On the structure of minimal winning coalitions in simple voting games

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Abstract

According to Coleman's index of collective power, a decision rule that generates a larger number of winning coalitions imparts the collectivity a higher a-priori power to act. By the virtue of the monotonicity conditions, a decision rule is totally characterized by the set of minimal winning coalitions. In this paper we investigate the structure of the families of minimal winning coalitions corresponding to maximal and proper simple voting games (SVG). We show that if the proper and maximal SVG is swap robust and all the minimal winning coalitions are of the same size, then the SVG is a specific (up to an isomorphism) system. We also provide examples of proper SVGs to show that the number of winning coalitions is not monotone with respect to the intuitively appealing system parameters like the number of blockers, number of non-dummies or the size of the minimal blocking set.

Keywords: simple voting games, proper games, swap robust, collective power

1 Introduction

In most decision making bodies, or equivalently collectivities, such as the United Nations Security Council and the Council of Ministers in the European Union, the decision whether to accept or reject a bill is taken by the process of voting. Typically, when a bill is presented before such a collectivity, each of its members votes either in favor of the bill ('yes') or against it ('no'). Thus, if we assume that there is no abstention, voting creates a *bi-partition* of the set of voters into a 'yes' camp and a 'no' camp. The voting procedure in each collectivity is governed by its own constitution whose role is to lay down the decision making rule that aggregates individual votes to determine the collective decision of the voting body.

The class of mathematical structures that is used to model such voting situations is called a *simple voting game*, SVG. Formally, an SVG, $\mathcal{F}$, is a collection of subsets of a set N corresponding

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to the set of voters, such that

1. $\phi \notin \mathcal{F}$ (if all voters vote against the bill, the bill is rejected),
2. $N \in \mathcal{F}$ (if all voters vote for the bill, the bill is passed), and
3. if $X \subseteq X'$ and $X \in \mathcal{F}$ then $X' \in \mathcal{F}$ (increased support for a bill cannot hurt passing the bill).

The sets in $\mathcal{F}$ are referred to as *winning coalitions.* A coalition $X \in \mathcal{F}$ is said to be a *minimal winning coalition* if no proper subset of it is winning, i.e., if for any $X' \subset X$, $X' \notin \mathcal{F}$. We denote the set of minimal winning coalitions in $\mathcal{F}$ by $\mathcal{F}^{\min}$. Because of a monotonicity requirement (3) on the decision rule, one can see that an SVG, $\mathcal{F}$, is totally characterized by the set $\mathcal{F}^{\min}$. An SVG, $\mathcal{F}$, is called *proper* if for any $X, X' \in \mathcal{F}$, $X \cap X' \neq \emptyset$. For details on the importance of proper games see Felsenthal and Machover (1998). An SVG is *swap robust* if for any two winning coalitions X, Y, and $x \in X \setminus Y$, $y \in Y \setminus X$, either $X \setminus \{x\} \cup \{y\}$ or $Y \setminus \{y\} \cup \{x\}$ is a winning coalition. That is, an SVG is swap robust if one could swap two members in two winning coalitions and get at least one winning coalition as a result. This concept, introduced by Taylor and Zwicker (1993), is particularly important since weighted voting games, most commonly used in real life situations, are swap robust.

Often there arises a need for evaluating decision rules according to several criteria, e.g, whether the rule is too resistant, or how sensitive it is to fluctuations in the voter's wishes etc. For detailed discussions, see Felsenthal and Machover (1998), Barua et al. (2004). A very important criterion for assessing a decision rule is the degree to which it empowers the collectivity as a whole as a decision maker. Alternatively put, a decision rule is often evaluated in terms of its a-priori bias in favor of a change in the status-quo. Thus, while a decision rule with a very high a-priori bias in favor of change might make approval of individual bills *too* easy, another decision rule that has a small a-priori bias towards change may be criticized as being too resistant, thus tending to "engender immobilism". Coleman (1971) proposed a measure to quantify this bias in favor of change that a collectivity has, solely by the virtue of the decision rule itself. Given an SVG, $\mathcal{F}$ defined on a set N, Coleman's measure of the *power of a collectivity to act* is given by

$$C(\mathcal{F}) = \frac{|\mathcal{F}|}{2^{|N|}}.$$

Since $|\mathcal{F}|$ is the total number of winning coalitions and $2^{|N|}$ is the total number of coalitions (including the empty one) in the game $\mathcal{F}$, $C(\mathcal{F})$ is the prior probability that a resolution will be adopted by the collectivity, assuming all coalitions are equi-probable. It therefore follows that according to Coleman's measure, given two SVGs on the same set of voters, the one with more winning coalitions has a higher a-priori collective power to act.

The purpose of this paper is two-fold. In the first part of the paper we concentrate on *swap robust* SVGs. Swap robust games constitute a significant and a big class of SVGs for which the influence relation (see section 2 below) induces a complete ordering of the voters (also see Isbell (1956), Taylor (1995)). These games are also known in the literature as *complete* SVGs or *linear* SVGs. A lot of work has been done on swap robust games (see Freixas and Puente

(1998), Freixas and Pons (2008), Freixas and Molinero (2009)), a special area of interest being the structure of these games (for e.g., see Carreras and Freixas (1996)). In this paper we use the theory of hypergraphs and extremal set theory to investigate the structure of the family of minimal winning coalitions, $\mathcal{F}^{\min}$, corresponding to proper decision rules for which $C(\mathcal{F})$ is maximum. Garcia-Molina and Barbara (1985) have shown that if $\mathcal{F}$ is maximal intersecting family (i.e. the one with $C(\mathcal{F}) = 1/2$), then $\mathcal{F}^{min}$ corresponds to the set of hyper-edges of a 3-chromatic hypergraph (also see Loeb and Conway (2000)). We use this result to prove that if a proper and maximal SVG $\mathcal{F}$ is swap robust and such that all coalitions in $\mathcal{F}^{\min}$ have the same size, then the family $\mathcal{F}^{\min}$ is unique up to an isomorphism. Moreover the elements of $\mathcal{F}^{\min}$ are all r-element subsets of a $(2r-1)$-element set. Thus if a proper SVG is maximal and swap robust with all the minimal winning coalitions of the same size, then all the non-dummy voters (that is, voters who belong to at least one coalition in $\mathcal{F}^{\min}$) are symmetric. The condition that SVG is swap robust is essential. If the system is maximal and proper but not swap robust, then $\mathcal{F}^{\min}$ could correspond to other set systems, for example to the edges of a Fano plane.

Second, we investigate the following question. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two proper SVGs, such that $|\mathcal{F}_1^{\min}| = |\mathcal{F}_2^{\min}|$ and each minimal winning coalition in both systems has r voters. When can we say that $|\mathcal{F}_1| \geq |\mathcal{F}_2|$? We use the celebrated Kruskal-Katona theorem to show in this case that $|\mathcal{F}_1| \geq |\mathcal{F}_2|$ if the elements of $\mathcal{F}_2^{\min}$ are the last (in co-lex order) elements among all r-element sets. Furthermore, $|\mathcal{F}_1| \geq |\mathcal{F}_2|$ if the family $\mathcal{F}_2^{\min}$ can be obtained from $\mathcal{F}_1^{\min}$ by shifting. However, we were not able to find a "single" parameter which would reveal whether $|\mathcal{F}_1| \geq |\mathcal{F}_2|$. For this purpose we considered the intuitively appealing parameters such as the number of blockers, number of non-dummies, the size of the smallest blocking set. Though our hunt for a parameter that completely characterizes the size of winning coalitions may have turned up negative results, an important implication of this finding is to reemphasize the pitfalls against which we must take care. Thus if a collectivity A has (say) 3 blockers each of whom can unilaterally stall any of its actions, and another collectivity B has just 1 blocker, we must refrain from making the "logical" conclusion that B has a larger a-priori power to act than A.

The rest of the paper is organized as follows. We will present some preliminaries and definitions in section 2. In section 3 we prove our main result for maximal and proper swap robust SVGs and prove other related statements. In section 4 we use Kruskal-Katona's theorem to show that given two SVGs $\mathcal{F}_1$ and $\mathcal{F}_2$, if $\mathcal{F}_1^{min}$ is obtained from $\mathcal{F}_2^{min}$ by shifting, then $|\mathcal{F}_1| \leq |\mathcal{F}_2|$. In the same section we also describe smallest SVGs and give constructions which show that the size of an SVG is not a monotone function of some major parameters such as number of blockers and number of non-dummies. Section 5 concludes.

2 Definitions

Many definitions used in the theory of voting games have parallel definitions in hypergraph and extremal set theory. We include both in this section. Since most of tool we use come from mathematics, we use hypergraph and set theory terminology in proofs. We state the main results in terms of voting games language.

General definitions.
A *set system* or a *family of sets* on a set N (or with elements in N) is simply a collection of subsets of N. For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$. For a finite set N, and a set system, $\mathcal{F}$, of subsets of N, a pair $(N, \mathcal{F})$ is called a hypergraph on vertex set N with edge-set $\mathcal{F}$, see Berge (1988) for more notations on hypergraphs. The elements of N are referred to as vertices. Observe that SVG $\mathcal{F}$ forms the set of hyperedges of a hypergraph $(N, \mathcal{F})$ and the voters of the SVG are the vertices of the hypergraph. In order to avoid confusion, while proving our main results we will refer to an SVG, $\mathcal{F}$, as a set system or a family of sets, voters or players as vertices and coalitions as edges or sets. We will also assume that the sets in the families have size at least 2 and there are at least two sets in the families considered.

Set system parameters.
For a set system $\mathcal{F}$, let $upset(\mathcal{F}) = \{S : S \supseteq X, X \in \mathcal{F}\}$, $downset(\mathcal{F}) = \{S : S \subseteq X, X \in \mathcal{F}\}$. If $\mathcal{F}$ is an SVG, we say that $\mathcal{F}^{min}$ generates $\mathcal{F}$ since $upset(\mathcal{F}^{min}) = \mathcal{F}$. For a set system $\mathcal{F}$ on elements N, we say that a subset of N is a *transversal* of $\mathcal{F}$ if it nontrivially intersects each set of $\mathcal{F}$. A transversal is also known as a *blocking set*. Let $tr(\mathcal{F})$ be the size of the smallest transversal of $\mathcal{F}$, let $support(\mathcal{F}) = \cup_{X \in \mathcal{F}^{min}} X$. An element $b \in N$ is called a *blocker* in an SVG, $\mathcal{F}$, if it belongs to every set in $\mathcal{F}$; the number of blockers in $\mathcal{F}$ is denoted by $b(\mathcal{F})$. In the language of simple voting games, a blocker is also known as a *veto player*. It may be noted here that if there exists an element $i \in N$ in an SVG $\mathcal{F}$ such that $\mathcal{F}^{min} = \{i\}$, then i is referred to as a *dictator*. An element $i \in N$ is called a *non-dummy* of an SVG $\mathcal{F}$ if it belongs to at least one set in $\mathcal{F}^{min}$, otherwise it is called a *dummy*. Note that $support(\mathcal{F})$ is the set of non-dummies of $\mathcal{F}$.

Types of set systems.
An SVG $\mathcal{F}$ is *r-trivial* if the sets in $\mathcal{F}^{min}$ are all r-element subsets of a $(2r-1)$-element set. An *r-trivial* SVG is a *simple majority game* if $|support(\mathcal{F})| = 2r - 1$. An SVG $\mathcal{F}$ is *swap robust* or *weakly robust* if for any $A, B \in \mathcal{F}$ and any $i \in A \setminus B$, $j \in B \setminus A$, $A \setminus \{i\} \cup \{j\} \in \mathcal{F}$ **or** $B \setminus \{j\} \cup \{i\} \in \mathcal{F}$. A family of sets is called *Sperner* if no set is a subset of another set, see Engel (1997). A system of sets is *intersecting* if any two sets of the family intersect by a nonempty set. In particular, a proper SVG $\mathcal{F}$ is an intersecting set system and the family of minimal winning coalitions $\mathcal{F}^{min}$ is an intersecting Sperner set system. A set system in which all sets have size r is called r-uniform, or simply uniform. For a set X, $\binom{X}{r}$ denotes the set of all r-element subsets

of X. Given a set $X \subseteq [n]$, we denote the complement of X by X^c, i.e., $X^c = [n] \setminus X$. For a set system $\mathcal{F}$, let $\mathcal{F}^c = \{X^c : X \in \mathcal{F}\}$. An SVG $\mathcal{F}$ on the set N is called *strong* iff for all subsets $S \subseteq N$, either $S \in \mathcal{F}$ or $S^c \in \mathcal{F}$. An SVG that is proper and strong is called *decisive* or *self-dual*.

A *proper coloring of a hypergraph* is an assignment of labels (called colors) to the vertices of a hypergraph such that every edge has at least two distinct labels on its vertices. A hypergraph for which there exists a proper coloring with k colors is called *k-colorable*. A k-colorable hypergraph which is not $(k-1)$-colorable is called *k-chromatic*. We say that a family of sets, $\mathcal{F}$, is k-colorable (k-chromatic), if a hypergraph with hyper-edge set $\mathcal{F}$ is k-colorable (k-chromatic). For properties of 3-chromatic hypergraphs, see Erdős and Lovász (1975).

Shifted and swap robust families.
Let $\mathcal{F}$ be a family of subsets of $[n]$. For $i, j \in [n]$, $E \in \mathcal{F}$, let

$$E(i,j) = \begin{cases} E \setminus \{i\} \cup \{j\}, & \text{if} \quad i \in E, \quad j \notin E, \quad E \setminus \{i\} \cup \{j\} \notin \mathcal{F}, \\ E, & \text{otherwise.} \end{cases}$$

A *shift* of $\mathcal{F}$ with indices i, j, is $\mathcal{F}(i,j) = \{E(i,j) : E \in \mathcal{F}\}$, i.e., it is a set system where each set E is replaced by $E(i,j)$. Note that $|\mathcal{F}| = |\mathcal{F}(i,j)|$. In a weakly robust family $\mathcal{F}$, we say that a vertex x has smaller or equal influence than a vertex y and write $x \leq y$ if for each edge E containing x but not y, there is an edge $E' = E \setminus \{x\} \cup \{y\}$. We say that x has smaller influence than y if $x \leq y$ and there is a set E containing y and not x such that $E \setminus \{y\} \cup \{x\} \notin \mathcal{F}$. Two vertices $x, y \in N$ are called *twins* if $x \leq y$ and $y \leq x$, we write $x \sim y$.

Let $\mathcal{F}$ be a swap robust family. We write $E \preceq E'$, $E, E' \in \mathcal{F}$ if for some $x \leq y$, $x \in E$, $y \notin E$, $E' = E \setminus \{x\} \cup \{y\}$. We say that $E \leq_s E'$ *in shift order* if there is a sequence of sets $E = E_1 \preceq E_2 \preceq \cdots \preceq E_m = E'$. The swap robustness property implies that if $E \leq_s E'$ and $E \in \mathcal{F}$ then $E' \in \mathcal{F}$. A set E is less than a set E' in *co-lex* order with respect to ordering of elements $(1, 2, \ldots, n)$ if either $E \subset E'$ or $\max(E \Delta E') \in E'$. Note that the shift order is not the same as co-lex order. Indeed, consider a swap robust game on the set $N = \{1, 2, 3, 4, 5, 6\}$ and let the ordering of the elements be as follows: $1 < 2 < 3 < 4 < 5 < 6$. Consider the following two subsets of N: $A = \{1, 2, 6\}$, $B = \{3, 4, 5\}$. Then $B < A$ in co-lex order with respect to this ordering of the elements, but clearly $B \not\leq A$ in shift order.

Please see the table comparing set (hypergraph) theory and voting theory notions in Section 5.

3 Weakly robust families

This section deals with swap robust and proper SVGs, or weakly robust intersecting families of sets; specifically those having the largest size, thus the largest possible Coleman index. In

Proposition 1 we state the known or easy results about proper *maximal* SVGs. It may be worthwhile to note here that proper and maximal SVGs are decisive or self-dual games. In Proposition 2 we state results about *swap-robust* families. Finally, we combine these results to prove the main result stated below as Theorem 1.

In general, the number of maximal intersecting families on an n-element set is large, it is at least $2^{\binom{n-1}{\lfloor n/2 \rfloor}}$, see Erdős and Hindman (1984). In Theorem 1, we prove that if, in addition, the family is swap robust with minimal sets of the same size, then it is unique up to an isomorphism.

Recall that an SVG $\mathcal{F}$ is called r**-trivial** if the sets in $\mathcal{F}^{min}$ are all r-element subsets of a $(2r-1)$-element set.

Theorem 1. *If $\mathcal{F}$ is a proper maximal swap robust SVG such that all sets in $\mathcal{F}^{min}$ have size r then $\mathcal{F}$ is r-trivial.*

Next, we state, for completeness, some known results which are considered "folklore".

Proposition 1.

1. *Each intersecting family of subsets of $[n]$ has at most 2^{n-1} members. It has exactly 2^{n-1} members if and only if for any $X \subseteq [n]$ either X or X^c is in the family.*
2. *[Garcia-Molina, Barbara (1985)] An intersecting Sperner system, $\mathcal{F}$ of subsets of $[n]$, has size 2^{n-1} if and only if $\mathcal{F}^{min}$ corresponds to a 3-chromatic hypergraph.*
3. *If $\mathcal{F}$ is an intersecting 3-chromatic Sperner family with no isolated vertices then for any $E \in \mathcal{F}$ and any $x \in E$, there is $E' \in \mathcal{F}$ such that $E' \cap E = \{x\}$. Moreover, for any two vertices x, y, $x \neq y$, there is $E \in \mathcal{F}$ such that $x \in E$ and $y \notin E$.*

Proof. 1. Let $F \in \mathcal{F}$, then $F^c \notin \mathcal{F}$. Let $\mathcal{F}^c = \{F^c : F \in \mathcal{F}\}$. Then $\mathcal{F} \cup \mathcal{F}^c \subseteq 2^{[n]}$ and $\mathcal{F} \cap \mathcal{F}^c = \emptyset$, $|\mathcal{F}| = |\mathcal{F}^c|$, so $|\mathcal{F}| \leq |2^{[n]}|/2 = 2^{n-1}$.

2. Let $\mathcal{F}$ be an intersecting family of size 2^{n-1}. Let $A \in \mathcal{F}^{min}$, $x \in A$. Color $A \setminus \{x\}$ red, $\{x\}$ green and the rest of vertices blue. Since the family is intersecting, each set has a red vertex or a green vertex. Since the family $\mathcal{F}^{min}$ is Sperner, each set, except for A has a blue vertex. We also have that A has two colors on its vertices. So, $\mathcal{F}^{min}$ is 3-colorable. Assume that $\mathcal{F}^{min}$ is 2-colorable. Let the set of red vertices in such a proper 2-coloring be R and the set of blue vertices be $B = R^c$. If $R \in \mathcal{F}$ then some its subset $R' \subseteq R$ is in $\mathcal{F}^{min}$, a contradiction. Thus $R \notin \mathcal{F}$, therefore its complement $B = R^c \in \mathcal{F}$, so some subset B' of B is in $\mathcal{F}^{min}$, a contradiction since there is no monochromatic set in $\mathcal{F}^{min}$.

Now, lets consider $\mathcal{F}'$, a 3-chromatic intersecting hypergraph which is Sperner. Consider $\mathcal{F} = upset(\mathcal{F}')$. We need to show that for each $A \in 2^{[n]}$ either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$. Assume not and there is $A \notin \mathcal{F}$ and $A^c \notin \mathcal{F}$. Color the vertices of A red and the vertices of A^c blue. Then there is no set S from $\mathcal{F}'$ such that $S \subseteq A$ or $S \subseteq A^c$ otherwise without loss of generality $S \subseteq A \in \mathcal{F}$, a contradiction. Thus, $\mathcal{F}'$ is 2-colorable, a contradiction.

3. Consider E and color all its vertices except for x red, color the rest of vertices blue. Then since $\mathcal{F}$ is not 2-colorable, we must have a monochromatic edge, E^*, in this coloring. E^* is not red, since otherwise it would be a proper subset of E, which would contradict that $\mathcal{F}$ is a Sperner family. Thus E^* is blue. Moreover, since $\mathcal{F}$ is intersecting, $E \cap E^* \neq \emptyset$, and the vertices in this intersection must be blue. Thus $E \cap E^* = \{x\}$. To conclude, let $E' = E^*$.

□

Next, we investigate the properties of swap robust families used to prove Theorem 1.

Proposition 2.

1. *[Taylor (1995)] Vertices of a swap robust family $\mathcal{F}$ can be ordered $x_1, x_2, \dots, x_n$ such that $x_i \leq x_j$ for all $1 \leq i < j \leq n$.*

2. *The family $\mathcal{F}$ is swap robust if and only if there is an ordering of vertices $1, 2, \dots, n$ such that any right shift does not change a family, i.e., $\mathcal{F}(i, j) = \mathcal{F}$ for all $1 \leq i < j \leq n$.*

Proof. 1. Observe first that in any swap robust family, for any two vertices x, y, either $x \leq y$ or $y \leq x$.

Indeed, otherwise $x \not\leq y$ and $y \not\leq x$. Thus there are sets X, Y, $x \in X$, $y \in Y$, $x \notin Y$, $y \notin X$ such that $X \setminus \{x\} \cup \{y\} \notin \mathcal{F}$ and $Y \setminus \{y\} \cup \{x\} \notin \mathcal{F}$. But this contradicts the definition of swap robustness. Observe that if $x \leq y$ and $y \leq z$ then $x \leq z$. Indeed assume that $x \leq y$ and $y \leq z$. Lets show that for any set E, $x \in E, z \notin E$, we have that $E \setminus \{x\} \cup \{z\} \in \mathcal{F}$. If $y \notin E$, we have that $E' = E \setminus \{x\} \cup \{y\} \in \mathcal{F}$, thus $E' \setminus \{y\} \cup \{z\} = E \setminus \{x\} \cup \{z\} \in \mathcal{F}$. If $y \in E$, we have that $E' = E \setminus \{y\} \cup \{z\} \in \mathcal{F}$. Thus $E' \setminus \{x\} \cup \{y\} = E \setminus \{x\} \cup \{z\} \in \mathcal{F}$. We also have that if $x < y$ then $y \not< x$ (this would contradict weak robustness again). This shows that $\leq$ is a partial order, moreover, it is a total order.

2. Let $\mathcal{F}$ be a swap robust family. Consider the ordering $x_1, \dots, x_n$ of vertices such that $x_i \leq x_{i+1}$, $i = 1, \dots, n-1$. If $\mathcal{F}(x_i, x_j) \neq \mathcal{F}$, $i < j$, then for some set E, with $x_i \in E$ and $x_j \notin E$, $E \setminus \{x_i\} \cup \{x_j\} \notin \mathcal{F}$, but this is a contradiction to the fact that $x_i \leq x_j$.

Now, assume that there is an ordering $x_1, \dots, x_n$ of vertices such that no right shift results in a new family. Assume that $\mathcal{F}$ is not weakly robust, then there are two edges, A, B

and two vertices a, b such that $a \in A \setminus B$ and $b \in B \setminus A$ and $A \setminus \{a\} \cup \{b\} \notin \mathcal{F}$ and $B \setminus \{b\} \cup \{a\} \notin \mathcal{F}$. Assume, without loss of generality that $a = x_i$ and $b = x_j$, $i < j$. Then the right shift $\mathcal{F}(x_i, x_j)$ will result in a new set $A \setminus \{a\} \cup \{b\}$, a contradiction.

□

Proof of Theorem 1. Let $\mathcal{F}$ be a maximal proper swap robust SVG on set of voters $[n]$, with minimal winning coalitions of size r. Note that if $r = 1$, then $\mathcal{F}^{min}$ consists of one set, say $\{1\}$, and then $\mathcal{F}$ is the family of all sets containing $\{1\}$. So, we could assume that $r \geq 2$. Then, by Proposition 1.2, we see that $\mathcal{F}^{min}$ corresponds to an r-uniform 3-chromatic hypergraph. Let the vertex set of this hypergraph be $X = support(\mathcal{F})$ and edge set be $\mathcal{F}^{min}$. Let $n' = |support(\mathcal{F})|$. We also have then that $\mathcal{F}$ corresponds to a 3-chromatic hypergraph.

If $n' \leq 2r - 2$ then $\mathcal{F}^{min}$ is 2-colorable. If $n' = 2r - 1$ and $\mathcal{F}^{min}$ is not complete (i.e. it is not isomorphic to $\binom{[n']}{r}$)), then it is also 2 colorable. So either $\mathcal{F}^{min}$ is isomorphic to $\binom{[2r-1]}{r}$ or $n' \geq 2r$. We shall show that the latter is impossible. Assume that $n' \geq 2r$. Lets order the vertices of X, $x_1, x_2, \ldots, x_{n'}$ such that $x_i \leq x_j$ for all $1 \leq i < j \leq n'$. Such an ordering exists by Proposition 2.1. Let $X' = \{x_1, \ldots, x_{n'-2r+2}\}$.

Claim 1. For any $x, y \in X'$, $x \sim y$.
Fix a vertex x and $E', E'' \in \mathcal{F}^{min}$ such that $E' \cap E'' = \{x\}$ (such exist by Proposition 1.3). Consider $y \notin E' \cup E''$, and $E \in \mathcal{F}^{min}$ such that $y \in E$ and $x \notin E$. Consider E, E'', x, y and apply swap robustness property. Then either $E \setminus \{y\} \cup \{x\} \in \mathcal{F}^{min}$ or $E'' \setminus \{x\} \cup \{y\} \in \mathcal{F}^{min}$. However the latter is impossible since $(E'' \setminus \{x\} \cup \{y\}) \cap E' = \emptyset$. Thus $E \setminus \{y\} \cup \{x\} \in \mathcal{F}^{min}$. This implies that $y \leq x$. Moreover, we see that for any x, there are at least $n' - 2r + 1$ vertices which have smaller of equal influence as x (since y was chosen arbitrarily from $N \setminus (E \cup E')$). Consider the smallest index j such that $x_1 < x_j$, i.e., in particular $x_{j-1} < x_j$ and $x_i \sim x_{i'}$ if $i, i' < j$. Then there are $j - 2$ vertices y such that $y \leq x_{j-1}$. On the other hand we know that there are at least $n' - 2r + 1$ vertices y such that $y \leq x_{j-1}$. Thus $n' - 2r + 1 \leq j - 2$, so $j \geq n' - 2r + 3$ and therefore any two vertices in $\{x_1, \ldots, x_{j-1}\} \subseteq \{x_1, \ldots, x_{n'-2r+2}\}$ are twins. This proves the claim.

Claim 2. Let $X'' = \{x_{n'}, x_{n'-1}, \ldots, x_{n'-r+2}\}$. For any $E \in \mathcal{F}^{min}$, $E \cap X'' \neq \emptyset$.
Assume that $E \cap X'' = \emptyset$. Let $A = E \cap X'$, $B = E \setminus A$. Note that $A \neq \emptyset$ because otherwise $E \subseteq X - X' - X''$, a contradiction since $|X - X' - X''| = n' - (n' - 2r + 2) - (r - 1) = r - 1$. By reordering vertices of X', we might assume that $A = \{x_1, \ldots, x_i\}$. Let $v \in N \setminus (E \cup X'')$. Note that such vertex exists since $n' \geq 2r$. Let $Q = \{v\} \cup X''$. We have that $E \leq_s Q$, thus $Q \in \mathcal{F}^{min}$. But $E \cap Q = \emptyset$, a contradiction. This proves Claim 2.

Lets color the set of last $r - 1$ vertices, X'', red, rest blue. There is no red set in $\mathcal{F}^{min}$ since each such set has r vertices; there is no blue set in $\mathcal{F}^{min}$ since by Claim 2 each such set intersects X'' by at least one vertex. Thus, $\mathcal{F}$ is 2-colorable, a contradiction.

□

4 Relating the size of a family to characteristics of its minimal members

This section addresses the following question. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two proper SVGs such that $|\mathcal{F}_1^{min}| = |\mathcal{F}_2^{min}| = k$ and such that all minimal winning coalitions in both SVGs have size r. We call such two SVGs ***r*-similar of size** k. When can we say that $|\mathcal{F}_1| > |\mathcal{F}_2|$?

We prove that $|\mathcal{F}_1| \geq |\mathcal{F}_2|$ if $\mathcal{F}_2^{min}$ is obtained from $\mathcal{F}_1^{min}$ by shifting, as stated in Theorem 2 below. Further, using Kruskal-Katona theorem, we prove in Proposition 3, that among all r-similar SVGs, $\mathcal{F}$ has the smallest size if sets in $\mathcal{F}^{min}$ are the last ones in co-lex order of all r-element subsets. We also seek a system parameter allowing to answer the original question of when $|\mathcal{F}_1| > |\mathcal{F}_2|$ for two r-similar SVGs. One might think that the number of blockers might be such a parameter. Since a blocker is a voter whose 'yes' vote is necessary for the bill to be passed, even if all the other voters vote in favor of the bill, the voting body will be unable to sanction any collective action if the blocker votes 'no'. Therefore it might seem rational to believe that if a voting game has less blockers than another, then its a-priori power to act should be greater, i.e., if $b(\mathcal{F}_1) \leq b(\mathcal{F}_2)$, then $|\mathcal{F}_1| \geq |\mathcal{F}_2|$. The other possible criteria are the number of non-dummies or the size of the smallest blocking set. Theorem 3 shows that, in general, the size of a family is not a monotone function with respect to size of the smallest blocking set, number of blockers, or number of non-dummies.

Theorem 2. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two r-similar SVGs such that $\mathcal{F}_2^{min} = \mathcal{F}_1^{min}(i,j)$ for some $i, j \in N$. Then $|\mathcal{F}_2| \leq |\mathcal{F}_1|$.*

Proposition 3. *Among all r-similar SVGs of size k, an SVG whose minimal winning coalitions are last k sets in co-lex order with respect to some ordering of the players, has the smallest size.*

Example The co-lex order of 2-elt subsets of [4] is

$$\{1,2\} < \{1,3\} < \{2,3\} < \{1,4\} < \{2,4\} < \{3,4\}.$$

If $\mathcal{F}^{min}$ consists of three first sets in this order then

$$\mathcal{F} = \{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}\},$$

if $\mathcal{F}^{min}$ consists of the last three sets in this order then

$$\mathcal{F} = \{\{1,4\},\{2,4\},\{3,4\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}\}.$$

So, the last three sets in co-lex order generate smaller set system.

Theorem 3. *There are r-similar proper weakly robust SVGs $\mathcal{F}_i$ and $\mathcal{F}'_i$ such that $|\mathcal{F}_i| < |\mathcal{F}'_i|$, $i = 1, 2, 3, 4$ and such that*

- $|support(\mathcal{F}_1)| > |support(\mathcal{F}'_1)|$ *and* $|support(\mathcal{F}_2)| < |support(\mathcal{F}'_2)|$,
- $|support(\mathcal{F}_i)| = |support(\mathcal{F}'_i)|$, $i = 3, 4$,
- $b(\mathcal{F}_3) > b(\mathcal{F}'_3)$ *and* $b(\mathcal{F}_4) < b(\mathcal{F}'_4)$,
- $tr(\mathcal{F}_3) < tr(\mathcal{F}'_3)$ *and* $tr(\mathcal{F}_4) > tr(\mathcal{F}'_4)$.

We need some more definitions in this section. For a set system $\mathcal{F} \subseteq \binom{[n]}{r}$, let the *down-shadow* of $\mathcal{F}$ be

$$\partial_l(\mathcal{F}) = \{S \in \binom{[n]}{r-1} : S \subset X, X \in \mathcal{F}\}.$$

Let the *upper shadow* of $\mathcal{F}$ be

$$\partial_u(\mathcal{F}) = \{S \in \binom{[n]}{r+1} : S \supset X, X \in \mathcal{F}\}.$$

Recall that the downset of $\mathcal{F}$ is $downset(\mathcal{F}) = \{S : S \subseteq X, X \in \mathcal{F}\}$, the upset of $\mathcal{F}$ is $upset(\mathcal{F}) = \{S : S \supseteq X, X \in \mathcal{F}\}$. The main tool of this section is a celebrated theorem of Kruskal and Katona which we state in the elegant formulation by Lovász. Another tool is the shifting technique by Frankl (1987).

Theorem 4 (Kruskal (1963), Katona (1968), Lovász (1979), Frankl (1984))**.** *Let $\mathcal{F} \subseteq \binom{[n]}{r}$, $|\mathcal{F}| = \binom{m}{r}$, for a real number $m \geq r$. Then $|\partial_l(\mathcal{F})| \geq \binom{m}{r-1}$.*

Proposition 4 (Frankl (1991))**.** *Let $\mathcal{F} \subseteq \binom{[n]}{r}$ then $\partial_l(\mathcal{F}(i,j)) \subseteq (\partial_l(\mathcal{F}))(i,j)$. Moreover, if $\mathcal{F}$ is a family of the k first sets of $\binom{[n]}{r}$ in co-lex order, then it has the smallest shadow among all families of r-element sets with k sets in the family.*

Corollary 1. *$downset(\mathcal{F}(i,j)) \subseteq (downset(\mathcal{F}))(i,j)$.*

Proof. Observe first that $downset(\mathcal{F}(i,j)) = \mathcal{F}(i,j) \cup \partial_l(\mathcal{F}(i,j)) \cup \partial_l(\partial_l(\mathcal{F}(i,j)) \cup \cdots$. Similarly $(downset(\mathcal{F}))(i,j) = \mathcal{F}(i,j) \cup (\partial_l(\mathcal{F}))(i,j) \cup (\partial_l(\partial_l(\mathcal{F})))(i,j) \cup \cdots$. We have that

$$\partial_l(\partial_l(\partial_l(\cdots\partial_l(\mathcal{F}(i,j))\cdots))) \subseteq \partial_l(\partial_l(\cdots(\partial_l(\mathcal{F}))(i,j)\cdots)) \subseteq \partial_l((\partial_l(\partial_l(\cdots\partial_l(\mathcal{F})\cdots)))(i,j)) \subseteq (\partial_l(\partial_l(\cdots\partial_l(\mathcal{F}))))(i,j).$$

□

Proposition 5. *$upset(\mathcal{F}(i,j)) \subseteq (upset(\mathcal{F}))(i,j)$.*

Proof. Proposition 4 implies that $downset(\mathcal{F}(i,j)) \subseteq (downset(\mathcal{F}))(i,j)$. We shall use complementation to use the results on down-sets to obtain results about upsets. We have that $upset(\mathcal{F}) = \{S : S \supseteq X, X \in \mathcal{F}\} = \{X \cup X' : X' \subseteq X^c, X \in \mathcal{F}\} = \{(X^c \setminus X')^c : X' \subseteq X^c, X \in \mathcal{F}\} = downset(\mathcal{F}^c)^c$. This implies that $upset(\mathcal{F}) = (downset(\mathcal{F}^c))^c$. Thus $upset(\mathcal{F}(i,j)) = (downset(\mathcal{F}(i,j)^c))^c = (downset(\mathcal{F}^c(j,i)))^c \subseteq ((downset(\mathcal{F}^c))(j,i))^c = ((upset(\mathcal{F}))^c(j,i))^c = (upset(\mathcal{F}))(i,j)$. □

Now we come to the proofs of the results stated in the beginning of this section.

Proof of Theorem 2. The theorem follows immediately from the Proposition 5 using the fact that $|upset(\mathcal{F})| = |upset(\mathcal{F})(i,j)|$ and the fact that shifting operation does not change the cardinality of a set system. □

Proof of Proposition 3. First, we observe that if a set X_1 immediately precedes X_2 in co-lex order, then X_2^c immediately precedes X_1^c in colex order. Indeed, let i be the largest element of $X_1 \Delta X_2$. We have that $i \in X_2$. Since $X_1 \Delta X_2 = X_1^c \Delta X_2^c$, and $i \in X_1^c$, we have that X_2^c is less then X_1^c in co-lex order. If there is a set Y, such that it is between X_2^c and X_1^c in co-lex order, then we have that Y^c is between X_1 and X_2 in co-lex order. Thus X_2^c immediately precedes X_1^c in co-lex order.

We have that $upset(\mathcal{F}) = downset(\mathcal{F}^c)^c$. Since $\mathcal{F}$ is the set of the last k sets in co-lex order, $\mathcal{F}^c$ is the set of the first k sets in co-lex order. Kruskal-Katona theorem claims that if $\mathcal{F} \subseteq \binom{[n]}{r}$ is formed by $|\mathcal{F}|$ smallest sets in co-lex order then $\partial_l(\mathcal{F})$ is the smallest among all set systems of size $|\mathcal{F}|$. Observing that if the sets in $\mathcal{F}$ are the smallest r-element sets in co-lex order then $\partial_l(\mathcal{F})$ are also smallest $(r-1)$-element sets in co-lex order, we have that if $\mathcal{F}$ has smallest r-element sets in co-lex order then $downset(\mathcal{F})$ is the smallest among all families of r-elements sets of size $|\mathcal{F}|$. We have that $downset(\mathcal{F}^c)$ is smallest among all families of k sets of size $n-r$, thus $upset(\mathcal{F})$ is also smallest among all families of k sets of size r. □

Proof of Theorem 3. To prove this theorem, we provide an explicit construction. Let $\mathcal{F}_i$s and $\mathcal{F}_i'$s, $i = 1,2,3,4$, be swap robust SVGs defined by the sets of their minimal winning coalitions as follows.

$\mathcal{F}_1 \subseteq 2^{[7]}$,
$\mathcal{F}_1^{min} = \{\{1,i\} : i = 2,\dots,7\} \cup \{\{2,i\} : i = 3,4,5\} \cup \{\{3,i\} : i = 4,5\}$.

$\mathcal{F}_1' \subseteq 2^{[7]}$,
$(\mathcal{F}_1')^{min} = \{\{1,i\} : i = 2,\dots,6\} \cup \{\{2,i\} : i = 3,\dots,6\} \cup \{\{3,i\}, i = 4,5\}$.

$\mathcal{F}_2 \subseteq 2^{[8]}$,
$\mathcal{F}_2^{min} = \{\{1,i\} : i = 2,\dots,7\} \cup \{\{2,i\} : i = 3,\dots,7\} \cup \{3,4\}$.

$\mathcal{F}'_2 \subseteq 2^{[8]}$,
$(\mathcal{F}'_2)^{min} = \{\{1, i\} : i = 2, \ldots, 8\} \cup \{\{2, i\} : i = 3, 4, 5\} \cup \{\{3, i\} : i = 4, 5\}$.

$\mathcal{F}_3 \subseteq 2^{[6]}$,
$\mathcal{F}_3^{min} = \{\{1, 2, i\} : i = 3, \ldots, 6\} \cup \{\{1, 3, i\} : i = 4, 5\} \cup \{1, 4, 5\}$.

$\mathcal{F}'_3 \subseteq 2^{[6]}$,
$(\mathcal{F}'_3)^{min} = \{\{1, 2, i\} : i = 3, \ldots, 6\} \cup \{\{1, 3, i\} : i = 4, 5\} \cup \{2, 3, 4\}$.

$\mathcal{F}_4 \subseteq 2^{[12]}$,
$\mathcal{F}_4^{min} = \{\{1, 2, i\} : i = 3, \ldots, 12\} \cup \{\{1, 3, i\} : i = 4, \ldots, 12\} \cup \{2, 3, 4\}$.

$\mathcal{F}'_4 \subseteq 2^{[12]}$,
$(\mathcal{F}'_4)^{min} = \{\{1, 2, i\} : i = 3, \ldots, 12\} \cup \{\{1, 3, i\} : i = 4, \ldots, 7\} \cup$
$\{\{1, 4, i\} : i = 5, \ldots, 7\} \cup \{\{1, 5, i\} : i = 6, 7\} \cup \{1, 6, 7\}$.

Note that $|(\mathcal{F}_1)^{min}| = |(\mathcal{F}'_1)^{min}| = 11$, $|(\mathcal{F}_2)^{min}| = |(\mathcal{F}'_2)^{min}| = 12$, $|(\mathcal{F}_3)^{min}| = |(\mathcal{F}'_3)^{min}| = 7$, $|(\mathcal{F}_4)^{min}| = |(\mathcal{F}'_4)^{min}| = 20$. An explicit calculation will show that $|\mathcal{F}_1| = 103$, $|\mathcal{F}'_1| = 104$, $|\mathcal{F}_2| = 204$, $|\mathcal{F}'_2| = 207$, $|\mathcal{F}_3| = 23$, $|\mathcal{F}'_3| = 25$, $|\mathcal{F}_4| = 1790$ and $|\mathcal{F}'_4| = 1855$. Note also that all these families could be made proper by adding the same set of voters to each winning coalition. This will not change the relative amount of blockers or non-dummies.

□

5 Table of corresponding voting theory and hypergraph terms

players, voters	vertices
coalitions	edges, hyperedges, sets
simple voting game (SVG)	a family of sets satisfying the conditions stated in page 2 proper filter
proper SVG	intersecting proper filter
blocking set	transversal
the set of non-dummy players	support
dummy player	vertex not in any minimal (by inclusion) set

6 Conclusions

In this paper we prove that if the minimal winning coalitions corresponding to a proper, maximal and swap robust SVG, $\mathcal{F}$, are all of the same size, then $\mathcal{F}$ is unique up to an isomorphism.

Furthermore, it is an $r-$trivial SVG. However, the result does not hold if the SVG is not swap robust. We also provide examples to show that the size of an SVG cannot be characterized by its simple parameters like the number of blockers, the number of non-dummies, or the size of the minimal blocking set. This shows that a collectivity with a larger number of blockers or a bigger blocking set does not necessarily have a smaller a-priori bias in favor of changing the status-quo than another collectivity with fewer number of blockers or a smaller blocking set. Thus one must refrain from making fallacious naive assumptions.

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References

[1] Barua, R., Chakravarty, S. R., Roy, S. and Sarkar, P., 2004. A characterization and some properties of the Banzhaf-Coleman-Dubey-Shapley sensitivity index. Games and Economic Behavior 49, 31-48.

[2] Berge, C., 1988. Hypergraphs. Selected topics in graph theory, 3, 189–206, Academic Press, San Diego, CA.

[3] Carreras, F. and Freixas, J., 1996. Complete simple games. Mathematical Social Sciences 32, 139–155.

[4] Coleman, J. S., 1971. Control of collectivities and the power of a collectivity to act. In: Leberman, B., (Ed.), Social Choice. Gordon and Breach, New York.

[5] Engel, K., 1997. Sperner Theory. Cambridge University Press.

[6] Erdős, P. and Hindman, N., 1984. Enumeration of intersecting families. Discrete Mathematics 48, no. 1, 61–65.

[7] Erdős, P. and Lovász, L., 1975. Problems and results on 3-chromatic hypergraphs and some related questions. Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P.

Erdős on his 60th birthday), Vol. II, pp. 609–627. Colloq. Math. Soc. Janos Bolyai, Vol. 10, North-Holland, Amsterdam.

[8] Felsenthal, D. S. and Machover, M., 1998. The measurement of voting power: theory and practice, problems and paradoxes. Edward Elgar, Cheltenham.

[9] Frankl, P., 1987. The shifting technique in extremal set theory, in: Surveys in Combinatorics. Lond. Math. Soc. Lect. Note Ser., 123, 81–110.

[10] Frankl, P., 1991. Shadows and shifting. Graphs and Combinatorics 7, 23–29.

[11] Frankl, P., 1984. A new short proof of Kruskal-Katona theorem. Discrete Mathematics 48, 327–329.

[12] Freixas, J. and Molinero, X., 2009. On the existence of a minimum integer representation for weighted voting systems. Annals of Operations Research 166, 243–260.

[13] Freixas, J. and Pons, M., 2008. Achievable hierarchies in simple games. Theory and Decision DOI: 10.1007/s11238-008-9108-0.

[14] Freixas, J. and Puente, M., 1998. Complete games with minimum. Annals of Operation Research 84, 97–109.

[15] Garcia-Molina, H. and Barbara, D., 1985. How to assign votes in a distributed system. Journal of the Association for Computing Machinery 32, 841–860.

[16] Isbell, J. R., 1956. A class of majority games. Quarterly Journal of Mathematics Oxford Ser. 7, 183–187.

[17] Katona, G. O. H., 1968. A theorem of finite sets. In: Proceedings of Tihany Conference, 1966; 187–207, Budapest.

[18] Kruskal, J. B., 1963. The number of simplices in a complex. In: Mathematical Optimization Techniques, 251–278. Berkeley-Los Angeles, University of California Press.

[19] Loeb, D. and Conway, A., 2000. Voting fairly: transitive maximal intersecting families of sets, In memory of Gian-Carlo Rota. Journal of Combinatorial. Theory Ser. A 91, no. 1-2, 386–410.

[20] Lovász, L., 1979. Combinatorial problems and exercises. Academiai Kiadó, (Budapest) - North Holland Publ. (Amsterdam). Problem 13.31.

[21] Taylor, A. D. and Zwicker, W. S., 1993. Weighted voting, multicameral representation, and power. Games and Economic Behavior 5, 170–181.

[22] Taylor, A. D., 1995. Mathematics and Politics, Springer, Berlin.