and T_ℓ be trees on k and ℓ vertices respectively, $k < \ell$. Note that $\mathrm{ex}(n, T_\ell) \geq \frac{\ell-2}{2} n - \ell^2$. Indeed, just take $\lfloor \frac{n}{\ell-1} \rfloor$ vertex disjoint copies of $K_{\ell-1}$. Then

$$\mathrm{ex}(n, T_k) \leq \frac{k-1-\epsilon}{2} n = \sqrt{n} \left(\frac{k-1-\epsilon}{2} \sqrt{n} \right) < \sqrt{n} \left(\frac{\ell-2}{2} \sqrt{n} - \ell^2 \right) \leq \sqrt{n}\, \mathrm{ex}(\sqrt{n}, T_\ell),$$

for sufficiently large n. Thus $\mathrm{ex}(n, T_k) < \mathrm{ex}(\sqrt{n}, T_\ell)\sqrt{n}$ and Lemma 19 implies that $T_k \overset{R}{\not\sim} T_\ell$.

Now, we shall prove the second statement of Theorem 10 without assuming the validity of Conjecture 9. Let T_k be a balanced tree on k vertices and T_ℓ be any tree on $\ell \geq k+1$

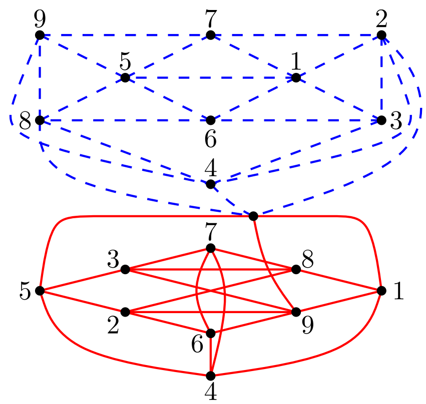

Figure 13: An edge-coloring of $H_{9,6}$ without monochromatic H_3 is obtained by identifying vertices of the same label.

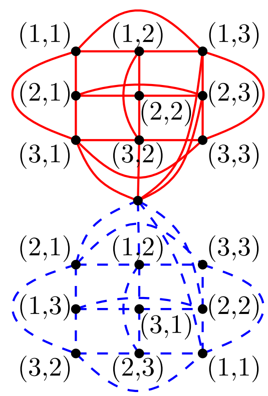

Figure 14: An edge-coloring of $H_{9,8}$ without monochromatic H_4 is obtained by identifying vertices of the same label.

vertices. Let G be a k-regular graph of girth at least k, which is known to exist [25]. We construct a bipartite k-regular graph B of girth at least k from G by taking for each v in G two vertices v_1, v_2 in B and for every edge uv in G the edges u_1v_2 and u_2v_1 in B. Finally, let $F = L(B)$ be the line graph of B. We shall show that $F \not\to T_\ell$ and $F \to T_k$.
As B is bipartite, $F = L(B)$ is a union of two graphs F_1, F_2, each is a vertex disjoint union of copies of K_k, where each clique in F_i corresponds to a set of edges incident to a vertex in the ith partite set of B, $i = 1, 2$. Note that a clique in F_1 intersects a clique in F_2 by at most one vertex and that each vertex in F belongs to two cliques, one from F_1 and one from F_2.
Coloring F_1 red and F_2 blue gives no monochromatic T_ℓ since each monochromatic connected component has $k < \ell$ vertices. Thus $F \not\to T_\ell$.

Next, we show that $F \to T_k$. Let vw be an edge of T_k such that the components of $T_k - vw$ rooted at v and w have order at most $\lceil \frac{k+1}{2} \rceil$. Consider any edge-coloring of F with colors red and blue. Note that $|V(F)| = |E(B)| = \frac{k}{2}|V(B)|$ and $|E(F)| = \binom{k}{2}|V(B)| = (k-1)|V(F)|$. Hence there are at least $\frac{k-1}{2}|V(F)|$ red edges or at least $\frac{k-1}{2}|V(F)|$ blue edges. (Note that Conjecture 9, if true, would imply that there is a red or blue copy of T_k, independent of the girth of B and whether T_k is balanced.) Assume without loss of generality that there are at least $\frac{k-1}{2}|V(F)|$ red edges. Consider the red subgraph G_r of F and a subgraph G of G_r of highest average degree. It follows that $\delta(G) \geq \lceil \frac{k-1}{2} \rceil$ and $|E(G)| \geq \frac{k-1}{2}|V(G)|$, and so $\Delta(G) \geq k-1$. If $\Delta(G) = k-1$, then G is $(k-1)$-regular and we can embed T_k into G greedily. So without loss of generality we have $\Delta(G) \geq k$.
Let x be a vertex of maximum degree in G, i.e., $\deg_G(x) \geq k$. It follows that x has incident red edges in both corresponding maximum cliques C_1, C_2 in F. Without loss of generality x has at least $\lceil \frac{k-1}{2} \rceil$ incident red edges in C_1. We embed v onto x, w onto a neighbor of x in G_r in C_2 and all neighbors of v different from w onto neighbors of x in G_r in C_1. Now we can greedily embed the subtrees $T_1, \ldots, T_a$ of $T_k - v$ with their roots at the designated vertices in G_r. Say T_1 is the subtree rooted at w. As $\delta(G) \geq \lceil \frac{k-1}{2} \rceil \geq |V(T_1)| - 1, \sum_{i=2}^{a} |V(T_i)|$ and B has girth greater than k, the embeddings of T_1 and $\bigcup_{i=2}^{a} T_i$ are in disjoint sets of cliques. It follows that $F \to T_k$. $\square$

4.6 Proof of Theorem 11 (Multicolor Ramsey numbers)

We prove the first part of the theorem. Let $m = R(G, G, F)$. Consider a 3-edge-coloring c of K_m without red or blue H and without green F, which exists as $m < R(H, H, F)$. Let Γ denote the graph obtained from K_m by removing all green edges under c. Thus $\Gamma \not\to H$ due to the coloring c restricted to Γ. But $\Gamma \to G$, since any 2-edge-coloring of Γ without monochromatic G can be extended by the green edges of c to an edge-coloring of K_m without red or blue G and without green F.

We prove the second statement by induction on k with $k = 2$ being obvious.
Let Γ be a graph such that $\Gamma \to_k G$, but $\Gamma \not\to_k H$, $k \geq 3$. Let c be a k-edge-coloring of Γ with no monochromatic H. Let a graph Γ' be obtained from Γ by deleting the edges of color 1. We have that $\Gamma' \not\to_{k-1} H$ since c restricted to Γ' is a $(k-1)$-coloring with no monochromatic H. We claim that $\Gamma' \to_{k-1} G$, which, if true, gives $G \overset{R}{\not\sim}_{k-1} H$ and by induction $G \overset{R}{\not\sim} H$, as desired.
Let us assume for the sake of contradiction that $\Gamma' \not\to_{k-1} G$, i.e., there is a $(k-1)$-edge-coloring c' of Γ' without monochromatic G. We see that there is a copy of G in color 1 of c, otherwise the coloring c'' of Γ that is the same as c' on Γ' and that colors all other edges with color 1 has no monochromatic G, a contradiction to the fact that $\Gamma \to_k G$. Repeating the argument above to all colors in c, we see that each of them contains G. More generally, we see that any edge-coloring of Γ with k colors avoiding monochromatic H must have a monochromatic G in each color. However, since $G \subseteq H$, Γ' has no monochromatic H under c', and hence the coloring c'' of Γ has no monochromatic H. Thus c'' must have monochromatic G in each color, however there is no monochromatic G in any of the colors $2, \ldots, k$, a contradiction. □

5 Conclusions

This paper addresses Ramsey equivalence of graphs and gives a negative answer to the question of Fox *et al.* [10]: "Are there two connected non-isomorphic graphs that are Ramsey equivalent?" for wide families of graphs determined by so-called "clique splitting" properties and chromatic number. In particular, we find an infinite family of graphs that are not Ramsey equivalent to any other connected graphs. This extends the only such known family consisting of all cliques, paths, and stars.

Replacing ω with any other "nice" Ramsey parameter, s, generalizes Theorems 3 and 5. Here, we say that a parameter s is a "nice" Ramsey parameter, if for any graph H and any Ramsey graph Γ for H, we have $s(\Gamma) \geq s(H)$ and for all $\epsilon > 0$ equality is attained for at least one Γ with $\Gamma \overset{\epsilon}{\to} H$. So, both ω and $-g_0$ (the negative of the odd girth) are nice Ramsey parameters.

There are many questions that remain open in this area. Even the following weaker question is very far from being understood: "Are there other graph parameters that distinguish graphs in a Ramsey sense?", i.e., is there a parameter s such that $s(G) \neq s(H)$ implies that $R(G) \overset{R}{\not\sim} R(H)$? Here, we showed that the chromatic number, χ, is very likely to be such a distinguishing parameter by proving this implication for graphs satisfying some additional properties. Interestingly enough, it is not clear, but most likely not true that $\chi(G) \neq \chi(H)$ implies that $\mathcal{R}_\chi(G) \neq \mathcal{R}_\chi(H)$. Indeed, $\mathcal{R}_\chi(K_4) = R(K_4) = 18$, but the positive answer to the Burr-Erdős-Lovász-Conjecture shows that there is a 5-chromatic graph G with $R_\chi(G) = 4^2 + 1 = 17$, so $\chi(K_4) < \chi(G)$ but $\mathcal{R}_\chi(K_4) > \mathcal{R}_\chi(G)$. We believe that there are infinitely many pairs of graphs G, H of different chromatic number and the same value for $\mathcal{R}_\chi$.

In this paper, we addressed the relation between other types of Ramsey numbers and Ramsey equivalence and got results in terms of multicolor Ramsey numbers. The following questions are open. If $R(G, F) \neq R(H, F)$ for some graph F, does this imply $G \overset{R}{\not\sim} H$? For any two non-isomorphic graphs G, H, is there an integer k such that $R_k(G) \neq R_k(H)$? Here $R_k(G)$ is the smallest integer n such that any coloring of edges of K_n with k colors contains a monochromatic copy of G. For example, we see that $R(P_4 + e) = R(K_{1,3}) = 6$, and $R_k(P_4 + e) > 2k + 2 = R_k(K_{1,3})$, for odd $k > 3$. Another question is whether the fact that $R_k(G) \neq R_k(H)$ for some k implies that $G \overset{R}{\not\sim} H$. We answered the last question in positive only when G is a subgraph of H.

Cliques play a special role in Ramsey theory and got a particular attention in Ramsey equivalence. Still, it is not clear for what graphs is a clique a minimal Ramsey graph. Specifically, if the size Ramsey number $R_e(H)$ is less than $\binom{R(H)}{2}$, does it imply that $K_{R(H)}$ is not a minimal Ramsey graph for H?

A positive answer to the following question would immediately give a negative answer to the question of Fox *et al.*: "Is there a graph in the Ramsey class of any connected graph G, that does not belong to the Ramsey class of any other connected graph, except for subgraphs of G?"

It is also not clear how small could be a distinguishing graph for two not Ramsey equivalent graphs. Is there a function f such that for any two not Ramsey equivalent graphs G, H, the smallest order of their distinguishing graph is at most $f(R(G), R(H))$?

Finally, we show that two trees of different order are not Ramsey equivalent provided that the Erdős-Sós-Conjecture is true or if one of the trees is balanced. We do not know whether there are two Ramsey equivalent non-isomorphic trees on the same number of vertices.

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A Small Distinguishing Graphs

Lemma 20. $H_{5,2} \to P_4 + e$.

Proof. Let u denote the vertex of degree 2 in $H_{5,2}$ and let let $e = vw$ denote the edge incident to both neighbors of u. Let x, y, z denote the other vertices. Assume there is a 2-edge-coloring of $H_{5,2}$ without monochromatic $P_4 + e$. Without loss of generality assume the edge uv is blue.
If (**Case 1**) e and two more edges vx, vy incident to v are red, then either there is a red $P_4 + e$ containing these edges or zx, zy, zw and uw are blue. But then these form a blue copy of $P_4 + e$.
If (**Case 2**) e and at most one other edge incident to v is red, then assume vx, vy are blue. Then the edges wx, wy, zx, zy and uw must be red, but these contain a copy of $P_4 + e$.
If (**Case 3**) e and vx are blue, then wy, wz, xy and xz must be red. But then the edges vy, vz and uw must be blue, but contain a copy of $P_4 + e$.
If (**Case 4**) e is blue but all the edges vx, vy, vz are red, then wx, wy, wz are blue. But together with uv and vw this is a blue $P_4 + e$. □

Lemma 21. $H_{5,3} \to P_5$.

Proof. Assume there is 2-edge-coloring of $H_{5,3}$ without monochromatic P_5. Let u denote the vertex of degree 3 in $H_{5,3}$, let x, y, z denote its neighbors and v, w the remaining vertices. There are two edges of the same color incident to u, assume ux, uy are red. Since there is no monochromatic $K_{2,3}$ (it contains P_5) there is at least one edge from $\{x, y\}$ to $\{v, w, z\}$ in red.
If (**Case 1**) there are $r, r' \in \{v, w, z\}$ such that the edges xr, yr' are red, then $r = r'$. Then all edges from $\{x, y\}$ to $\{v, w, z\} \setminus \{r\}$ are blue. But then the edge from r to $\{v, w, z\} \setminus \{r\}$ can be neither red nor blue.
If (**Case 2**), without loss of generality, x has only blue edges to $\{v, w, z\}$, then yp is red for some $p \in \{v, w, z\}$ and all edges from p to $\{v, w, z\} \setminus \{p\}$ are blue. This yields a blue C_4 on $\{x, v, w, z\}$. Since all edges which are incident to this C_4 (but not contained) are red we can find a red P_5. □

Lemma 22. $K_{5,5} \to C_4$.

Proof. Consider a 2-edge-coloring of the edges of $K_{5,5}$ and vertex v. Let V denote the partite set of $K_{5,5}$ containing v. Then v is incident to three edges vx, vy, vz of the same color, say red. From each of the four vertices in $V \setminus \{v\}$ at most one edge to $\{x, y, z\}$ is red, otherwise there is a red $K_{2,2}$. But then there are two vertices in $\{x, y, z\}$ and two vertices in $V \setminus \{v\}$ forming a blue $K_{2,2}$ by pigeonhole principle. □

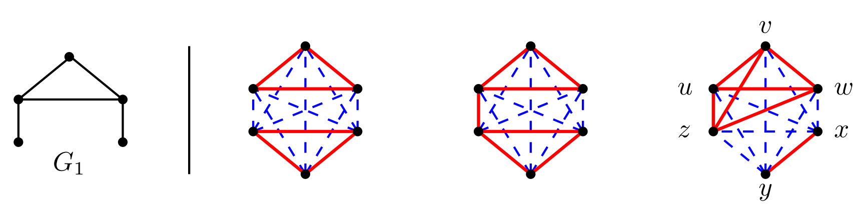

Figure 15: All possible 2-colorings of K_6 without monochromatic G_1.

Lemma 23. *Each 2-edge-coloring of K_6 without monochromatic G_1 equals to one of the colorings given in Figure 15, up to isomorphism and renaming colors.*

Proof. Consider a 2-edge-coloring of K_6 on vertices u, v, w, x, y, z without monochromatic G_1. We may assume that $K = \{u, v, w\}$ forms a red K_3 since $R(K_3) = 6$. Clearly there are no two independent red edges between K and $K^c = \{x, y, z\}$. If there is a vertex in K incident to at least two red edges to K^c, then all edges between the two other vertices in K and K^c are blue. Thus these blue edges form a blue $K_{2,3}$, and hence no edge in K^c is blue. But if all edges in K^c are red, then there is a red K_3 with two vertices in K^c and one vertex in K and a pending red edge in K and a pending in K^c. So we may assume that at most one vertex in K^c is adjacent to K in red, say z. Then $\{x, y\}$ and K form a blue $K_{2,3}$. This shows that xy is red. We consider the cases how many edges between x, y and z are red.
If (**Case 1**) xy is the only red edge, then $\{x, y\}$ and $\{u, v, w, z\}$ induce a blue $K_{2,4}$ and any additional blue edge within this $K_{2,4}$ yields a blue G_1. Hence the red edges form a K_4 plus disjoint K_2, which corresponds to the rightmost coloring of Figure 15.
If (**Case 2**) there are at least two red edges, then z is not part of any red K_3 on $\{u, v, w, z\}$, since this K_3 would have a red pending edge in K and another one in K^c. Hence there is at most one red edge from z to K. Thus there is a blue copy of $K_{3,3} - e$ between K and K^c. This shows that no edge in K^c is blue and the coloring corresponds to the left or the middle coloring in Figure 15. □

Lemma 24. $\Gamma \to G_1$.

Proof. Assume c is a 2-coloring of the edges of Γ, labeled like in Figure 9, without monochromatic G_1. Let K denote the copy of K_6 in Γ. Due to Lemma 23, c restricted to K is isomorphic to one of three colorings of K_6 given in Figure 15. We will distinguish cases based on the coloring of K under c.
(**Case 1**:) The red subgraph of K under c consists of two disjoint K_3 and the blue edges in K form a copy of $K_{3,3}$. If one of these blue edges is contained in a blue K_3 with a vertex from $\{x, y, z\}$, then there is blue G_1. Due to the construction of Γ we can find three vertex disjoint copies of K_3 each with exactly one vertex from each of the red K_3 in K and exactly one vertex from $\{x, y, z\}$. Since there is a red edge from K to $\{x, y, z\}$ in each of these, one of the red K_3 in K has two independent pending red edges. This gives a red G_1, a contradiction.
(**Case 2**:) The red subgraph of K under c consists of two disjoint K_3 connected by a single edge e. Then all edges from K to $\{x, y, z\}$ are blue if not adjacent to e. Then there are two vertices in K, not incident to e, each having two blue edges to $\{x, y, z\}$ but only one common neighbor in $\{x, y, z\}$. Since they are connected by an blue edge in K, this gives a blue G_1, a contradiction.
(**Case 3**:) The red subgraph of K_6 consists of K_4 and a single edge e. Then all edges from this K_4 to $\{x, y, z\}$ are blue. The blue edges in K form $K_{2,4}$. Again, if one of these blue edges in K forms a blue triangle with a vertex from $\{x, y, z\}$, then there is a blue G_1. Thus all edges from e to $\{x, y, z\}$ are red. If $e \notin \{v_1v_2, u_1u_2, w_1w_2\}$, then e together with $\{x, y, z\}$ forms a red copy of G_1. So assume $e = w_1w_2$. If the edge xy is blue, then $\{x, y, u_1, u_2, v_1\}$ gives a blue G_1. If it is red, then $\{x, y, z, w_1, w_2\}$ gives a red G_1, a contradiction.
Altogether we proved that there is no 2-edge-coloring of Γ without monochromatic G_1. □

Lemma 25. *Each 2-edge-coloring of K_6 without monochromatic G_3 equals to one of the colorings given in Figure 16, up to isomorphism and renaming colors.*

Proof. Consider a 2-edge-coloring of K_6 on vertices u, v, w, x, y, z without monochromatic G_3. We may assume that $K = \{u, v, w\}$ forms a red K_3.

If all edges from K to K^c are blue, then no edge among K^c is blue. Thus the coloring corresponds to the left one in Figure 16. So assume the edge uz is red. If $\{u, v, w, z\}$ forms a red K_4, then all edges from this K_4 to $\{x, y\}$ are blue. No matter which color is assigned to xy, the coloring has no monochromatic G_3 and corresponds to the middle or right color in Figure 16.

So assume further that $\{u, v, w, z\}$ is not a red K_4 (but uz is still red), without loss of generality wz is blue. Since uz is red, xz and yz are blue.

If (**Case 1**) wx is blue, then $\{w, x, z\}$ is a blue K_3 with pending blue edge yz. Thus uy, vy are red. But then wy needs to be blue (otherwise $\{v, w, y\}$ is red K_3 with pending red path vuz) and there is a blue K_4. Thus the coloring corresponds to the middle or right coloring of Figure 16 with switched colors, as argued above.

If (**Case 2**) wx is red, then xy is blue and vx, vz are blue. Then vy needs to be red, since $\{v, x, y\}$ is a blue K_3 with pending blue path xzw otherwise. Then wy is blue, since otherwise $\{v, w, y\}$ is a red K_3 with pending red path vuz otherwise. But now $\{w, y, z\}$ is a blue K_3 with pending blue path zxv, a contradiction. □

Lemma 26. $\Gamma \to G_3$.

Proof. Assume c is a 2-coloring of the edges of Γ, labeled like in Figure 9, without monochromatic G_3. Let K denote the copy of K_6 in Γ. Due to Lemma 25, c restricted to K is isomorphic to one of three colorings of K_6 given in Figure 16. We will distinguish cases based on the coloring of K under c.

(**Case 1**:) The red subgraph of K_6 consists of two disjoint K_3. Then the blue edges in K form a copy of $K_{3,3}$. If one of these blue edges is contained in a blue K_3 with a vertex from $\{x, y, z\}$, then there is blue G_3. On the other hand no vertex in $K^c = \{x, y, z\}$ sends a red edge to each of the red K_3s in K. Since there are 4 edges from each vertex in $\{x, y, z\}$ to K, each is incident to at least one red edge and one blue edge. If one of the edges induced by $\{x, y, z\}$ is red, then there is a red G_3 with a red K_3 from K and an edge between them. So $\{x, y, z\}$ induces a blue K_3 which forms a blue G_3 with an edge to K and another contained in K.

(**Case 2**:) The red subgraph of K_6 consists of a red K_4 only. Then no edge incident to this K_4 is red. Let a, b denote the vertices in K not contained in the red K_4. Then every blue edge between $\{x, y, z\}$ and the red K_4 is part of a blue G_3 together with a, b, and another vertex in K.

(**Case 3**:) The red subgraph of K_6 consists of a red K_4 and a disjoint red edge e. Again all edges incident to the red K_4 are blue and no blue edge in K is contained in a blue K_3 with a vertex from K^c. Thus all edges from e to K^c are red. Assume $e \notin \{v_1v_2, u_1u_2, w_1w_2\}$, say it

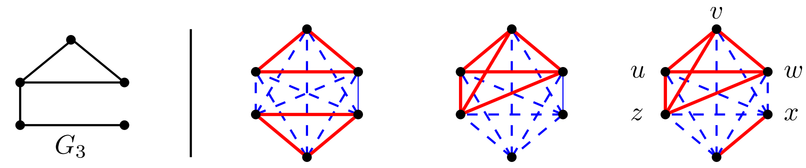

Figure 16: All possible 2-colorings of K_6 without monochromatic G_3.

is v_2w_2. If xy is blue then $\{x, y, u_1, v_1, z\}$ forms a blue G_3. If xy is red then $\{x, y, v_2, w_2, z\}$ gives a red G_3. So assume $e = w_1w_2$. If the edge xy is blue, then $\{x, y, u_1, v_1, z\}$ forms a blue G_3. If it is red, then $\{x, y, z, w_1, w_2\}$ gives a red G_3. □

Lemma 27. *For each $i \in \{1, 2, 3\}$ a 2-coloring of K_8 does not have a monochromatic G_i, if and only if one of the color classes induces two vertex disjoint copies of K_4.*

Proof. First of all note that an 2-edge-coloring of K_8 with one color class inducing two vertex disjoint K_4's does not contain a monochromatic copy of G_i for each $i \in \{1, 2, 3\}$.

On the other hand, consider an arbitrary 2-edge-coloring of K_8 without a monochromatic copy of G_i for a fixed $i \in \{1, 2, 3\}$. There is a monochromatic copy K of $H_{3,1}$, say in red (i.e. a red copy of K_3 with a pending edge), since $R(H_{3,1}) = 7$.

Suppose there is no monochromatic G_1. Then none of the two vertices of degree 2 in K is incident to another red edge in K_8. Thus there is a blue copy of $K_{2,4}$. Then the part with four vertices in this $K_{2,4}$ contains no further blue edge and induces a red K_4. But then no edge incident to this K_4 is red, and there is a blue copy of $K_{4,4}$. Since no other edge might be blue then, there are two disjoint red K_4.

Suppose there is no monochromatic G_2. The vertex of degree 3 in K has no other incident red edge. So it is the center of a blue $K_{1,4}$. The degree 1 vertices in this copy of $K_{1,4}$ do not induce a blue edge, so they induce a red K_4. But then no edge incident to this K_4 is red and there is a blue $K_{4,4}$ between K and the other vertices. As argued above the red edges form two disjoint K_4s and the blue edges form $K_{4,4}$.

Suppose there is no monochromatic G_3. Let K^c denote the set of vertices not in K. Then any edge connecting the vertex v of degree 1 in K to a vertex in K^c is blue. Assume there is a red edge from K to a vertex $u \in K^c$. Then any edge connecting u to a vertex in $K^c \setminus \{u\}$ is blue. Then each edge e within $K^c \setminus \{u\}$ or from $K^c \setminus \{u\}$ to $K \setminus \{v\}$ is red, since otherwise there is a blue G_3 spanned by u, v and e. But then there is red G_3, a contradiction.
So all edges between K and K^c are blue. Then there is no other blue edge and the red edges form two disjoint copies of K_4. □

Lemma 28. *$H_{8,5} \to G_i$ for each $i \in \{1, 2, 3\}$.*

Proof. Assume there is a 2-edge-coloring of $H_{8,5}$ without monochromatic G_i for some $i \in \{1, 2, 3\}$. We may assume that within the copy of K_8 the red edges form two disjoint K_4 with a blue $K_{4,4}$ in-between by Lemma 27. Let v denote the vertex of degree 5. Then v has only blue incident edges since every neighbor of v is part of a red K_4. But v has a neighbor in each of the red K_4's. Thus v together with these two vertices forms a blue K_3 which is contained in a blue copy of G_i for all $i \in \{1, 2, 3\}$, a contradiction. □

Lemma 29. *A 2-edge-coloring of K_7 does not have monochromatic G_1, if and only if one of the color classes induces vertex disjoint copies of K_3 and K_4.*

Proof. The proof is very similar to the proof of Lemma 27. □

Lemma 30. *$\Gamma' \to G_1$.*

Proof. Assume there is a 2-edge-coloring of Γ' without monochromatic G_1. Due to Lemma 29 we may assume that the copy of K_7 in Γ' is colored such that the blue edges induce a copy of $K_{3,4}$ and red consists of two disjoint copies of K_4 and K_3. Let K denote the red K_3 and u, v the two vertices of degree 5 in Γ'. Then each edge from $\{u, v\}$ to the red K_4 is blue and there are at least two such edges incident to each of u, v. Thus each edge from u or v to K is red, since there is a blue G_1 otherwise. Due to construction of Γ' there are two independent edges from K to $\{u, v\}$ and thus a red G_1, a contradiction. □

Conjecture 31. *A 2-edge-coloring of K_8 does not have monochromatic G_4, if and only if one of the color classes induces two disjoint copies of K_4 with at most one edge of same color in-between (i.e. the other color spans $K_{4,4}$ or $K_{4,4} - e$).*

Lemma 32. *If Conjecture 31 holds, then $H_{8,6} \to G_4$.*

Proof. Assume there is a 2-edge-coloring of $H_{8,6}$ without monochromatic G_4. We assume that the coloring of K_8 in $H_{8,6}$ has two disjoint red K_4 connected by at most one red edge according to Conjecture 31. Let v denote the vertex of degree 6. It is incident to at most one red edge to to each of the red K_4. Thus there is a blue K_3 with v and one vertex from each red K_4. But this forms a blue G_4 together with some of the other blue edges, a contradiction. □

Lemma 33. $K_{12,12} \to K_{2,3}$.

Proof. Consider a 2-coloring of the edges of $K_{12,12}$ and vertex u. Let V denote the partite set of $K_{12,12}$ containing u. Then u is incident to five edges uv, uw, ux, uy, uz of the same color, say red. From each of the 11 vertices in $V \setminus \{u\}$ at most two of the edges to $\{v, w, x, y, z\}$ are red, otherwise there is a red $K_{2,3}$. This means that there are three blue edges between each of the vertices in $V \setminus \{u\}$ and $\{v, w, x, y, z\}$. There are 10 sets of size 3 in $\{v, w, x, y, z\}$ and 11 vertices in $V \setminus \{v\}$. Hence there are two vertices in $V \setminus \{v\}$ and three vertices in $\{v, w, x, y, z\}$ forming a blue $K_{2,3}$ by pigeonhole principle. □

Lemma 34. *A 2-edge-coloring of K_9 does not have a monochromatic copy of $H_{3,2} = K_4 - e$ if and only if each color class is isomorphic to the Cartesian product $K_3 \times K_3$.*

Proof. First of all observe that $K_3 \times K_3$ does not contain a copy of $H_{3,2}$ since every edge is contained in exactly one copy of K_3. Moreover the complement of $K_3 \times K_3$ (as a subgraph of K_9) is isomorphic to $K_3 \times K_3$. Hence the edges of K_9 can be 2-colored without monochromatic $H_{3,2}$ using two edge disjoint copies of $K_3 \times K_3$.
On the other hand consider a 2-edge-coloring c of K_9 without monochromatic $H_{3,2} = K_4 - e$. We will assign labels $v_{i,j}$, $1 \leq i, j \leq 3$, to the vertices of K_9 such that this labeling corresponds to an arrangement of the vertices in a 3×3 grid where the red subgraph spans all rows and columns and all other edges are blue.
There is a monochromatic copy of G_5 under c, say in red, since $R(G_5) = 9$, see Figure 7. Let $K = \{v_{1,1}, v_{1,2}, v_{1,3}, v_{2,1}, v_{3,1}\}$ denote the vertices of this G_5 such that $v_{1,1}$ is the vertex of degree 4 and the edges of this red G_5 span the first row and first column in the grid, see Figure 18. Observe that no edge spanned by K is red except for the edges in the red G_5. Indeed if another edge is red, then there is a red $H_{3,2}$ in K. Let K^c denote the vertices not in K. We will use the following claim.

Claim 1. *If C is a red copy of K_3 and uv is a vertex disjoint blue edge, then there is a vertex x in C such that xu and xv are blue, and there are two independent red and two independent blue edges between $C - x$ and uv. See Figure 17 for an illustration.*

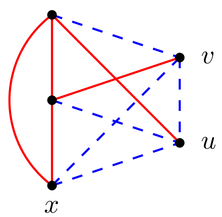

Figure 17: The unique 2-coloring of K_5 (up to isomorphism) without monochromatic $H_{3,2}$ provided there is a red K_3 (left) and a disjoint blue edge (right).

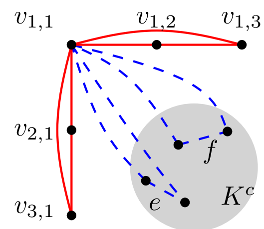

Figure 18: The partial labeling of vertices of K_9 under a 2-coloring without monochromatic $H_{3,2}$ in the proof of Lemma 34, with solid red and dashed blue edges.

Indeed, there is at most one red edge between each vertex in $\{u, v\}$ and C and for at most one vertex in C both edges to uv are blue. Hence for exactly one vertex in C both edges to uv are blue and there are exactly two further independent blue edges between C and uv. This proves Claim 1.
By assumption the four vertices in K^c do not induce a monochromatic $H_{3,2}$ and hence there are at least two blue edges e, f. Let C_1, C_2 denote the red copies of K_3 in K. We will apply Claim 1 to each of the pairs $\{e, C_1\}$, $\{e, C_2\}$, $\{f, C_1\}$, $\{f, C_2\}$. There is a vertex x_i in C_i, $i = 1, 2$, such that both edges between x_i and a blue edge in K^c are blue by Claim 1. Then $x_1 = x_2 = v_{1,1}$, since otherwise there is blue copy of $H_{3,2}$. Hence the blue edges in K^c are independent, since two adjacent blue edges together with $v_{1,1}$ form a blue $H_{3,2}$. Thus e and f are the only blue edges in K^c. See Figure 18 for the partial labeling. Furthermore there are two independent red edges and two independent blue edges from each of the edges e and f to each $C_i - v_{1,1}$, $i = 1, 2$, by Claim 1. It remains to find labels for the vertices in e and f.

Claim 2. *For any two vertices $u \in \{v_{1,2}, v_{1,3}\}$, $v \in \{v_{2,1}, v_{3,1}\}$ there is exactly one vertex w in K^c such that uw and vw are red.*

Indeed, assume there are two such vertices w, w' in K^c for some pair u,v. Then the edge ww' is red by Claim 1 and there is a red $H_{3,2}$. Thus there is at most one such vertex. Assume there is no such vertex in K^c for some pair. Then there is a red $H_{3,2}$, since there are two independent red edges between each of e and f and each C_i, $i = 1, 2$, by Claim 1, a contradiction. This proves Claim 2.
Let $v_{2,2}$ denote the vertex which is adjacent to $v_{1,2}$ and $v_{2,1}$ in red which exists by Claim 2. Without loss of generality assume $v_{2,2}$ is incident to e. Let $v_{3,3}$ denote the other vertex incident to e. Due to Claim 1 applied to e and C_1 and C_2, the edges $v_{3,3}v_{1,3}$ and $v_{3,3}v_{3,1}$ are red and the edges $v_{2,2}v_{1,3}$, $v_{2,2}v_{3,1}$, $v_{3,3}v_{1,2}$ and $v_{3,3}v_{2,1}$ are blue. With the same arguments we choose $f = v_{3,2}v_{2,3}$ accordingly. This shows that the red color class is isomorphic to $K_3 \times K_3$. □

Lemma 35. $H_{9,6} \to H$ *for each* $H \in \{H_{3,2}, H_1, H_2\}$.

Proof. Consider a 2-edge-coloring of $H_{9,6}$. Let v denote the vertex of degree 6. We shall show that it contains each of the graphs from $\{H_{3,2}, H_1, H_2\}$ as a monochromatic subgraph. Either there is a monochromatic copy of $H_{3,2}$ in the copy of K_9 in $H_{9,6}$ or we may assume by Lemma 34 that the K_9 in $H_{9,6}$ is an edge disjoint union of a red $K_3 \times K_3$ and a blue

$K_3 \times K_3$. Each edge in K_9 belongs to a unique monochromatic triangle. If v sends two blue edges to vertices u, w, where uw is blue, then the blue triangle containing uw in K_9 together with v form a blue $H_{3,2}$. Thus, we may assume that neighborhood of v via blue edges forms a red clique, and, similarly, its neighborhood via red edges forms a red clique. Since degree of v is 6, and the largest monochromatic clique in K_9 is a triangle, these cliques must be triangles. However, there are no two disjoint red and blue triangles in K_9, so we arrive at a contradiction. Thus, there is a monochromatic $H_{3,2}$.

Assume that the monochromatic copy K of $H_{3,2}$ is red. First, we assume that K does not contain v. Let K^c denote the set of vertices from K_9 that are not in K. If there is a red edge between a vertex of degree 3 of K and K^c, we have a monochromatic H_1. If there is a red edge between a vertex of degree 2 of K and K^c, we have a monochromatic H_2. If all edges between degree 3 vertices of K and K^c are blue and there are two adjacent blue edges in K^c, then there is a blue copy of H_1. If all edges between degree 3 vertices of K and K^c are blue and there are no two adjacent blue edges in K^c, then K^c forms a red K_5 minus a matching, and thus contains a copy of H_1. If all edges between degree 2 vertices of K and K^c are blue, then there is a blue copy of H_2 or there is no blue edge induced by K^c, In the latter case K_c induces a red K_5 that contains a red copy of H_2.
Now, assume that any monochromatic $H_{3,2}$ contains v, i.e., there is no monochromatic $H_{3,2}$ in a copy of K_9 of $H_{9,6}$. Hence the coloring of K_9 is like it is described in Lemma 34. Then, it is easy to see that K and an appropriate edge of K_9 form a monochromatic copy of H_1 and, similarly, a monochromatic copy of H_2. □