EXACT BOUNDS ON THE SIZES OF COVERING CODES

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ABSTRACT. A code $\mathcal{C}$ is a covering code of X with radius r if every element of X is within Hamming distance r from at least one codeword from $\mathcal{C}$. The minimum size of such a $\mathcal{C}$ is denoted by $c_r(X)$. Answering a question of Hämäläinen, Honkala, Litsyn and Östergård [10], we show further connections between Turán theory and constant weight covering codes. Our main tool is the theory of supersaturated hypergraphs. In particular, for $n > n_0(r)$ we give the exact minimum number of Hamming balls of radius r required to cover a Hamming ball of radius $r+2$ in $\{0,1\}^n$.

We prove that $c_r(B_n(\mathbf{0}, r+2)) = \sum_{1\le i\le r+1} \binom{\lfloor (n+i-1)/(r+1)\rfloor}{2} + \lfloor n/(r+1)\rfloor$ and that the centers of the covering balls $B(\mathbf{x}, r)$ can be obtained by taking all pairs in the parts of an $(r+1)$-partition of the n-set and by taking the singletons in one of the parts.

1. THE GENERALIZED COVERING RADIUS PROBLEM

Let Q be a (finite) q-ary alphabet. We usually identify Q with the set of integers $\{0, 1, 2, \ldots, q-1\}$. The set of sequences of length n, Q^n, is a metric space with the Hamming distance, $d(\mathbf{x}, \mathbf{y}) = |\{i : x_i \neq y_i\}|$ for $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, $\mathbf{x}, \mathbf{y} \in Q^n$. The *Hamming ball*, $B_{n,Q}(\mathbf{x}, r)$, about the center $\mathbf{x}$ is the set of all vectors $\mathbf{v} \in Q^n$ such that $d(\mathbf{x}, \mathbf{v}) \le r$. The subscript Q is usually omitted if it is clear from the context, especially when $Q = \{0,1\}$. We say that $\mathbf{x}$ has *weight* $d(\mathbf{x}, \mathbf{0})$, where $\mathbf{0} = (0, 0, \ldots, 0)$. We frequently call the set of codewords of $\{0,1\}^n$ of weight ℓ the ℓ-th layer of the hypercube. It is also identified with the set of all ℓ-subsets of the set $[n]$ denoted by $\binom{[n]}{\ell}$. Namely, $\mathbf{x} \in Q^n$ is identified with $X \subseteq [n]$ when $x_i = 1$ iff $i \in X$. If $\mathbf{x}, \mathbf{y} \in Q^n$ are identified with subsets $X, Y \subseteq [n]$ then $d(\mathbf{x}, \mathbf{y}) = |X \bigtriangleup Y|$.

A *covering code* $\mathcal{C}$ of a set $X \subseteq Q^n$ with radius r is a subset $\mathcal{C} \subseteq Q^n$ with the property that every element of X belongs to some Hamming ball with radius r centered about an element of $\mathcal{C}$, i.e.,

$$X \subseteq \cup_{\mathbf{y} \in \mathcal{C}} B_{n,Q}(\mathbf{y}, r).$$

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We often call a covering code of radius r an *r-cover*. The minimum size of such a code is denoted by $c_r(X)$ and is called the *r-covering number* of X. The subscript r will be omitted when the radius r is understood from the context, especially in the case $r = 1$. The 1-covering number is simply called the *covering number*. If X is the full set of codewords, $X = Q^n$, then minimizing $|\mathcal{C}|$ is called the *covering radius problem* and was studied, e.g., by Graham and Sloane [9]. More recent research deals not with the whole space but with smaller important subsets of it. For example, Hämäläinen, Honkala, Litsyn and Östergård [10] proposed to determine $c_r(B_n(\mathbf{0}, \ell))$.

The main goal of this paper is to show how the results of extremal hypergraph theory can be applied to find an (asymptotic) solution for this problem. We will show connections between Turán theory and constant weight covering codes.

2. Constant weight codes and Turán's theorem

An *(n, w, ℓ, r)-code* is a set, $\mathcal{C}$, of 0–1 vectors of length n with weight w such that every word of weight ℓ lies within Hamming distance r from at least one element of $\mathcal{C}$. In other words, the Hamming balls of radius r, centered about the elements of $\mathcal{C} \subseteq \binom{[n]}{w}$, cover all vectors of weight ℓ. The minimum size of such a code is denoted by $K(n, w, \ell, r)$ and was studied by several authors, e.g., see the survey by Etzion, Wei and Zhang [6].

If $w > \ell$, for example, then $K(n, w, \ell, w - \ell)$ is the well-known *setcover number*, $C(n, w, \ell)$. A code $\mathcal{C}$ is an $(n, w, \ell, w - \ell)$-code, if it is a family of w-subsets such that every ℓ-subset of $[n]$ is contained in at least one of the codewords. Thus its size is at least $\binom{n}{\ell}/\binom{w}{\ell}$. For fixed w and ℓ, Rödl [15] proved that, as n tends to infinity,

$$C(n, w, \ell) = (1 + o(1))\binom{n}{\ell}/\binom{w}{\ell}.$$

For $w < \ell$, the determination of the minimum $(n, w, \ell, \ell - w)$-code is still open in general. It is the *Turán problem*, and $K(n, w, \ell, \ell - w)$ is denoted by $T(n, w, \ell)$. Turán [18] (also see in [1]) proved that

$$T(n, 2, \ell) = \sum_{1 \leq i \leq \ell - 1} \binom{\lfloor (n + i - 1)/(\ell - 1) \rfloor}{2} = \frac{n^2}{2(\ell - 1)} + O(n). \tag{1}$$

Consider a partition of $[n]$ into $(\ell - 1)$ parts, $[n] = P_1 \cup \cdots \cup P_{\ell - 1}$, each part of size either $\lfloor n/(\ell - 1) \rfloor$ or $\lceil n/(\ell - 1) \rceil$. Take all pairs from the P_i's, $1 \leq i \leq \ell - 1$ as the centers of Hamming balls of radius

$\ell-2$, i.e., let the set of centers to be

$$\bigcup_{i=1}^{\ell-1}\binom{P_i}{2}.$$

Turán showed that this is the only way to get the extremal $(n,2,\ell,\ell-2)$-code. Note that this construction gives a cover not only for the ℓ-th layer, but almost for the whole Hamming ball, $B_n(\mathbf{0},\ell)$, too. We will see in Theorem 12 that the value of $c_{\ell-2}(B_n(\mathbf{0},\ell))$ is close to $T(n,2,\ell)$.

For the general case (i.e., for $w>2$), not even the limit $\lim_{n\to\infty}T(n,w,\ell)\binom{n}{w}^{-1}$, is known. The existence of this limit was shown by Katona, Nemetz and Simonovits [11], see also [6]. The current best lower bound for the Turán number, due to de Caen [2], is $T(n,w,\ell)\geq\binom{n}{w}\binom{\ell-1}{w-1}^{-1}(n-\ell+1)(n-w+1)^{-1}$. Together with the obvious upper bound $\binom{n}{w}$ we obtain that the order of magnitude of the Turán number (for fixed w and ℓ) is exactly n^w, i.e.,

$$T(n,w,\ell)=\Theta(n^w). \tag{2}$$

Here for two functions $f,g:\mathbb{R}\to\mathbb{R}$, $f=\Theta(g)$ if there are positive constants x_0, c_1 and c_2 such that $c_1\leq f(x)/g(x)\leq c_2$ for $x>x_0$.

For further results on the topic, see Sidorenko's survey [16].

3. The covering radius of Hamming balls

In this section we give the asymptotic value of the r-covering number $c_r(B_n(\mathbf{0},\ell))$ for all fixed r, ℓ with $\ell>r$. For $\ell\leq r$ the r-covering number is obviously 1. We give the exact bounds when $\ell=r+1$ and $\ell=r+2$ in the last section. Some of these bounds are corollaries of a more general result about covering products of graphs, which is interesting on its own and appears in the next sections.

Let $\mathcal{C}$ be an optimal $(n,\ell-r,\ell,r)$-code which is a family of $(\ell-r)$-element sets of size $T(n,\ell-r,\ell)$. The r-balls centered about the elements of $\mathcal{C}$ together with the r-balls centered about all the vectors of weight at most $(\ell-r-1)$ cover $B_n(\mathbf{0},\ell)$. Adding to $\mathcal{C}$ all the small sets (i.e., those having less than $(\ell-r)$ elements), we obtain

$$c_r(B_n(\mathbf{0},\ell))\leq T(n,\ell-r,\ell)+\sum_{i\leq\ell-r-1}\binom{n}{i}. \tag{3}$$

From (2) we have that the second term in (3) is negligible compared to the $T(n,\ell-r,\ell)$ as $n\to\infty$. We shall prove in Theorem 2 that this construction is asymptotically best possible. For this, we use the result on generalized Kneser graphs. A generalized Kneser graph is a graph $K(n,k,t)=(V,E)$, with vertex set $V = \binom{[n]}{k}$ and $E = \{\{F, F'\} \in V \times V : |F \cap F'| < t\}$. Let $\chi(G)$ denote the chromatic number of the graph G. Frankl and Füredi proved the following.

Theorem 1. [8] $\chi(K(n,k,t)) = (1 + o(1))T(n,t,k)$.

Next we determine $c_r(B_n(\mathbf{0},\ell))$ asymptotically.

Theorem 2. *For $\ell > r \geq 1$ fixed and $n \to \infty$ we have*

$$c_r(B_n(\mathbf{0},\ell)) = (1 + o(1))T(n, \ell - r, \ell).$$

Theorem 2 gives an asymptotic expression for $c_r\left(\binom{[n]}{\ell}\right)$ as follows.

$$\begin{aligned} K(n,\ell-r,\ell,r) = T(n,\ell-r,\ell) &\geq \\ c_r\left(\binom{[n]}{\ell}\right) &\geq c_r(B_n(\mathbf{0},\ell)) - \sum_{i \leq \ell-r-1} \binom{n}{i} = (1+o(1))T(n,\ell-r,\ell). \end{aligned}$$

The Turán configuration (i.e., an $(n, \ell - r, \ell, r)$-code) uses codewords (centers) located only at the $(\ell - r)$-th layer. The above inequalities imply that allowing centers outside of the $(\ell - r)$-th layer one can not make a (significantly) smaller r-cover of the ℓ-th layer.

Proof of Theorem 2. The upper bound follows from (3). For the lower bound we show that $c_r(B_n(\mathbf{0},\ell)) \geq \chi(K(n,\ell,\ell-r))$, then Theorem 2 follows from Theorem 1.

Let us assume, on the contrary, that $\mathcal{C} = \{U_1, U_2, \ldots, U_c\}$ is an r-cover of $B_n(\mathbf{0},\ell))$ with $c < \chi = \chi(K(n,\ell,\ell-r))$. Let

$$\binom{[n]}{\ell} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_c$$

such that

$$\mathcal{A}_i = \{F \in \binom{[n]}{\ell} : |F \triangle U_i| \leq r\}.$$

Next let

$$\mathcal{B}_1 = \mathcal{A}_1, \quad \mathcal{B}_2 = \mathcal{A}_2 \setminus \mathcal{A}_1, \quad \ldots, \quad \mathcal{B}_c = \mathcal{A}_c \setminus (\mathcal{A}_1 \cup \ldots \mathcal{A}_{c-1}).$$

The sets $\mathcal{B}_i$, $i = 1, \ldots, c$ form a partition of $\binom{[n]}{\ell}$. As $c < \chi$, there is a $\mathcal{B}_i$ and $F, F' \in \mathcal{B}_i$ such that $|F \cap F'| < \ell - r$. We have

$$|F \triangle U_i| \leq r, \quad |F' \triangle U_i| \leq r.$$

$$|F \setminus F'| > r, \quad |F' \setminus F| > r.$$

However, from the triangle inequality we have

$$2r < |F \triangle F'| \leq |F \triangle U_i| + |F' \triangle U_i| \leq 2r,$$

a contradiction. □

One can also note that Theorem 2 gives an asymptotic expression in terms of the Turán numbers although these numbers are not known in general. For the cases when $T(n, \ell - r, \ell)$ are known we can show more.

4. Supersaturated hypergraphs

In the previous section we determined, in the asymptotic sense, the r-covering numbers of the Hamming ball $B_n(\mathbf{0}, \ell)$ and the ℓ-th layer of the hypercube. In this section we give a direct proof.

Theorem 3. $c_r(\binom{[n]}{\ell}) = T(n, \ell - r, \ell) + o(n^{\ell - r})$.

Before formally proceeding with the proof we recall some definitions and results from hypergraph theory.

A *hypergraph* is a pair $H = (V, E)$ with vertex set V and set of edges $E \subseteq 2^V$. The number of its edges is denoted by $e(H)$. A hypergraph with all edges having cardinality h is an *h-uniform* hypergraph. The *complete* h-uniform hypergraph, K_h^ℓ, has ℓ vertices and $\binom{\ell}{h}$ edges. The complement of an h-uniform hypergraph $H = (V, E)$ is an h-uniform hypergraph $\overline{H} = (V, \binom{V}{h} \setminus E)$.

Let L be an h-uniform hypergraph. The Turán function $\mathrm{ex}(n, L)$ is the maximal number of edges in an h-uniform hypergraph on n vertices with no subhypergraph isomorphic to L. Considering complements one can see that

$$T(n, w, \ell) = \binom{n}{w} - \mathrm{ex}(n, K_w^\ell).$$

An h-uniform hypergraph G on n vertices is *supersaturated* (with respect to L) if $e(G) > \mathrm{ex}(n, L)$. We are going to use the following theorem of Erdős and Simonovits.

Theorem 4. [5] *For any given real number $c > 0$ and an h-uniform hypergraph L there exists a $c' = c'(c, L) > 0$ such that the following holds:*

If G is an h-uniform hypergraph on n vertices with $e(G) > \mathrm{ex}(n, L) + cn^h$ then G contains at least $c'n^t$ copies of L, where $t = |V(L)|$.

The number of common elements of the ℓ-th layer and a Hamming ball $B_n(\mathbf{x}, r)$ can be easily computed, it depends only on the weight of $\mathbf{x}$. We obtain that there exists a constant $\gamma_\ell > 0$ depending only on ℓ such that for every $\mathbf{x}$ with weight other than $\ell - r$

$$\left| B_n(\mathbf{x}, r) \cap \binom{[n]}{\ell} \right| \leq \gamma_\ell n^{r-1}. \tag{4}$$

Actually, the right hand side is at most $O(n^{r-\lceil i/2 \rceil})$ for any $\mathbf{x}$ of weight $\ell - r + i$, $(0 \leq i \leq 2r)$. Another version of the above inequality is used in the next section as Lemma 1.

Proof of Theorem 3. The upper bound for the r-covering number follows from the definition of the Turán number, or from (3). To prove the lower bound consider an r-cover $\mathcal{C}$ of $\binom{[n]}{\ell}$ and suppose that $|\mathcal{C}| \leq T(n, \ell - r, \ell) - cn^{\ell - r}$ for some $c > 0$. We are going to show that $n < n_1(c)$.

Let H be the $(\ell - r)$-uniform hypergraph with edges corresponding to the codewords of $\mathcal{C}$ of weight $\ell - r$ and let $\mathcal{C}'$ denote the rest of the code, $\mathcal{C}' = \mathcal{C} - E(H)$. Let $\overline{H}$ denote the complement of H. For the size of $\overline{H}$ we have

$$e(\overline{H}) \geq \binom{n}{\ell - r} - |\mathcal{C}| \geq \mathrm{ex}(n, K^\ell_{\ell - r}) + cn^{\ell - r}.$$

Thus, by Theorem 4, there are at least $c'n^\ell$ copies of the complete hypergraph $K^\ell_{\ell-r}$ in $\overline{H}$. Thus the $(\ell - r)$-element sets of $\mathcal{C}$ leave $c'n^\ell$ uncovered ℓ-sets. These ℓ-sets must be covered by the members of $\mathcal{C}'$. However each member of $\mathcal{C}'$ covers at most $\gamma_\ell n^{r-1}$ ℓ-element set by (4). Therefore we have $|\mathcal{C}'| \geq c'n^\ell / (\gamma_\ell n^{r-1})$, implying

$$\binom{n}{\ell - r} \geq T(n, \ell - r, \ell) > |\mathcal{C}| \geq |\mathcal{C}'| \geq \frac{c'n^\ell}{\gamma_\ell n^{r-1}}.$$

Hence $n < \gamma_\ell / c'$ follows. □

5. The covering radius of graph products

In this section we consider a covering problem in a power of a graph. In fact, we are going to determine (or estimate) the r-covering number of a certain set X in the usual q-ary space; only in the definition of X we use a q-vertex graph G. To avoid confusion between graph-theoretic distance and the Hamming distance, in this section we shall denote the Hamming distance by d_{Hamm} and the distance in a graph G by d_G. Let $G = (V(G), E(G))$ be a graph with q vertices and let $a \in V(G)$ be a vertex of degree δ. Denote the set of its neighbors by N. Denote the set of vertices of distance

2 from a by $N^\star$ (the set of second neighbors), i.e., $N^\star = \{v \in V(G) : d_G(a,v) = 2\}$, and its size by Δ. Define the vertex set of the product of n copies of G, G^n as follows.

$$V(G^n) = V(G)^n = \{(v_1, ..., v_n) : v_i \in V(G), \quad i = 1, ..., n\}.$$

As usual, we also abbreviate $(v_1, \dots, v_n)$ as $\mathbf{v}$, and the special vertex $(a, a, \dots, a)$ as $\mathbf{a}$. We say that a vertex has *weight* i if its Hamming distance is i from the special vertex $\mathbf{a}$. As before, we say that $\mathcal{C}$ is an r-cover of $X \subseteq V(G^n)$ if every element of X is within Hamming distance r from at least one element of $\mathcal{C}$. Define the edge set of G^n as follows.

$$E(G^n) = \{\{\mathbf{u}, \mathbf{v}\} : \{u_k, v_k\} \in E(G) \text{ for some } k, \text{ and } u_l = v_l, \text{ for all other } l \neq k\}.$$

In general, the graph distance between the vertices $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_n\}$ is $\sum_{1 \le i \le n} d_G(u_i, v_i)$, and thus the Hamming distance does not exceed it. Let the i-th *neighborhood* of $\mathbf{v} \in V(G^n)$ be the set of vertices in G^n at graph distance exactly i from $\mathbf{v}$. Let $G^n(\mathbf{v}, l)$ be the union of i-th neighborhoods of $\mathbf{v}$ for $0 \le i \le l$.

We study the minimum size of an r-cover of $G^n(\mathbf{a}, r+2)$ by proving that the following two constructions give asymptotically optimal r-covers. In general this yields a stronger result as to study simply the coverings of the Hamming balls in $V(G)^n$, because the i-neighborhood of $\mathbf{a}$ in G^n is usually a small fraction of the Hamming ball with center $\mathbf{a}$ and radius i.

First consider a partition of $[n]$, expressing $[n]$ as a disjoint union of $P_1, \dots, P_{r+1}$ such that

$$\lfloor n/(r+1) \rfloor \le |P_i| \le |P_{i+1}| \le \lceil n/(r+1) \rceil, \quad \text{for} \quad 1 \le i \le r.$$

Construction 1. (An r-cover of $G^n(\mathbf{a}, r+2)$). Let

$$S = \{\mathbf{v} = (v_1, \dots, v_n) : d_{\text{Hamm}}(\mathbf{v}, \mathbf{a}) = 2 \text{ with } v_i, v_j \in N \text{ for some } i, j \in P_s, \ i \neq j, \ 1 \le s \le r+1\}$$

and for $1 \le \sigma \le r+1$

$$S_\sigma = \{\mathbf{v} = (v_1, \dots, v_n) : d_{\text{Hamm}}(\mathbf{v}, \mathbf{a}) = 1 \text{ with } v_i \in N \text{ for some } i \in P_\sigma\},$$

$$S^\star = \{\mathbf{v} = (v_1, \dots, v_n) : d_{\text{Hamm}}(\mathbf{v}, \mathbf{a}) = 1 \text{ with } v_i \in N^\star \text{ for some } i \in [n]\}.$$

Define

$$\mathcal{C} = S \cup S_1 \cup S^\star \cup \{\mathbf{a}\}.$$

Claim. $\mathcal{C}$ is an r-cover of $G^n(\mathbf{a}, r+2)$ of size

$$\delta^2 T(n,2,r+2) + \delta\lfloor n/(r+1)\rfloor + \Delta n + 1. \tag{5}$$

Indeed, let $\mathbf{v} \in G^n(\mathbf{a}, r+2)$. As $d_{G^n}(\mathbf{v}, \mathbf{a}) = \sum_{1\le i\le n} d_G(v_i, a)$, so $d_{G^n}(\mathbf{v}, \mathbf{a}) \le r+2$ implies that $d_{\mathrm{Hamm}}(\mathbf{v}, \mathbf{a}) \le r+2$. First, consider the case when $\mathbf{v} \in G^n(\mathbf{a}, r+2)$ and $d_{\mathrm{Hamm}}(\mathbf{v}, \mathbf{a}) = r+2$. Then $n-(r+2)$ coordinates of $\mathbf{v}$ are equal to a, and the other $r+2$ are from N. Since there are only $r+1$ P_i's, at least two of these coordinates, lie in some P_s. Therefore $\mathbf{v}$ is contained in a ball of radius r about a center from S. These centers, together with $\mathbf{a}$, cover all points of $\mathbf{v} \in G^n(\mathbf{a}, r+2)$ except some of those with $d_{\mathrm{Hamm}}(\mathbf{v}, \mathbf{a}) = r+1$ and $d_{G^n}(\mathbf{v}, \mathbf{a}) = r+1$ or $r+2$. These are covered by the rest of $\mathcal{C}$.

To calculate the size of this cover, notice that we have obtained the members of S by changing the coordinates of $\mathbf{a}$ at exactly two places (from a set P_s) and the new coordinates are independently chosen from N, thus $|S| = |N|^2 T(n,2,r+2)$ giving us the first term in (5). Similarly, the members of S_σ are obtained by changing a single coordinate in P_σ, thus $|S_1| = |N|\lfloor n/(r+1)\rfloor$, and we obtain the second term in (5). Finally, the members of $S^\star$ are obtained by changing any single coordinate of $\mathbf{a}$, we obtain the third term. This concludes the proof of the Claim.

With a very similar argument we have the following slightly different cover.

Construction 2. (An r-cover of $G^n(\mathbf{a}, r+2)$). Define

$$\mathcal{C} = S \cup S_1 \cup S_2 \cup \{\mathbf{a}\},$$

then $\mathcal{C}$ is an r-cover of $G^n(\mathbf{a}, r+2)$ of size

$$\delta^2 T(n,2,r+2) + \delta\left(\lfloor n/(r+1)\rfloor + \lfloor (n+1)/(r+1)\rfloor\right) + 1.$$

The following Theorem shows that the Constructions 1 and 2 are asymptotically optimal.

Theorem 5. *For fixed $r \ge 1$ and $n > n_0(q,r)$*

$$\delta^2\, T(n,2,r+2) \le c_r(G^n(\mathbf{a}, r+2)) \le \delta^2\, T(n,2,r+2) + \alpha n + 1,$$

where $\alpha = \min\{\Delta + (\delta/(r+1)), 2\delta/(r+1)\}$.

To prove this Theorem we need the following lemma.

Lemma 1. *Let $Q = \{0, 1, \ldots, q-1\}$, $X = Q^n$ be the space of codewords with the Hamming distance, and let L be the set of words from X of weight $r+2$. Suppose that $w(\mathbf{x}) > 2$, then for $n \geq 4r+3$*

$$|B_{n,Q}(\mathbf{x}, r) \cap L| \leq 5\binom{n-3}{r-1}(q-1)^{r+2} .$$

Proof. Let $w := w(\mathbf{x})$. Count $\mathbf{y} \in B(\mathbf{x}, r) \cap L$ according to the number of common elements of the supports of $\mathbf{x}$ and $\mathbf{y}$. Denote it by $j+k$, where j is the number of coordinates i such that $y_i = x_i \neq 0$, and k is the number of coordinates such that $y_i \neq x_i$ and both are non-zero. These coordinates can be selected in $\binom{w}{j}\binom{w-j}{k}$ ways from the support of $\mathbf{x}$ and there are $\binom{n-w}{r+2-j-k}$ ways for the rest of the support of $\mathbf{y}$. Notice that $d(\mathbf{y}, \mathbf{x}) = (r+2-j-k) + k + (w-j-k)$, thus $d(\mathbf{y}, \mathbf{x}) \leq r$ implies $w+2 \leq k+2j$, and we have that $j+k \geq \lceil \frac{w+2}{2} \rceil$. Each coordinate of the support can take $q-1$ or $q-2$ distinct values, thus we obtain

$$|B(\mathbf{x}, r) \cap L| = \sum_j \sum_k \binom{w}{j}\binom{w-j}{k}\binom{n-w}{r+2-j-k}(q-2)^k (q-1)^{(r+2)-j-k} ,$$

where the sum is taken for all pairs of $j \geq 0$ and $k \geq 0$ such that $w+2 \leq k+2j$ (and of course $k+j \leq w$). (For $q=2$, $k=0$ the value of the expression $(q-2)^k$ is taken to be 1.) Denote the above sum by $W(n, q, w, r)$. Writing $\binom{w}{j}\binom{w-j}{k}$ as $\binom{w}{j+k}\binom{j+k}{k}$, substituting $j+k=s$ and taking the sum for *all* k's one has the upper bound

$$W(n, q, w, r) \leq \sum_{\lceil \frac{w+2}{2} \rceil \leq s \leq \min\{w, r+2\}} \binom{w}{s}\binom{n-w}{r+2-s}(q-1)^{r+2} . \tag{6}$$

As a polynomial of n the highest degree corresponds to the case when s is minimum, and for $n \geq 4r$ the first term dominates the sum. Thus we obtain

$$\frac{W(n, q, w, r)}{(q-1)^{r+2}} \leq \begin{cases} \binom{3}{3}\binom{n-3}{r-1}, & w = 3 \\ \binom{4}{4}\binom{n-4}{r-2}, & w = 4 \\ 2\binom{w}{\lceil \frac{w+2}{2} \rceil}\binom{n-w}{r+2-\lceil \frac{w+2}{2} \rceil}, & w > 4 \end{cases} \tag{7}$$

$$\leq 5\binom{n-3}{r-1} \quad \text{for } n \geq 4r+3.$$

We give the details of these calculations in Appendix A.

□

We also need the following result of Lovász and Simonovits.

Theorem 6. [12] *Let F be a graph on n vertices with $\binom{n}{2} - \frac{n^2}{2}(1-\frac{1}{t})$ edges, where $t \geq r+1$ is a real number. Let $\omega_{r+2}(F)$ denote the number of complete $(r+2)$-graphs in the complement of F. Then*

$$\omega_{r+2}(F) \geq \frac{t\,(t-1)\ldots(t-r-1)}{(r+2)!}\frac{n^{r+2}}{t^{r+2}} \geq \frac{t-r-1}{t}\frac{n^{r+2}}{(r+2)(r+1)^{r+1}}. \tag{8}$$

The above Theorem gives a very good lower bound for $\omega_{r+2}(F)$ if $e(F) < \binom{n}{2} - \frac{n^2}{2}(1-\frac{1}{r+1})$. However, $T(n,2,r+2)$ can be slightly larger. It is easy to see using extremal construction for $T(n,2,r+2)$, that

$$\binom{n}{2} - \frac{n^2}{2}\left(1-\frac{1}{r+1}\right) \leq T(n,2,r+2) = \binom{n}{2} - \mathrm{ex}(n,K_{r+2}) \leq \binom{n}{2} - \frac{n^2}{2}\left(1-\frac{1}{r+1}\right) + r. \tag{9}$$

The next theorem of Erdős takes care of supersaturated graphs with slightly more edges than $\mathrm{ex}(n,K_{r+2})$.

Theorem 7. [3] *There is a positive constant κ_r (for any given r) such that the following holds. If F is a graph on n vertices with $e(F) = \binom{n}{2} - \mathrm{ex}(n,K_{r+2}) - x$ edges, where $0 \leq x < \kappa_r n$, then*

$$\omega_{r+2}(F) \geq x \prod_{1\leq i\leq r} \left\lfloor \frac{n-1+i}{r+1} \right\rfloor \geq \frac{x}{2}\left(\frac{n}{r+1}\right)^r. \tag{10}$$

Proof of Theorem 5. The upper bound for the covering number follows from Constructions 1 and 2. For the lower bound we are going to use the same ideas as in the proof of Theorem 3 but with more precise tools. Define

$$n_0(q,r) := \max\{10(q-1)^{r+2}\delta^2\frac{(r+1)^{r+1}}{(r-1)!},\ \frac{2r}{\kappa_r}\}\ , \tag{11}$$

where κ_r is a function of r coming from Theorem 7. Assume that $n > n_0(q,r)$. Let X be the $(r+2)$nd neighborhood of $\mathbf{a}$ in G^n with all coordinates in $N \cup \{a\}$:

$$X = \{\mathbf{v} = (v_1,\ldots,v_n) \in G^n :\ d_{\mathrm{Hamm}}(\mathbf{v},\mathbf{a}) = d_{G^n}(\mathbf{v},\mathbf{a}) = r+2,\ v_i \in N\cup\{a\},\ \ 1\leq i\leq n\}.$$

We shall give a lower bound on the r-covering number of X which is a lower bound on an r-covering number of $G^n(\mathbf{a}, r+2)$ as well. Let $\mathcal{C} \subset V(G)^n$ be a minimal r-cover of X, and assume that

$$|\mathcal{C}| \leq \delta^2\, T(n,2,r+2). \tag{12}$$

We are going to prove in two steps that equality holds here. Let

$$\mathcal{C}_2 = \{\mathbf{x} = (x_1,\ldots,x_n) \in \mathcal{C} :\ d_{\mathrm{Hamm}}(\mathbf{x},\mathbf{a}) = 2,\ x_i \in N\cup\{a\},\ \text{for all } 1\leq i\leq n\}.$$

First, we will show that $|\mathcal{C}_2|/\delta^2 \geq T(n,2,r+2)-2r$ and, as a second step, that $|\mathcal{C}_2|/\delta^2 \geq T(n,2,r+2)$.

Define the following graph $H = (V(H), E(H))$, corresponding to $\mathcal{C}_2$. The vertex set is $[n] \times N$ and two vertices (i,s) and (j,t) are joined if $i \neq j$ and there is a $\mathbf{v} \in \mathcal{C}_2$ such that $v_i = s$, $v_j = t$. We shall show that if $|\mathcal{C}_2|$ is not large enough then we have many uncovered elements of X which could not be covered by $\mathcal{C} \setminus \mathcal{C}_2$. For this, we count the number of K_{r+2}'s in the complement of H.

Now define all "transversal" graphs H_η for all $\eta \in N^n$, $\eta = (v_1, \ldots, v_n)$ as follows:

$$V(H_\eta) = \{(1,v_1), \ldots, (n,v_n)\}, \quad E(H_\eta) = E(H|H_\eta).$$

Let, for some $\lambda \in N^n$, H_λ have the minimal number of edges among the δ^n transversal graphs. Then

$$|\mathcal{C}_2| = |E(H)| = \frac{1}{\delta^{n-2}} \sum_{\eta \in N^n} |E(H_\eta)| \geq \delta^2 |E(H_\lambda)| \tag{13}$$

If $|E(H_\lambda)| = \binom{n}{2} - \frac{n^2}{2}\left(1 - \frac{1}{t}\right)$ for any $t < r+1$ then, using (9) and (13), $|\mathcal{C}| \geq |\mathcal{C}_2| = |E(H)| \geq \delta^2 T(n,2,r+2)$ and we are done. Suppose that for some $t \geq r+1$ one has

$$|E(H_\lambda)| = \binom{n}{2} - \frac{n^2}{2}\left(1 - \frac{1}{t}\right). \tag{14}$$

Then

$$E(H) \geq \delta^2 \left(\binom{n}{2} - \frac{n^2}{2}\left(1 - \frac{1}{t}\right)\right) = \delta^2 \left(\binom{n}{2} - \frac{n^2}{2}\left(1 - \frac{1}{r+1}\right)\right) - n^2 \frac{\delta^2}{2(r+1)} \times \frac{t-r-1}{t}. \tag{15}$$

Applying (8) for $E(H_\lambda)$ we obtain

$$\omega_{r+2}(H_\lambda) \geq \frac{t-r-1}{t} \frac{n^{r+2}}{(r+2)(r+1)^{r+1}}, \tag{16}$$

where $\omega_{r+2}(H_\lambda)$ is the number of K_{r+2}'s in $\overline{H}_\lambda$. These $\omega_{r+2}(H_\lambda)$ members of X must be covered by $\mathcal{C} \setminus \mathcal{C}_2$, thus

$$\omega_{r+2}(H_\lambda) \leq \sum_{\mathbf{x} \in \mathcal{C} \setminus \mathcal{C}_2} |X \cap B(\mathbf{x}, r)| \leq |\mathcal{C} \setminus \mathcal{C}_2| \max_{\mathbf{x} \in \mathcal{C} \setminus \mathcal{C}_2} |X \cap B(\mathbf{x}, r)|. \tag{17}$$

We may suppose that $\mathcal{C}$ is a minimal r-cover of X, so it contains no element $\mathbf{x} \in V(G^n)$ such that the Hamming ball $B(\mathbf{x}, r)$ is disjoint to X. This implies that for all $\mathbf{x} \in \mathcal{C} \setminus \mathcal{C}_2$, $d_{\mathrm{Hamm}}(\mathbf{a}, \mathbf{x}) > 2$, therefore one can use Lemma 1. We have

$$|X \cap B(\mathbf{x}, r)| \leq 5(q-1)^{r+2} \binom{n-3}{r-1} < n^{r-1} \frac{5(q-1)^{r+2}}{(r-1)!}. \tag{18}$$

The above three inequalities (16), (17) and (18) give

$$|\mathcal{C}\setminus\mathcal{C}_2| \geq \frac{t-r-1}{t} \times \frac{n^3(r-1)!}{(r+2)(r+1)^{r+1}5(q-1)^{r+2}}. \tag{19}$$

Combining (15) and (19) we have

$$|\mathcal{C}| = |\mathcal{C}_2| + |\mathcal{C}\setminus\mathcal{C}_2| \geq \delta^2\left(\binom{n}{2} - \frac{n^2}{2}\left(1-\frac{1}{r+1}\right)\right) +$$

$$n^2 \times \frac{t-r-1}{t}\left(-\frac{\delta^2}{2(r+1)} + \frac{n(r-1)!}{(r+2)(r+1)^{r+1}5(q-1)^{r+2}}\right).$$

Here for $n > n_0(q,r)$ (cf. (11)) the expression in the parantheses in the last term is not only positive but at least $\delta^2/2(r+1)$. However, (9) and (12) give that

$$|\mathcal{C}| \leq \delta^2\left(\binom{n}{2} - \frac{n^2}{2}\left(1-\frac{1}{r+1}\right)\right) + \delta^2 r,$$

so we obtain that

$$n^2 \times \frac{t-r-1}{t} \times \frac{\delta^2}{2(r+1)} \leq \delta^2 r.$$

Rearranging this yields that

$$\frac{n^2}{2}\left(1-\frac{1}{t}\right) \leq \frac{n^2}{2}\left(1-\frac{1}{r+1}\right) + r.$$

This together with (9) implies in (14) that

$$|E(H_\lambda)| = \binom{n}{2} - \frac{n^2}{2}\left(1-\frac{1}{t}\right) \geq \binom{n}{2} - \frac{n^2}{2}\left(1-\frac{1}{r+1}\right) - r \geq T(n,2,r+2) - 2r\ .$$

Thus

$$|E(H_\eta)| \geq T(n,2,r+2) - 2r \tag{20}$$

holds for every $\eta \in N^n$. Note that, using (13), we have already obtained that $|\mathcal{C}| \geq |\mathcal{C}_2| \geq \delta^2 T(n,2,r+2) - 2r\delta^2$.

The second part of the proof is very similar to the first one, except we will use the even more exact Theorem 7. We start with (20), and define again H_λ as the minimal transversal graph. We have

$$|E(H_\lambda)| = T(n,2,r+2) - x$$

for some $0 \leq x \leq 2r$. Also (13) becomes

$$|\mathcal{C}_2| \geq \delta^2|E(H_\lambda)| = \delta^2 T(n,2,r+2) - \delta^2 x. \tag{21}$$

Applying (10) for $E(H_\lambda)$ we obtain

$$\omega_{r+2}(H_\lambda) \geq \frac{x}{2}\frac{n^r}{(r+1)^r}.$$

Then (19) becomes

$$|\mathcal{C} \setminus \mathcal{C}_2| \geq \frac{x}{2} \times n \times \frac{(r-1)!}{(r+1)^r 5(q-1)^{r+2}}. \tag{22}$$

Combining (21) and (22) we have

$$|\mathcal{C}| = |\mathcal{C}_2| + |\mathcal{C} \setminus \mathcal{C}_2| \geq \delta^2 T(n,2,r+2) + \frac{x}{2} \times \left(-2\delta^2 + n\frac{(r-1)!}{(r+1)^r 5(q-1)^{r+2}}\right).$$

Here for $n > n_0(q,r)$ (cf. (11)) the last term in the parentheses is positive. Thus the equality holds in (12). □

6. Exact bounds for $c_r(B_n(\mathbf{0}, r+1))$, $c_r(B_n(\mathbf{0}, r+2))$ and $c_1(B_n(\mathbf{0}, 3))$

One can reduce the problem of covering several layers of a hypercube to the problem of covering a single layer using the following lemma.

Lemma 2. $c_r\left(\binom{[n]}{k} \cup \binom{[n]}{k-1}\right) \geq c_r\left(\binom{[n+1]}{k}\right)$.

Proof. Let $\mathcal{C}$ be an r-cover of $\binom{[n]}{k} \cup \binom{[n]}{k-1}$. We construct a new code $\mathcal{C}_{\text{new}}$ of length $n+1$ of the same size, $|\mathcal{C}_{\text{new}}| = |\mathcal{C}|$, and prove that it is an r-cover of $\binom{[n+1]}{k}$. We define

$$\mathcal{C}_{\text{new}} = \{C : C \in \mathcal{C} \text{ and } |C| \equiv r+k \pmod 2\} \cup \{C \cup \{n+1\} : C \in \mathcal{C} \text{ and } |C| + r + k \text{ is odd}\}.$$

Suppose $U \in \binom{[n+1]}{k}$. We shall show that there is a $C' \in \mathcal{C}_{\text{new}}$ such that $|C' \triangle U| \leq r$, thus proving that U is covered by some r-ball centered about an element of our cover.

There exists a $C \in \mathcal{C}$ such that $d(C, U \setminus \{n+1\}) \leq r$. As either C or $C \cup \{n+1\}$ belongs to $\mathcal{C}_{\text{new}}$ and both distances $d(C,U)$ and $d(C \cup \{n+1\}, U)$ are at most $d(C, U \setminus \{n+1\}) + 1$ we are done for the case $d(C, U \setminus \{n+1\}) < r$.

Now suppose that $d(C, U \setminus \{n+1\}) = r$. We have that $|I| + |J| + d((I,J)$ is even for any sets I and J. So, in the case $(n+1) \notin U$, we have $d(C,U) = r$, thus $|C| + r + k = |C| + d(C,U) + |U|$ is even, so $C \in \mathcal{C}$ and we are done. Finally, in the case $(n+1) \in U$, the equation $d(C, U \setminus \{n+1\}) = r$ implies that $|C| + r + k = |C| + d(C,U) - 1 + |U|$ is odd, so $C \cup \{n+1\} \in \mathcal{C}_{\text{new}}$ and we are done again. □

The following construction gives an optimal r-cover of $B_n(\mathbf{0}, r+1)$.

Construction 3. (An r-cover of $B_n(\mathbf{0}, r+1)$) Let $n > 2r$ and define

$$\mathcal{C} = \{\{1\}, \{2\}, \dots, \{n-2r\}, \{n-2r+1, \dots, n\}\}.$$

Then $\mathcal{C}$ is an r-cover of $B_n(\mathbf{0}, r+1)$ of size $n - 2r + 1$.

We shall use the following form of a theorem of Lovász.

Theorem 8. [13] *If the family of all the $(r+1)$-subsets of an $(n+1)$-set is partitioned into $n - 2r$ classes ($n \geq 2(r+1)$), then there is a class containing two disjoint $(r+1)$-sets.*

Theorem 9. *For $n > 2r$ we have $c_r(B_n(\mathbf{0}, r+1)) = n - 2r + 1$.*

Proof of Theorem 9. The upper bound for the r-covering number follows from Construction 3. The case $n = 2r + 1$ is obvious, so from now on we suppose that $n \geq 2r + 2$. Applying Lemma 2 with $k = r + 1$, we have

$$c_r(B_n(\mathbf{0}, r+1)) \geq c_r\left(\binom{[n]}{r+1} \cup \binom{[n]}{r}\right) \geq c_r\left(\binom{[n+1]}{r+1}\right).$$

Now we proceed as in the proof of Theorem 2. Consider an r-cover $\mathcal{C} = \{U_1, \dots, U_c\}$ of $\binom{[n+1]}{r+1}$. Let $\mathcal{A}_i$ be the set of $(r+1)$-element sets covered by the r-ball centered about U_i.

$$\mathcal{A}_i := \{F \in \binom{[n+1]}{r+1} : \ |F \triangle U_i| \leq r\}, \ i = 1, \dots, c$$

and

$$\mathcal{B}_1 = \mathcal{A}_1, \ \mathcal{B}_2 = \mathcal{A}_2 \setminus \mathcal{A}_1, \ \dots, \ \mathcal{B}_c = \mathcal{A}_c \setminus (\mathcal{A}_1 \cup \dots \mathcal{A}_{c-1}).$$

The sets $\mathcal{B}_i$, $i = 1, \dots, c$ form a partition of $\binom{[n+1]}{r+1}$. For $c \leq n - 2r$, Theorem 8 implies that there is a class $\mathcal{B}_i$ containing two disjoint $(r+1)$-sets F, F'. These two disjoint $(r+1)$-sets could not belong to any Hamming ball with radius r. Therefore $c \geq n - 2r + 1$ and the lower bound follows. □

For $n > n_0(r)$ we give the exact minimum number of Hamming balls of radius r required to cover the Hamming ball of radius $r + 2$. The result of Tort addresses the case $r = 1$.

Theorem 10. [17] *For $n \geq 6$*

$$c_1\left(\binom{[n]}{3}\right) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor.$$

Using this, we find $c_1\Big(B_n(\mathbf{0},3)\Big)$.

Theorem 11. *For $n>1$*

$$c_1\Big(B_n(\mathbf{0},3)\Big)=\left\lfloor\frac{n^2}{4}\right\rfloor.$$

Moreover, for $n>4$

$$c_1\Big(B_n(\mathbf{0},3)\Big)=c_1\left(\binom{[n]}{3}\cup\binom{[n]}{2}\right).$$

Proof of Theorem 11. As in the previous constructions, consider a partition of the vertex set into two parts of sizes $\lfloor n/2\rfloor$ and $\lceil n/2\rceil$, and as a cover, take single element sets from the smaller part and all pairs from each of the partite set. Thus every 3-element set is covered by some pair. Every 2-element set either belongs to a cover itself, or is covered by some single element set. Similarly, every 1-element set is covered by some pair. Empty set is covered by any single element set. This cover gives us the upper bound for the covering number.

$$c_1\left(\binom{[n]}{3}\cup\binom{[n]}{2}\right)\le c_1\Big(B_n(\mathbf{0},3)\Big)\le T(n,2,3)+\left\lfloor\frac{n}{2}\right\rfloor=T(n+1,2,3)=\left\lfloor\frac{n^2}{4}\right\rfloor.$$

The lower bound for $c_1\left(\binom{[n]}{3}\cup\binom{[n]}{2}\right)$ is provided by Tort's result [17] using Lemma 2 for $n\ge 5$.

$$c_1\left(\binom{[n]}{3}\cup\binom{[n]}{2}\right)\ge c_1\left(\binom{[n+1]}{3}\right)\ge\left\lfloor\frac{n^2}{4}\right\rfloor.$$

Note that for small n, for example for $n=4$, there are other optimal covers, e.g.,

$$\{(0,0,0,0),(1,1,1,0),(0,1,1,1),(1,0,0,1)\}.$$

A simple case analysis shows that $c_1\Big(B_4(\mathbf{0},3)\Big)=4$. In fact

$$c_1\left(\binom{[4]}{1}\cup\binom{[4]}{2}\cup\binom{[4]}{3}\right)=4.$$

However, $c_1\left(\binom{[4]}{3}\cup\binom{[4]}{2}\right)\le 3$ since $\mathcal{C}=\{(1,0,0,0),(0,1,1,1),(1,1,1,1)\}$ is a cover. The cases when $n=2$ or 3 are trivial. $\square$

Applying Theorem 5 with $G = K_2$, we obtain the following important special case. Let $P_1, \dots, P_{r+1}$ be a partition of $[n]$ with $\lfloor n/(r+1) \rfloor \le |P_1| \le |P_i| \le \lceil n/(r+1) \rceil$ for $1 \le i \le r+1$.

Construction 4. (An r-cover of $B_n(\mathbf{0}, r+2)$). Suppose that $n \ge 2r+2$ and let

$$\mathcal{C} = \bigcup_{i=1}^{r+1} \binom{P_i}{2} \bigcup \{\{u\} : u \in P_1\}.$$

This is indeed an r-cover. If $Y \in B_n(\mathbf{0}, r+2)$ and $|Y| = r+2$ then at least two elements of Y belong to some P_i and, thus, Y is covered by a corresponding pair from P_i. A similar argument works when $|Y| = r+1$ and there are two elements of Y in the same P_i. Otherwise Y must have an element in each P_i, in particular in P_1 and thus it is covered by a corresponding one-element set. If $|Y| = r$ and $Y \cap P_i \ne \emptyset$, it is covered by some pair from P_i. If $|Y| < r$ then Y is covered by any single element set. The size of this cover is $T(n, 2, r+2) + \lfloor n/(r+1) \rfloor = T(n+1, 2, r+2)$.

Theorem 12. *For $n > n_0(r)$ we have $c_r(B_n(\mathbf{0}, r+2)) = T(n+1, 2, r+2)$.*

Proof of Theorem 12. The upper bound for the covering number follows from Construction 4. For the lower bound consider Theorem 5 for $G = K_2$. We apply Lemma 2 (with $k = r+2$) and the lower bound on X (as defined in the proof of Theorem 5) to show that

$$(B_n(\mathbf{0}, r+2)) \ge c_r\left(\binom{[n]}{r+1} \cup \binom{[n]}{r+2}\right) \ge c_r\left(\binom{[n+1]}{r+2}\right) \ge T(n+1, 2, r+2).$$

□

7. Conclusions

We believe that the value of $n_0(r)$ in Theorem 5 could be improved, and in particular we conjecture the following. Theorem 5 holds for all $n > cr^3$, where $c = c(q)$. We also believe that one could improve the lower bound for $c_r(G^n(\mathbf{a}, r+2))$ by $\Theta(n)$ using the fact that the unique construction achieving optimal coefficient of the n^2 term uses elements of weight 2 only.

8. Appendix A

First we consider the function $W = W(n, q, w, r)$ of Lemma 1. Here we prove in detail the fact that each term in (6) is at most a half of the previous one. Observe that

$$\binom{w}{s}\binom{n-w}{r+2-s} = x\binom{w}{s+1}\binom{n-w}{r+2-(s+1)},$$

where

$$x = \frac{(s+1)(n-w-r-1+s)}{(w-s)(r+2-s)}.$$

If $n \geq 4r$ then

$$x \geq \frac{(s+1)(3r-w-1+s)}{(w-s)(r+2-s)} \geq$$

$$\frac{(\lceil \frac{w+2}{2} \rceil + 1)(3r-w-1+\lceil \frac{w+2}{2} \rceil)}{(w-\lceil \frac{w+2}{2} \rceil)(r+2-\lceil \frac{w+2}{2} \rceil)} \geq 2.$$

Second, we prove the very last inequality of Lemma 1 in detail, i.e., for $w > 4$ and $n \geq 4r+3$, $r \geq 2$ one has

$$2\binom{w}{\lceil \frac{w+2}{2} \rceil}\binom{n-w}{r+2-\lceil \frac{w+2}{2} \rceil} \leq 5\binom{n-3}{r-1}.$$

In the case $w = 5$ the ratio of the left hand side and the right hand side is

$$2\binom{5}{4}\binom{n-5}{r-2}\frac{1}{5}\binom{n-3}{r-1}^{-1} = \frac{2(r-1)(n-r-2)}{(n-3)(n-4)}$$

which is less than 1 for $n > 2r$. Similarly for $w = 6$ we have

$$2\binom{6}{4}\binom{n-6}{r-2}\frac{1}{5}\binom{n-3}{r-1}^{-1} = \frac{6(r-1)(n-r-2)(n-r-1)}{(n-3)(n-4)(n-5)}$$

which is again at most 1 for integers $n \geq 4r+3$, $r \geq 2$.

In general, we have that for $w > 4$

$$\binom{w+2}{\lceil \frac{w+4}{2} \rceil}\binom{n-w-2}{r+2-\lceil \frac{w+4}{2} \rceil} \leq \binom{w}{\lceil \frac{w+2}{2} \rceil}\binom{n-w}{r+2-\lceil \frac{w+2}{2} \rceil}.$$

Indeed, the ratio of the left hand side and the right-hand side is

$$\frac{(w+2)(w+1)}{\lceil \frac{w+4}{2} \rceil \lfloor \frac{w}{2} \rfloor} \times \frac{(r+2-\lceil \frac{w+2}{2} \rceil)(n-w-r-2+\lceil \frac{w+2}{2} \rceil)}{(n-w)(n-w-1)}$$

and this is at most 1 in the range given.

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