SUBSETS OF VERTICES OF THE SAME SIZE AND THE SAME MAXIMUM DISTANCE

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Abstract. For a connected graph $G = (V, E)$ and a subset X of its vertices, let $d^*(X) = \max\{\mathrm{dist}_G(x, y) : x, y \in X\}$ and let $h^*(G)$ be the largest k such that there are disjoint vertex subsets A and B of G, each of size k such that $d^*(A) = d^*(B)$. Let $h^*(n) = \min\{h^*(G) : |V(G)| = n\}$. We prove that $h^*(n) = \lfloor (n + 1)/3 \rfloor$, for $n \geq 6$. This solves the homometric set problem restricted to the largest distance exactly. In addition we compare $h^*(G)$ with a respective function $h_{\mathrm{diam}}(G)$, where $d^*(A)$ is replaced with $\mathrm{diam}(G[A])$.

1. Introduction

For a subset X of vertices of a graph G, let $d^*(X) = \max\{\mathrm{dist}_G(x, y) : x, y \in X\}$, where dist_G is the distance in G. Two subsets of vertices $A, B \subseteq V$ are *weakly homometric* if $|A| = |B|$, $A \cap B = \emptyset$, and $d^*(A) = d^*(B)$. Let $h^*(G)$ be the largest k such that G has weakly homometric sets of size k each. Let $h^*(n)$ be the smallest value of $h^*(G)$ over all connected n-vertex graphs. Informally, any connected graph G on n vertices has two disjoint subsets of vertices of the same size at least $h^*(n)$ that have the same largest distance (in G) between their vertices.

The notion of weakly homometric sets originates from the notion of homometric sets introduced by Albertson et al. [1]. For a subset of vertices X, let $d(X)$ be a multiset of pairwise distances between the vertices of X. Two disjoint sets of vertices A and B are called homometric, if $d(A) = d(B)$. Let $h(G)$ be the largest k such that G has two homometric sets of size k each. Let $h(n)$ be the largest value of $h(G)$ among all connected n-vertex graphs. The best known bounds on $h(n)$ are as follows:

$$c\left(\frac{\log n}{\log\log n}\right)^2 \leq h(n) \leq n/4 - c' \log\log n,$$

for positive contants c, c', where the lower bound is due to Alon [2], and the upper one is due to Axenovich and Özkahya [3], both of the bounds are slight improvements of the original bounds by Albertson et al. [1]. There are much better bounds on $h(G)$ known when G is a tree or when G has diameter 2, see Fulek and Mitrović [6] and Bollobás et al. [4], see also an earlier paper by Caro and Yuster [5]. Weakly homometric sets are concerned only with one, the largest, distance. In this note we find $h^*(n)$ exactly.

Theorem 1. *For any $n \geq 6$, $h^*(n) = \lfloor (n + 1)/3 \rfloor$.*

Note that considering connected graphs in the definition of h^* is not an essential restriction. Indeed, if a graph G is not connected and has at least two components of size at least two each, then by taking ∞ as a distance between any two vertices

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from different components, we see that $h^*(G) \geq \lfloor n/2 \rfloor$. Otherwise, G has two connected components, one of which is a single vertex. Thus by Theorem 1 applied to the larger component $h^*(G) \geq \lfloor n/3 \rfloor$.

When the distance is considered in a subgraph rather than in an original graph, we consider the following function that is of independent interest. For a graph G, $h_{\text{diam}}(G)$ is the largest integer k such that there are disjoint sets $A, B \subseteq V(G)$, each of size k and so that $\text{diam}(G[A]) = \text{diam}(G[B])$.

Theorem 2. *Let G be an n-vertex graph, then $h_{\text{diam}}(G) \geq \lfloor (n+1)/3 \rfloor$. Moreover if $\text{diam}(G) \geq 4$ or $\text{diam}(G) = 1$ then $h_{\text{diam}}(G) = \lfloor n/2 \rfloor$.*

In order to prove the main result, we consider an auxiliary coloring of the edges of a complete graph on the vertex set $V = V(G)$ with colors $1, 2, \ldots, \text{diam}(G)$ such that the color of xy is $\text{dist}_G(x, y)$, $x, y \in V$. The result follows from observations about the structure of the color classes. In fact, the proof allows for an algorithm determining large weakly homometric sets.

2. Proofs

Let, for a graph G and $X \subseteq V(G)$, $E_i(X) = \{xy : \ x, y \in X, \text{dist}_G(x, y) = i\}$, i.e., E_i is a set of pairs at distance i in G. We say that $E_i(X)$ is *good* is it contains two disjoint pairs $xy, x'y'$. Note that if a non-empty $E_i(X)$ is not good, i.e., bad, it is a triangle or a star in X. Further observe that if $X = A \cup B$, where A and B are weakly homometric in G, then $E_i(X)$ is good, for $i = d^*(A)$. We say that we *split* a set X of vertices if we form two disjoint subsets of X of size $\lfloor |X|/2 \rfloor$. We denote $d(xy) = \text{dist}_G(x, y)$, $x, y \in V(G)$. We denote the edge set of a star with center x and leaves set X as $S(x, X)$.

Lemma 3. *Let G be a graph, $X \subseteq V(G)$, $i = d^*(X)$. If $E_i(X)$ is good or $d^*(X) \leq 2$, then $h^*(G) \geq \lfloor (|X| - 1)/2 \rfloor$.*

Proof. Assume first that $xy, x'y' \in E_i(X)$ are disjoint pairs of vertices and $i = d^*(X)$. Split X such that x, y are in one part and x', y' in another part. The resulting sets are weakly homometric sets. If $d^*(X) = 2$, then either $E_2(X)$ is good implying $h^*(G) \geq \lfloor |X|/2 \rfloor$ or non-edges form a star or a triangle, so deleting one vertex allows to split the remaining vertices of X in two sets each inducing a clique. Thus $h^*(G) \geq \lfloor (|X| - 1)/2 \rfloor$ in this case. If $d^*(X) = 1$, then X induces a clique and $h^*(G) \geq \lfloor |X|/2 \rfloor$. □

Proof of Theorem 1. First we shall show the lower bound on $h^*(n)$. Consider a connected graph G on n vertices. Let $d = \text{diam}(G)$. If $d = 2$, the lower bound follows from the Lemma 3. So, we assume that $d \geq 3$. If $E_d(V)$ is good, then by Lemma 3 $h^*(G) \geq \lfloor (n-1)/2 \rfloor \geq \lfloor (n+1)/3 \rfloor$. If $E_d(V)$ is bad, it is either forms a triangle or a star.

Case 1 $E_d(V)$ forms a triangle xyz.

Let x' and y' be distinct vertices such that $d(xx') = d(yy') = d - 1$. Such x', y' could be chosen on a shortest xy-path. Let A and B be disjoint subsets of $V - z$, each of size $\lfloor (n+1)/3 \rfloor$, A containing x and x', B containing y and y'. We see that A and B are weakly homometric with maximum distance $d - 1$.

Case 2 $E_d(V)$ forms a star.
Let $E_d(V) = S(x_0, Y)$, forming a star with center x_0 and leaves set Y. Let $x_d \in Y$, i.e., $d(x_0x_d) = d$. Consider a shortest x_0-x_d path $x_0, \dots, x_d$ of length d.

Case 2.1 $|Y| \le n - \lfloor (n+1)/3 \rfloor - 1$.
Let A and B be disjoint sets such that $|A| = |B| = \lfloor (n+1)/3 \rfloor$, $A \subseteq V - Y - \{x_1\}$, A contains x_0, x_{d-1}, B contains x_1, x_d. Then A and B are weakly homometric with largest distance $d-1$.

Case 2.2 $|Y| \ge n - \lfloor (n+1)/3 \rfloor$.
In particular $d \le \lfloor (n+1)/3 \rfloor$. Let T be the breadth-first search tree with root x_0. Let L_i's be the layers of T, i.e., sets of vertices at distance i from x_0, $i = 1, \dots, d$. We have that $L_d = Y$, $L_0 = \{x_0\}$.

If T is a broom, i.e., all vertices of Y have a common neighbor, x_{d-1} in T, then $d^*(Y \cup \{x_{d-1}, x_{d-2}\}) = 2$ and by Lemma 3 $h^*(G) \ge \lfloor (|Y| + 2 - 1)/2 \rfloor \ge \lfloor (n - \lfloor (n+1)/3 \rfloor + 1)/2 \rfloor \ge \lfloor (n+1)/3 \rfloor$.

If T is not a broom, then some layer L_i, $i < d$, has more than one vertex and $d \le \lfloor (n+1)/3 \rfloor - 1$. Let i be the smallest such index, i.e., $L_j = \{x_j\}$ for $j < i$. Then we see that $S(x_j, Y) \subseteq E_{d-j}(V)$, $j < i$. Let $V_j = V - \{x_0, \dots, x_{j-1}\}$, $j = 1, \dots, d$.

We consider $E_{d-1}(V_1), E_{d-2}(V_2), \dots$ in order and show that each of these sets $E_j(V_j)$ is either good, allowing to use Lemma 3, or is a star with center x_j.

If for $0 < j < i$, $S(x_j, Y) \ne E_{d-j}(V_j)$, then for smallest such j, $E_{d-j}(V_j)$ is good and $d^*(V_j) = d - j$, so by Lemma 3, $h^*(G) \ge \lfloor (n - j - 1)/2 \rfloor \ge \lfloor (n - (d-2) - 1)/2 \rfloor \ge \lfloor (n - \lfloor (n+1)/3 \rfloor + 2)/2 \rfloor \ge \lfloor (n+1)/3 \rfloor$. Thus, we have that $S(x_j, Y) = E_{d-j}(V_j)$, $j = 1, \dots, i-1$. Consider $x_i, x_i' \in L_i$. We have that $d(x_ix_d) = d - i$, and the largest distance $d^*(V_i) = d - i$. Moreover, we claim that $d(x_i'y) = d - i$ for each $y \in Y$. Assume not and $d(x_i'y) < d - i$. Then $d(x_{i-1}y) < d - i + 1$, a contradiction. Thus $E_{d-i}(V_i)$ is good, and by Lemma 3, we have $h^*(G) \ge \lfloor (n-i)/2 \rfloor \ge \lfloor (n - \lfloor (n+1)/3 \rfloor + 1)/2 \rfloor \rfloor \ge \lfloor (n+1)/3 \rfloor$. In all these cases we have that $h^*(G) \ge \lfloor (n+1)/3 \rfloor$.

For the upper bound on $h^*(n)$, let $k = \lfloor (n+1)/3 \rfloor$. Consider a graph G that is a union of a clique K on $n-k$ vertices and a path P on $k+1$ vertices such that K and P share exactly one vertex x that is an end-point of P. Consider two weakly homometric sets A and B in $V(G)$ that have the largest possible size $h^*(G)$. If $(A \cup B) \subseteq V(K)$ then $h^*(G) \le \lfloor (n-k)/2 \rfloor$. So, let's assume that $x' \in V(P) \cap (A \cup B)$ such that x' has the largest distance from x among the vertices of $A \cup B$. Assume further that $x' \in A$ and let $i = d(x'x)$. Then $E_{i+1}(G)$ consists of all pairs $x'y$, $y \in V(K) - \{x\}$ and pairs containing vertices from P that are further from x as x' (if any). Since there are no such vertices in $A \cup B$, we see that E_{i+1} restricted to $A \cup B$ is a star, so $i + 1 \ne d^*(A)$. Thus $A \subseteq V(P)$. If $d^*(A) > 1$ then $d^*(B) > 1$ and $B \setminus V(K) \ne \emptyset$. Thus at least one vertex in P is from B, so $|A| \le |V(P)| - 1 = k$. If $d^*(A) = 1$, then $|A| = 2$. Thus $h^*(G) \le \max\{\lfloor (n-k)/2 \rfloor, k, 2\} \le \lfloor (n+1)/3 \rfloor$, for $n \ge 6$. □

Proof of Theorem 2. Let G be a graph on n vertices and let $k = \lfloor (n+1)/3 \rfloor$. Assign a color $c(A) = \text{diam}(G[A])$ to each k-element subset A of vertices of G. Then $c(A) \in \{1, 2, \dots, k-1, \infty\}$. So, there are at most k colors used in this coloring. The coloring c corresponds to a coloring of vertices of the Kneser graph $K(n,k)$. Since the chromatic number $\chi(K(n,k)) = n - 2k + 2$, see Lovász [7], and the number k of colors used is less than the chromatic number $n - 2k + 2$, we see that c is not a proper coloring, so there are two disjoint sets A and B of the same color. Thus $h_{\text{diam}}(G) \geq k$. In particular, $h_{\text{diam}}(G) \geq \lfloor (n+1)/3 \rfloor$.

If $\text{diam}(G) = 1$ then G is a complete graph and the conclusion follows trivially. If $\text{diam}(G) \geq 4$, we consider a vertex v that is at distance at least 4 to some other vertex. Consider a breadth first search tree with a root v. Let V_i, $i = 0, 1, 2, \dots, q$ be the layers of that tree, i.e., V_i is a set of vertices at distance i from v, $V_0 = \{v\}$, $q \geq 4$. We see that there are no edges between any two non-consecutive layers. We shall build two disjoint sets A and B such that $G[A]$ and $G[B]$ are both disconnected, i.e., have diameter ∞.

If each layer has size less than $n/2$, put v and V_2 in A, put V_1 in B and split the remaining vertices (except maybe one) between A and B such that $|A| = |B|$. We see that v is not adjacent to any other vertex of A and we see that any vertex of V_2 is not adjacent to any vertex from $B \setminus V_2$.

If there is a layer, L, of size at least $n/2$ then the total number of vertices in all other layers is less than $n/2$. Consider the layers other than L, in order, and assign all vertices of each layer to the same set, A or B, in an alternating fashion. Split the vertices of L between A and B such that $|A| = |B| = \lfloor n/2 \rfloor$. More precisely, let $\{V_0, V_1, \dots\} \setminus L = \{V_{i_1}, V_{i_2}, \dots\}$, where $i_1 < i_2 < \cdots$. Put vertices of V_{i_k} in A if k is even, put vertices of V_{i_k} in B if k is odd. We see that there is always a full layer in A between some two vertices of B and there is a full layer of B between two vertices of A. So, $G[A]$ and $G[B]$ are disconnected. □

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