Graphs having small number of sizes on induced k-subgraphs

Maria Axenovich

Department of Mathematics
Iowa State University
USA
axenovic@math.iastate.edu

and

József Balogh*

Department of Mathematics
University of Illinois at Urbana-Champaign
USA
jobal@math.uiuc.edu

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Abstract

Let G be a graph on n vertices, k, ℓ are integers such that $2\ell \leq k \leq n - 2\ell$, n is large enough. Let

$$\nu_k(G) = |\{|E(H)| : H \text{ is an induced subgraph of } G \text{ on } k \text{ vertices}\}|.$$

We show that if $\nu_k(G) \leq \ell$ then G has a complete or an empty subgraph on at least $n - \ell + 1$ vertices, and a homogeneous set of order at least $n - 2\ell + 2$. These results are sharp.

1 Introduction

Let G be a graph on n vertices, let k be an integer, $1 \leq k \leq n$. A *k-subgraph* of G is an induced subgraph of order k. Let $i(G)$ be a total number of isomorphism classes on induced subgraphs of G. Let

$$\nu_k(G) = |\{|E(H)| : H \text{ is a } k\text{-subgraph of } G\}|.$$

A *trivial* set is a subset of vertices in a graph inducing either an empty or a complete graph. The parameter $i(G)$ was investigated in multiple papers in attempts to find the

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maximum of $i(G)$ over all graphs on n vertices, see [8], [9], and in research determining the structure of graphs G, for which $i(G)$ is small. It has been shown in [2], [11] that graphs with "small" $i(G)$ have a large trivial subset of vertices, in particular, if $\varepsilon < 10^{-21}$ and $i(G) \leq \varepsilon n^2$ then $t(G) \geq (1-4\varepsilon)n$. When all k-subgraphs are isomorphic or simply have the same size, one can determine the structure of G as follows.

Proposition 1 ([1], [6]). *Let G be a graph on n vertices such that $\nu_k(G) = 1$. If $2 \leq k \leq n-2$ then G is either a complete or an empty graph.*

In this note, we continue investigating the structure of graphs G such that $\nu_k(G)$ is small. We show that it exhibits the same behavior, as the structure of graphs with small $i(G)$. In particular, we show that G in this case must have a large trivial subset, where "large" is $|V(G)| - c$, for a constant $c = c(\nu_k(G))$. Moreover, we prove the existence of a large *homogeneous* set, i.e., a subset, X, of vertices such that for any $u, v \in X$, $N(u) \setminus \{v\} = N(v) \setminus \{u\}$. Our main result is the following.

Theorem 1. *Let G be a graph on n vertices, let ℓ be a positive integer, and k be any integer satisfying $2\ell \leq k \leq n-2\ell$. Then for large enough n, if $\nu_k(G) \leq \ell$ then G has a trivial vertex set of size at least $n-\ell+1$ and a homogeneous vertex set of size at least $n-2\ell+2$. These results are sharp.*

The graph with $\ell - 1$ edges and maximal degree 1 (or its complement) shows the bounds on the sizes of trivial and homogeneous sets are best possible. The n-vertex graph which is a disjoint union of $\lfloor n/2 \rfloor$ edges shows that the condition $2\ell \leq k \leq n-2\ell$ is necessary in order to obtain the above bounds on the sizes of trivial and homogeneous subsets of vertices. When $\ell = 2$, Theorem 1 gives a precise structural result for large n. We prove the same result in the Appendix for all n, see the following.

Theorem 2. *Let G be a graph on n vertices. Let k be an integer, $4 \leq k \leq n-4$, and let $\nu_k(G) = 2$. Then G is either a star, a disjoint union of an edge and $n-2$ vertices, or the complement of one of these graphs.*

2 Proofs

We start with some definitions. For two disjoint sets A, B of vertices in a graph G, we denote by (A, B) a bipartite subgraph of G containing all the edges of G with one endpoint in A and another in B. We define a relation $\sim$ on $V(G)$, the vertices u, v are related, i.e., $u \sim v$ iff $N(u) - \{v\} = N(v) - \{u\}$. It is easy to check that $\sim$ is an equivalence relation and the equivalence classes are the *homogeneous sets*. Note that a homogeneous set must span either a complete or an empty graph, furthermore if A, B are two homogeneous sets then (A, B) is either a complete or an empty bipartite graph. For two disjoint sets A, B, we write $A \sim B$ if (A, B) is a complete bipartite, $A \not\sim B$ if (A, B) is an empty bipartite graph. We say that a set of vertices A, and $G[A]$ are *trivial* if $G[A]$ is either a complete or an empty graph. Similarly, we say that (A, B) is *trivial* if it is either a complete or an empty bipartite graph. We say that the sets of vertices A and B are trivial of different types if one of them induces an empty graphs and the other one induces a complete graph. A *q-skewchain* is a bipartite graph with parts $A = \{a_1, \ldots, a_q\}$ and B, such that $N(a_i) \subsetneq N(a_{i+1})$, $i = 1, 2, \ldots, q-1$. A *k-subset* of vertices is a subset of size k. For all other definitions and notations, we refer the reader to [13].

Our main tool is the following reformulation of a result by Balogh, Bollobás and Weinreich [3] (for different variants of this theorem see [4] or [5]).

Theorem 3. *Let t be a fixed integer. Then there is a function $g(t)$ such that the following holds. Let G be a bipartite graph with partite sets A, B, $|A| = |B| = n$, where $n \geq g(t)$. Assume that the vertices in A all have distinct neighborhoods. Then G has either*
(i) an induced matching of size t or
(ii) a bipartite complement of an induced matching of size t or
(iii) an induced t-skewchain.

Let parameters n, k, ℓ satisfy the conditions of the Theorem 1. Let

$$f(\ell) = 2\ell R(g(R(8\ell^3))),$$

where $R(n) = R(n, n)$ is the classical symmetric Ramsey number [12] and $g(t)$ is the function from Theorem 3. The proof of the Theorem 1 will be based on three cases - when G has at least one "large" homogeneous set, when G has two "relatively large" homogeneous sets, and, finally, when G has many "small" distinct homogeneous sets. We shall consider corresponding Lemmas 1-3 and complete the proof based on them.

Lemma 1. *Let graph G have a homogeneous set of size at least $n - f(\ell)$. If $\nu_k(G) = \ell$ then G has a homogeneous set of size at least $n - 2\ell + 2$ and a trivial set of size at least $n - \ell + 1$.*

Proof. Let T_1 be a homogeneous set of size at least $n - f(\ell)$. We have that $V - T_1 = A \cup B$ such that $T_1 \sim A$ and $T_1 \not\sim B$. Without loss of generality, let T_1 be an independent set. Let $B = B_1 \cup B_2 \cup B_3$ such that $G[B_1]$ does not have isolated vertices, each vertex of $B_2 \subset B \setminus B_1$ is adjacent only to some but not all vertices of A and each vertex of B_3 is not adjacent to any other vertex of G, see Figure 2. Note that if $A = \emptyset$, then $B_2 = B_3 = \emptyset$, and G consists of edges induced by B_1 and isolated vertices. It is easy to see that $|B_1| \leq 2\ell - 2$. Therefore there is a homogeneous set of vertices of size at least $n - 2\ell + 2$. Next, we assume that $A \neq \emptyset$. We shall use the following definition. For sets F, C, D of vertices, we say that the sets $F_1, \ldots, F_t$ are obtained from $F = F_0$ by **(C,D)-exchange in t-steps**, for $t \leq s = \min\{|C \cup F|, |D \setminus F|\}$, if F_i is obtained from F_{i-1} by deleting a vertex from $F_{i-1} \cap C$ and adding a vertex from $D \setminus F_{i-1}$. If the number of steps t in an exchange is equal to s, we say that the exchange is *longest*.

Case 1. $n - 2f(\ell) \leq k \leq n - 2\ell$.

Let F_0 be a k-set containing all vertices of A, as many vertices from T_1 and as few vertices from $B_2 \cup B_3$ as possible. Since $n - |F_0| \geq 2\ell$, we can create $\min\{|B_3| + |B_2| + 1, 2\ell\}$ k-sets of distinct sizes on corresponding subgraphs by $(T_1, B_3 \cup B_2)$-exchange performed on F_0. Therefore $|B_2| + |B_3| \leq \ell - 1$.

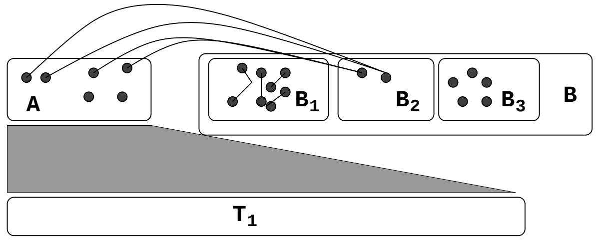

Figure 1: Sets T_1, A and $B = B_1 \cup B_2 \cup B_3$.

Therefore, it is possible to choose F_1, a k-set such that $A \cup B_1 \subseteq F_1$, $(B_2 \cup B_3) \cap F_1 = \emptyset$. First, perform longest (T_1, B_2)-exchange on F_1. Let the resulting set be F_2. Next perform (A, T_1)-exchange on F_2 in $|A| - 1$ steps. Let the last set obtained be F_3. We have that $B_1 \cup B_2 \subseteq F_3$ and $F_3 \cap A = \{a\}$. Next perform longest (T_1, B_3)-exchange on F_3. Finally, do $(\{a\}, T_1)$-exchange on F_3, followed by (B_1, T_1)-exchange producing as many distinct sizes on resulting subgraphs as possible. As a result of these exchanges, we have obtained k-subgraphs with nonincreasing sizes. The total number of such distinct sizes is at least $|B_2| + |A| - 1 + |B_3| + |B_1|/2 + 2$. Since this quantity is at most ℓ, we have that $|A| + |B_1| + |B_2| + |B_3| \leq 2\ell - 2$.

Case 2. $2\ell \leq k < n - 2f(\ell)$.
First we shall prove the following:

$$|B_1| + 2|A| \leq 2\ell - 1. \tag{1}$$

Assume that (1) does not hold, i.e., $|A| > \ell - (|B_1|+1)/2$. Let $F_0 \subseteq T_1$ be a set of size k. First construct sets from F_0 by (T_1, B_1)-exchange so that as many distinct sizes are obtained on corresponding k-subgraphs. Note, that we can build at least $\lceil (|B_1|+1)/2 \rceil$ such sets. We can assume, without loss of generality, that the last subset obtained is F_1, such that $B_1 \subseteq F_1$. Next, construct $|A| \geq \lfloor (2\ell - |B_1|)/2 \rfloor$ sets of distinct sizes on corresponding k-subgraphs from F_1 by (T_1, A)-exchange. We have all together at least $\lceil (|B_1| + 1)/2 \rceil + \lfloor (2\ell - |B_1|)/2 \rfloor > \ell$ distinct sizes, a contradiction. Thus (1) follows.

Next we create the sequence of k-subsets with decreasing sizes on corresponding subgraphs as follows. Since $|B_2|+|B_3| < f(\ell)$, using (1), we can find a k-set, F_0, containing all of $A \cup B_1$ and none of the vertices from $B_2 \cup B_3$. Let $H_0 = F_0 \cap T_1$. Consider (T_1, B_2)-exchange in s steps, where

$$s = \min\{|T_1 \cap F_0| - 1, |B_2|\}.$$

This gives us $s+1$ sets $F_0, F_1, \ldots, F_s$ with decreasing sizes on the corresponding k-subgraphs.

Case a. $s = |T_1 \cap F_0| - 1$.
We have that $F_0, F_1, \ldots, F_s$ induce $|H_0| = k - |A| - |B_1|$ k-subgraphs of distinct sizes. Let $F_{s+1} = (F_s \backslash A) \cup H_1$, $H_1 \subseteq T_1 \backslash F_s$. Create sets $F_{s+2}, F_{s+3}, \ldots$ from F_{s+1} by (B_1, T_1)-exchange such that as many distinct sizes on corresponding subgraphs occur as possible. The number

of k-subgraphs with distinct sizes constructed so far is at least $(k - |B_1| - |A|) + |B_1|/2 = k - |B_1|/2 - |A| \geq 2\ell - \ell + 1/2 > \ell$, a contradiction. Note that in the last inequality we used is (1).

Case b. $s = |B_2|$.

We have that $F_0, F_1, \ldots, F_s$ induce $|B_2| + 1$ k-subgraphs of distinct sizes. Note that $F_s = A \cup B_1 \cup B_2 \cup H_2$ for some $H_2 \subseteq T_1$. Let us perform a longest (T_1, B_3)-exchange on F_s. Let the last resulting set be F_p. Let $F_{p+1} = (F_p \setminus A) \cup H_3$, where $H_3 \subseteq T_1 \setminus F_p$. Finally, we perform a longest (B_1, T_1)-exchange on F_{p+1}.

If $|H_2| = |T_1 \cap F_s| \leq |B_3|$, then we have obtained all together at least $|B_2| + 1 + |H_2| + |B_1|/2 + 1 = 2 + |B_2| + k - |A| - |B_1| - |B_2| + |B_1|/2 = 2 + k - |A| - |B_1|/2 > \ell$ distinct sizes (here the last inequality follows from (1)). A contradiction.

If $|T_1 \cap F_s| > |B_3|$, then we have obtained at least $|B_2| + 1 + |B_3| + 1 + |B_1|/2 \leq \ell$ distinct sizes on corresponding subgraphs. Using (1) gives us that $|A| + |B_1| + |B_2| + |B_3| + 2 \leq 2\ell$. Thus $|T_1| \geq n - 2\ell + 2$.

Now, we are prepared to prove that there is a trivial subset of size at least $n - \ell + 1$. We have that $V - T_1 = A \cup B$ such that $T_1 \sim A$ and $T_1 \not\sim B$ and, without loss of generality, T_1 is inducing an empty graph. Let $|A| = a$, $|B| = b$. We have that $a + b \geq \ell$ otherwise we are done. Let r be the number of components in $G[B]$.

By taking a vertex from each component of $G[B]$, we have, together with T_1, a trivial set of size s,

$$s \geq (n - a - b) + r. \tag{2}$$

On the other hand, there are $\beta = b - r + 1$ subgraphs of distinct sizes in $G[B]$, call the class of vertex sets of these subgraphs $\mathcal{B}$. Now, consider k-subgraphs $F_1, \ldots, F_\beta$ spanned by sets from $\mathcal{B}$ and subsets of T_1. If $\beta \leq \ell$, consider the subgraphs $G_1, G_2, \ldots$, where $G_1 = F_\beta$ and G_{i+1} is spanned by $V(G_{i+1}) \setminus \{v\} \cup \{u\}$, where $v \in T_1$ and $u \in A$. Thus, we can create $|A| = a$ k-subgraphs with new distinct sizes. The total number of distinct sizes on k-subgraphs is t,

$$t \geq b - r + 1 + a. \tag{3}$$

Now, from (2) and (3), and the fact $t \leq \ell$, we have that $s \geq n - (a + b - r) \geq n - (t - 1) \geq n - \ell + 1$.

□

Lemma 2. *Let G have two distinct maximal homogeneous sets of sizes at least 2ℓ each. Then $\nu_k(G) > \ell$.*

Proof. Let T_1, T_2 be distinct maximal homogeneous set, $|T_i| \geq 2\ell$, $i = 1, 2$. Consider sets $A_1 \subseteq T_1$, $A_2 \subseteq T_2$, such that $|A_i| = 2\ell$, $i = 1, 2$. Since T_1 and T_2 are distinct homogeneous sets, there is a vertex v such that, without loss of generality, $\{v\} \sim A_1$ and $\{v\} \not\sim A_2$ and $v \notin A_1 \cup A_2$. Let $R_i \subseteq A_1$, $S_i \subseteq A_2$, $|R_i| = |S_i| = i$, $i = 0, 1, \ldots, 2\ell$. Let $X \subseteq V(G) - (A_1 \cup A_2 \cup \{v\})$, such that $|X| = k - 2\ell - 1$. Let $Y = X \cup \{v\}$. Note that such set X exists since $|V-(A_1\cup A_2)| = n-4\ell \geq k-2\ell$, for $k\leq n-2\ell$. Let $X_1\subseteq X$, $X_2\subseteq X$ such that $X_i\sim A_i, (X\setminus X_i)\not\sim A_i$, $i=1,2$. Let $|X_1|=r$, $|X_2|=s$.

(i) $G[A_1\cup A_2]$ is trivial.
Without loss of generality, we may assume that $G[A_1\cup A_2]$ is empty. Let

$$F_i = G[R_i\cup S_{2\ell-i}\cup X\cup\{v\}],\quad H_i = G[R_{i+1}\cup S_{2\ell-i}\cup X],\quad i=0,\dots,2\ell-1.$$

Then

$$|E(F_i)| = ir+(2\ell-i)s+|E(G[X\cup\{v\}])|+i,\quad |E(H_i)| = (i+1)r+(2\ell-i)s+|E(G[X])|,$$

$i=0,\dots,2\ell-1$. Simplifying these expressions, we get $|E(F_i)| = i(r-s+1)+(2\ell s+|E(G[X\cup\{v\}])|)$ and $|E(H_i)| = i(r-s)+(r+2\ell s+|E(G[X])|)$. Either $r-s\neq 0$ or $r-s+1\neq 0$, therefore either the sets H_i, $i=0,1,\dots,\ell$, or the sets F_i, $i=0,1,\dots,\ell$, give $\ell+1$ distinct sizes on corresponding subgraphs, a contradiction.

(ii) A_1, A_2 are trivial of different types.
Assume without loss of generality that A_1 induces a complete graph, A_2 induces an empty graph, and (A_1,A_2) is an empty bipartite graph. Let

$$F_i = G[R_i\cup S_{2\ell-i}\cup Y],\quad i=0,\dots,2\ell.$$

We have that

$$|E(F_i)| = (r+1)i+(2\ell-i)s+|E(G[Y])|+\binom{i}{2},\quad i=0,\dots,2\ell.$$

$|E(F_i)|$ is a quadratic function of i, thus for $2\ell+1$ arguments, it takes at least $\ell+1$ different values, a contradiction.

(iii) A_1, A_2 are trivial of the same types and (A_1,A_2) is trivial of a type different from the type of A_1. Let

$$F_i = G[R_i\cup S_{2\ell-i}\cup Y],\quad i=0,1,\dots,2\ell.$$

If A_1 and A_2 are empty then

$$|E(F_i)| = |E(G[Y])|+i(r+1)+s(2\ell-i)+i(2\ell-i),\quad i=0,\dots,2\ell.$$

If A_1 and A_2 are complete then

$$|E(F_i)| = |E(G[Y])|+\binom{i}{2}+\binom{2\ell-i}{2}+i(r+1)+(2\ell-i)s,\quad i=0,\dots,2\ell.$$

Each of these is a quadratic function of i, therefore gives at least $\ell+1$ different values for $i=0,1,\dots,2\ell$. □

Lemma 3. *Let G have at least $f(\ell)/2\ell$ distinct maximal homogeneous sets. Then $\nu_k(G)>\ell$.*

Proof. Let $T_1, T_2, \ldots, T_m$ be distinct maximal homogeneous sets in G, $m \geq f(\ell)/2\ell$. Let $v_i \in T_i$, $i = 1, \ldots, m$. Consider the largest subset Q of $\{v_1, \ldots, v_m\}$ inducing a trivial graph. The Ramsey theorem guarantees that $|Q| \geq g(R(8\ell^3))$. Apply Theorem 3 to a bipartite subgraph G' of G with partite sets $Q, V \setminus Q$, and edges of G with one endpoint in Q and another in $V \setminus Q$. We have that there are subsets $Q' \subseteq Q$ and $P' \subseteq V - Q$, $|Q'| = |P'| = R(8\ell^3)$, such that $Q' \cup P'$ either induces in G' a matching or its bipartite complement or q-skewchain for $q = |Q'|$. By applying Ramsey theorem to $G[P']$, we can find a trivial subset P'' in P', $|P''| = 8\ell^3$. Let $B = P''$ and let A be the set of vertices in Q' corresponding to P''.

Let $A = \{u_1, \ldots, u_{8\ell^3}\}$ and $B = \{v_1, \ldots, v_{8\ell^3}\}$. By taking graph complements and relabeling the vertices, we have the following possible structure induced by A and B: A and B are trivial and either
a) (A, B) is an induced matching $\{u_i\} \sim \{v_i\}$, $i = 1, \ldots, 8\ell^3$ or
b) (A, B) is an induced skew-chain with $\{u_i\} \sim \{v_i, v_{i+1}, \ldots, v_{8\ell^3}\}$, $i = 1, \ldots, 8\ell^3$.

If $k \leq 16\ell^3 - 2\ell - 1$ then we can easily find $\ell + 1$ k-subgraphs of $G[A \cup B]$ with distinct sizes. We can assume that $k \geq 16\ell^3 - 2\ell$. Let $X \subseteq V - A - B$ with $|X| = k - 16\ell^3 + 2\ell$. Let a_i be the number of neighbors of u_i in $X \cup B$ and let b_i be the number of neighbors of v_i in $X \cup A$, $i = 1, \ldots, 8\ell^3$.

By pigeonhole principle, we have one of the cases **(i)** or **(ii)** as follows.
(i) $|\{a_1, \ldots, a_{8\ell^3}\}| > 2\ell$, or $|\{b_1, \ldots, b_{8\ell^3}\}| > 2\ell$.
Assume, without loss of generality, that $a_1, \ldots, a_{2\ell+1}$ are all distinct integers. Let

$$F_j := X \cup A \cup B \setminus (\{v_1, \ldots, v_{2\ell+1}\} \setminus \{v_j\}), \quad j = 1, \ldots, 2\ell + 1.$$

The graphs induced by F_js have k vertices and $2\ell + 1$ distinct sizes.

(ii) There is a subset of 2ℓ indices, without loss of generality, $\{1, 2, \ldots, 2\ell\}$, such that $a_1 = \ldots = a_{2\ell}$ and $b_1 = \ldots = b_{2\ell}$.
Let $M = \{v_1, u_1, v_2, u_2, \ldots, v_{2\ell}, u_{2\ell}\}$. Now let

$$F_j = (X \cup A \cup B \setminus M) \cup \{u_1, \ldots, u_\ell, v_1, \ldots, v_j, v_{\ell+1}, \ldots, v_{2\ell-j}\},$$

$j = 1, \ldots, \ell - 1$. Let $F_0 = (X \cup A \cup B \setminus M) \cup \{u_1, \ldots, u_\ell, v_{\ell+1}, \ldots, v_{2\ell}\}$, and $F_\ell = (X \cup A \cup B \setminus M) \cup \{u_1, \ldots, u_\ell, v_1, \ldots, v_\ell\}$. The graphs induced by the sets F_j, $j = 0, \ldots, \ell$ have k vertices and have $\ell + 1$ distinct sizes.

□

Proof of Theorem 1. Consider a graph G on n vertices with $\nu_k(G) \leq \ell$. Let $T_1, T_2, \ldots, T_m$ be maximal homogeneous sets in G such that $|T_1| \geq |T_2| \geq \ldots \geq |T_m|$.
Case 1. $|T_1| > n - f(\ell)$.
In this case the conclusions of the Theorem follow immediately from Lemma 1.

Case 2. $|T_2| \geq 2\ell + 1$.
In this case we arrive at a contradiction using Lemma 2 with homogeneous sets T_1 and T_2.

Case 3. $|T_1| \le n - f(\ell)$ and $|T_2| \le 2\ell$.
The conditions $|T_2 \cup T_3 \cup \ldots \cup T_m| \ge f(\ell)$ and $|T_i| \le 2\ell$ for $i = 2, \ldots, m$ imply that $m \ge f(\ell)/2\ell$. Therefore, we arrive at a contradiction using Lemma 3.

□

3 Appendix - Proof of Theorem 2

Let G be a graph on n vertices such that each k-subgraph has size i_1 or i_2 for some integers i_1, i_2. We suppose that both values appear otherwise we are done by Proposition 1.
Case 1. $i_1 = 0$ or $i_1 = \binom{k}{2}$.
We may assume, by taking a complement of G if necessary, that $i_1 = 0$. We have that some of the k-subgraphs are empty and others have size $i = i_2$. Consider the largest independent set S of size at least k. Let $v \notin S$, then $N(v) \cap S = S$ or $|N(v) \cap S| = 1$, otherwise it is easy to find two nonempty k-subgraphs with distinct sizes containing v and $k-1$ vertices from S. We see, in particular, that $i \le k-1$, and, if $|N(v) \cap S| = 1$ for some v, then $i = 1$. It is obvious that if $i = 1$ and $k \ge 4$ then G must have exactly one edge. Thus, we may assume that for each $v \notin S$, $N(v) \cap S = S$. If there are two vertices $u, u' \notin S$ then consider u, u' and $k-2$ vertices of S. These k vertices induce a subgraph with at least $2(k-2) > k-1$ edges, for $k \ge 4$, a contradiction. Thus there is exactly one vertex not in S and G is a star.

Case 2. $i_1, i_2 \notin \{0, \binom{k}{2}\}$.
Let $i_1 < i_2$ and $i_2 - i_1 = \ell$, $\ell \le k-1$.

Case 2.1 There are vertices u, v, such that $|N(u) \setminus N(v)) \cap S| \ge 2$, for $S = V \setminus \{u, v\}$.

Let $Q = Q(u,v) = S \setminus N(u) \Delta N(v)$. Assume that $|(N(u) \setminus N(v)) \cap S| \ge |(N(v) \setminus N(u)) \cap S|$. Let us find subsets $U', U'' \subseteq (N(u) \setminus N(v)) \cap S$, $V' \subseteq (N(v) \setminus N(u)) \cap S$ such that $|V'| + 1 \le |U'| < |U''|$. Consider largest such subsets such that $|V'| + |U'| + 1 \le k$. Then choose $Q', Q'' \subseteq Q$ such that $|Q'| + |V'| + |U'| + 1 = k$ and $|Q''| + |V'| + |U''| + 1 = k$. Note that these subsets can be chosen unless $Q = \emptyset$ and $(N(v) \setminus N(u)) \cap S = \emptyset$. We have that the subgraphs induced by u, V', U', Q' and by v, V', U', Q' differ in size by $t = |U'| - |V'|$, $t > 0$ and the subgraphs induced by u, V', U'', Q'' and by v, V', U'', Q'' differ in size by $t' = |U''| - |V'| > t > 0$. Thus we have that $i_2 - i_1 = t$ and $i_2 - i_1 = t'$, a contradiction.

If $Q = \emptyset$ and $(N(v) \setminus N(u)) \cap S = \emptyset$ then $\nu_{k-1}(G[S]) = 1$, thus by Proposition 1, we have that S induces a trivial set. Thus G is one of the following: a) a star or its complement; b) a star and an isolated vertex; c) a complement of a star and an isolated vertex. Note that b) and c) are impossible since in that case $\nu_k(G) \ge 3$.

Case 2.2. For any two vertices $u, v \in V(G)$, if $S = V \setminus \{u, v\}$, then $|(N(u) \setminus N(v)) \cap S| \le 1$.

Then, in particular, it implies that the degrees of any two vertices differ by at most 1. Thus, $V(G) = V_d \cup V_{d+1}$ such that for each $v \in V_d$, $deg(v) = d$ and for each $v \in V_{d+1}$, $deg(v) = d+1$. Note also that

$$u \in V_d, \quad v \in V_{d+1}, \text{ then } \quad N(u) \setminus \{v\} \subseteq N(v) \setminus \{u\}. \tag{4}$$

Therefore, if $A \subseteq V_d$ induces a nontrivial connected graph in $G[V_d]$ then (A, V_{d+1}) forms a complete bipartite subgraph of G. Consider $A, B \subseteq V_d$ inducing two nontrivial components in $G[V_d]$. Let $a \in A, b \in B$. Then it is easy to see that $|N(a) \cap V_d| \leq 1$ and $|N(b) \cap V_d| \leq 1$. Therefore, either $G[V_d]$ is connected or each nontrivial connected component in $G[V_d]$ has maximum degree 1 and thus is an edge. Note that V_d cannot induce both edges and isolated vertices. Indeed, the degrees of vertices incident to edges in V_d are $|V_{d+1}|+1$ and the degrees of vertices isolated in $G[V_d]$ are at most $|V_{d+1}|$ which is impossible since all vertices in V_d have the same degree d.

Subcase a. V_d induces an empty set in G.
Let $v \in V_d$, $u \in N(v)$. We have by (4) that each $w \in V_{d+1}$ is adjacent to u. Thus $d+1 = deg(u) \geq |V_{d+1}|$. We also have that $d = deg(v) \leq |V_{d+1}|$. Therefore $d = |V_{d+1}|$ or $d = |V_{d+1}| - 1$. In the first case we have that (V_d, V_{d+1}) form a complete bipartite subgraph and V_{d+1} must induce a complete graph by (4). Therefore, $|V_d| = 2$ and $G = K_n \setminus e$, for an edge e. In the latter case, we again have that V_{d+1} induces a complete graph and (V_d, V_{d+1}) induces a complete bipartite graph with deleted stars of equal sizes centered in V_{d+1} and covering each vertex of V_d. If the number of these stars is ℓ and their sizes are k then $|V_d| = k\ell$, $d = n - k\ell - 2$, $d+1 = n-k-1$. Thus, $n - k\ell - 1 = n - k - 1$, $k\ell = k$, and $\ell = 1$. Therefore $G = E_n$, a contradiction.

Subcase b. V_d induces a matching.
In this case we have as before that V_{d+1} induces a complete graph and (V_d, V_{d+1}) forms a complete bipartite subgraph of G. Then $d = |V_{d+1}| + 1$, $d+1 = n-1$. Therefore, $|V_{d+1}| = n-3$, $|V_d| = 3$, a contradiction since then V_d cannot induce a matching. □

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