ON GENERALIZED RAMSEY THEORY: THE BIPARTITE CASE

MARIA AXENOVICH, ZOLTÁN FÜREDI AND DHRUV MUBAYI

Abstract. Given graphs G and H, a coloring of $E(G)$ is called an (H,q)-coloring if the edges of every copy of $H \subseteq G$ together receive at least q colors. Let $r(G,H,q)$ denote the minimum number of colors in an (H,q)-coloring of G. We determine, for fixed p, the smallest q for which $r(K_{n,n}, K_{p,p}, q)$ is linear in n, the smallest q for which it is quadratic in n. We also determine the smallest q for which $r(K_{n,n}, K_{p,p}, q) = n^2 - O(1)$, and the smallest q for which $r(K_{n,n}, K_{p,p}, q) = n^2 - O(n)$. Our results include showing that $r(K_{n,n}, K_{2,t+1}, 2)$ and $r(K_n, K_{2,t+1}, 2)$ are both $(1+o(1))\sqrt{n/t}$ as $n \to \infty$, thereby proving a special case of a conjecture of Chung and Graham. Finally, we determine the exact value of $r(K_{n,n}, K_{3,3}, 8)$, and prove that $2n/3 \le r(K_{n,n}, C_4, 3) \le n+1$. Several problems remain open.

1. Generalizing the Classical Problem for Multicolorings

The classical Ramsey problem asks for the minimum n such that every k-coloring of the edges of K_n yields a monochromatic K_p. For each n below this threshold, there is a k-coloring such that every K_p receives at least 2 colors. We may study the same problem by fixing n and asking for the minimum k such that $E(K_n)$ can be k-colored with each p-clique receiving at least 2 colors. For integers n, p, q, a (p,q)-coloring of K_n is a coloring of $E(K_n)$ in which the edges of every K_p together receive at least q colors. Let $f(n,p,q)$ denote the minimum number of colors in a (p,q)-coloring of K_n.

This function was first studied in this form by Elekes, Erdős, and Füredi (as described in Section 9 of [14]). Erdős and Gyárfás [15] improved the results about 15 years later, using the Local Lemma to prove an upper bound of $O(n^{c_{p,q}})$, where $c_{p,q} = (p-2)\big/\left(\binom{p}{2} - q + 1\right)$. They also determined, for each p, the smallest q such that $f(n,p,q)$ is linear in n and the smallest q such that $f(n,p,q)$ is quadratic in n. Many cases remain unresolved, most notably the growth rate of $f(n,4,3)$ and $f(n,5,9)$. In [23] it is shown that $f(n,4,3) < e^{O(\sqrt{\log n}\,)}$, thereby proving that $f(n,4,3)$ grows slower than any power of n, but it remains open whether $f(n,4,3)/\log n \to \infty$. In [4] it is shown that $\frac{1+\sqrt{5}}{2}n - 3 \le f(n,5,9) \le 2n^{1+c/\sqrt{\log n}}$, which still leaves the problem of determining the growth rate exactly.

In this paper we generalize this problem beyond cliques.

Definition. Given graphs G and H, and an integer $q \le |E(H)|$, an (H,q)-*coloring* of G is a coloring of $E(G)$ in which the edges of every copy of $H \subseteq G$ together receive at least q colors. Let $r(G,H,q)$ denote the minimum number of colors in an (H,q)-coloring of G.

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Note that determining $r(K_n, K_p, 2)$ is hopeless, since it is equivalent to determining the classical Ramsey numbers for multicolorings. Let $r_k(H)$ be the minimim n such that every k-coloring of $E(K_n)$ yields a monochromatic copy of the subgraph H. Then $r_k(K_p) = n$ is equivalent to the statements $r(K_n, K_p, 2) > k$, and $f(K_{n-1}, K_p, 2) = k$.

Although the function $r(K_n, H, q)$ was studied (in the form $r_k(H)$) by Erdős and Rado [16] as early as 1956, and the case $r(K_{n,n}, K_{p,p}, q)$ was considered by Chvátal [11] in relation to Zarankiewicz's problem, our results and techniques have a different flavor. In our investigation of $r(K_{n,n}, K_{p,p}, q)$, we always assume that p and q are fixed and $n \to \infty$.

In Section 2 we reprove a result of Chung and Graham [8] about $r(K_n, C_4, 2)$ and extend it to $r(K_n, K_{2,t+1}, 2)$ and $r(K_{n,n}, K_{2,t+1}, 2)$, thereby proving a special case of a conjecture of theirs. Both of these Ramsey numbers are asymptotic to $\sqrt{n/t}$ as $n \to \infty$. We also observe that a recent result of Alon, Rónyai, and Szabó [3] implies $r(K_{n,n}, K_{3,3}, 2) = n^{1/3}(1 + o(1))$.

Using the Local Lemma a very general upper bound is given in Section 3. Following the Erdős–Gyárfás results on cliques, for fixed p, in Section 4 we determine the smallest q for which $r(K_{n,n}, K_{p,p}, q)$ is linear in n, and the smallest q for which $r(K_{n,n}, K_{p,p}, q)$ is quadratic in n; these values are $q = p^2 - 2p + 3$ and $q = p^2 - p + 2$, respectively. In Section 5 we prove that the smallest q for which $r(K_{n,n}, K_{p,p}, q) = n^2 - O(1)$ is $q = p^2 - \lfloor p/2 \rfloor + 1$. In Section 6 we prove that the smallest q for which $r(K_{n,n}, K_{p,p}, q) = n^2 - O(n)$ is $q = p^2 - \lfloor (2p-1)/3 \rfloor + 1$.

In Section 7 we determine the exact value of $r(K_{n,n}, K_{3,3}, 8)$ by relating the allowable colorings to four-cycle packings of $K_{n,n}$; this value is $(3/4)n^2$ if n is even, and $\lceil (3/4)n^2 + n/4 \rceil$ if n is odd. Finally, in Section 8 we investigate $r(K_{n,n}, C_4, 3)$. We prove that $2n/3 \leq r(K_{n,n}, C_4, 3) \leq n + 1$, and study a related function defined by relaxing the requirements of a $(C_4, 3)$-coloring.

2. Multicolor Ramsey numbers ($q = 2$)

Let $\mathrm{ex}(G, H)$ be the maximal t such that there is a (not necessarily induced) subgraph of G with t edges not having H as a subgraph, i.e., the size of the largest H-free subgraph. Usually, $\mathrm{ex}(K_n, H)$ is called the *Turán number* of H, and $\mathrm{ex}(K_{n,n}, K_{a.b})$ is a symmetric version of the *Zarankiewicz number*. The classical upper bound for the Zarankiewicz number, due to Kővári, Sós and Turán [22] has been recently improved in [19], where it is shown that for $1 \leq a \leq b$

$$2\ \mathrm{ex}(K_n, K_{a,b}) \leq \mathrm{ex}(K_{n,n}, K_{a,b}) \leq (b - a + 1)^{1/a} n^{2-(1/a)} + a n^{2-(2/a)} + an. \tag{1}$$

These are believed to be asymptotically optimal as $n \to \infty$. Chung and Graham [8] noticed that the knowledge of the Turán number $\mathrm{ex}(K_n, G)$ can be used to deduce a lower bound on the multicolored Ramsey number $r(G, H, 2)$ through the following obvious inequality

$$r(G, H, 2) \geq \frac{|E(G)|}{\mathrm{ex}(G, H)} \geq \frac{|E(G)|}{\mathrm{ex}(K_n, H)}, \tag{2}$$

where $n = |V(G)|$. Summarizing (1) and (2) we obtain the following lower bound on $r(K_{n,n}, K_{p,p}, 2)$.

$$n^{1/p}(1 + o(1)) \leq \frac{n^2}{\mathrm{ex}(K_{n,n}, K_{p,p})} \leq r(K_{n,n}, K_{p,p}, 2). \tag{3}$$

It was pointed out by Spencer [8] that a standard probabilistic argument shows

$$r(G, H, 2) \leq r(K_n, H, 2) \leq \frac{n^2}{\mathrm{ex}(K_n, H)} \log n. \tag{4}$$

The following lemma connects the Ramsey numbers $r(K_{n,n}, K_{a,b}, 2)$ and $r(K_n, K_{a,b}, 2)$ in the same way as the Turán numbers $\mathrm{ex}(K_n, K_{a,b})$ and $\mathrm{ex}(K_{n,n}, K_{a,b})$ are related in (1) by Bollobás (cf. [5], p. 310). Note that $r(K_{n,n}, K_{a,b}, 2) \leq r(K_{2n}, K_{a,b}, 2)$ is obvious, but this lemma enables us to determine the asymptotic values of some $r(K_{n,n}, K_{a,b}, 2)$.

Lemma 2.1. *Suppose that $b \geq 2$. Then $r(K_{n,n}, K_{a,b}, 2) \leq r(K_n, K_{a,b}, 2) + 1$.*

Proof. Let $c : E(K_n) \to [m]$ be an edge-coloring of K_n without a monochromatic $K_{a,b}$. Let $V(K_n) = \{v_1, ..., v_n\}$ and $V(K_{n,n}) = A \cup B$, with $A = \{a_1, ..., a_n\}$, $B = \{b_1, ..., b_n\}$. Then the following edge-coloring $c' : E(K_{n,n}) \to [m+1]$ is a $(K_{a,b}, 2)$-coloring. Let $c'(a_i, b_j) = c(v_i, v_j)$ if $i \neq j$ and $m + 1$ if $i = j$. □

Corollary 2.2. $r(K_{n,n}, K_{2,2}, 2) = n^{1/2}(1 + o(1))$ *and* $r(K_{n,n}, K_{3,3}, 2) = n^{1/3}(1 + o(1))$.

Proof. The lower bounds for these Ramsey numbers follow from (3) while the upper bounds are implied by Lemma 2.1 and the following two asymptotics. Chung and Graham [8] proved $r(K_n, K_{2,2}, 2) = n^{1/2}(1 + o(1))$ by constructing a k-coloring of the edges of K_{k^2-k+1} if $k - 1$ is a power of a prime, such that no monochromatic C_4 occurs. A $(K_{3,3}, 2)$-coloring implying $r(K_n, K_{3,3}, 2) = n^{1/3}(1 + o(1))$ was recently given by Alon, Rónyai and Szabó [3]. □

Chung and Graham [8] conjectured that $r(K_{n,n}, K_{s,t+1}, 2)$ is asymptotic to $(n/t)^{1/s}$ for fixed $t + 1 \geq s \geq 2$ and proved it for $t + 1 = s = 2$. Chung [9] proved this for $s = 2$ and some special values of t using a complicated argument based on high-dimensional projective geometries over finite fields. In [8], the proof of the above conjecture for $s = t + 1 = 2$ used Singer's theorem on the existence of simple difference sets. Below we prove the conjecture for $s = 2$ and fixed $t \geq 1$ using simple self-contained argument.

Theorem 2.3. *Let t be a positive integer. Then the Ramsey numbers $r(K_{n,n}, K_{2,t+1}, 2)$ and $r(K_n, K_{2,t+1}, 2)$ are both asymptotic to $\sqrt{n/t}$ as $n \to \infty$.*

Proof. A lower bound $r(K_{n,n}, K_{2,t+1}, 2) > \sqrt{n/t} - O(n^{1/4})$ follows from (2) using (1) (or using the original bound by Kővári, Sós and Turán [22] which was extended to multicolored graphs by Chung and Graham [8].)

To prove $r(K_n, K_{2,t+1}, 2) \leq (1 + o(1))\sqrt{n/t}$ we give a coloring based on the construction from [18], where it was proved that $\mathrm{ex}(n, K_{2,t+1}) = \frac{1}{2}\sqrt{t}n^{3/2} + O(n^{4/3})$ for any fixed $t \geq 1$. Then the asymptotics for the Ramsey numbers follow from Lemma 2.1.

Let q be a prime power such that $(q-1)/t$ is an integer, and let $n = (q-1)^2/t$. We define a coloring c of the edges of K_n by $(q-1)/t + O(\sqrt{q}\log q)$ colors such that no monochromatic copy of $K_{2,t+1}$ occurs. Then the upper bound for the Ramsey number for all n follows from the fact that for every sufficiently large n there exists a prime q satisfying $q \equiv 1 (\mathrm{mod}\ t)$ and $\sqrt{nt} - n^{1/3} < q < \sqrt{nt}$ (see [21]).

Let $\mathbf{F}$ be the q-element finite field, $h \in \mathbf{F}$ an element of order t, $H = \{1, h, ..., h^{t-1}\}$. H is a t-element subgroup of $\mathbf{F}\setminus\{0\}$. Let $H_1, ..., H_{(q-1)/t}$ be the cosets of H. These cosets give the decomposition $\mathbf{F}\setminus\{0\} = H_1 \cup ... \cup H_{(q-1)/t}$. The vertices of K_n are labeled by the t-element orbits of $(\mathbf{F}\setminus\{0\}) \times (\mathbf{F}\setminus\{0\})$ under the action of multiplication by powers of H. Thus the vertex set consists of equivalence classes in $(\mathbf{F}\setminus\{0\}) \times (\mathbf{F}\setminus\{0\})$, $n = (q-1)^2/t$, where $(a, b) \sim (x, y)$ if there is an $\alpha \in H$ such that $a = \alpha x$ and $b = \alpha y$. The class represented by (a, b) is denoted by $\langle a, b\rangle$. Color the edge joining two classes $\langle a, b\rangle$ and $\langle x, y\rangle$ with color i if $ax + by \in H_i$. This relation is symmetric, and compatible with the equivalence classes, i.e., $ax + by \in H_i$, $(a, b) \sim (a', b')$, and $(x, y) \sim (x', y')$ imply $a'x' + b'y' \in H_i$. Note that the edges $(\langle a, b\rangle, \langle x, y\rangle)$ with $ax + by = 0$ are still uncolored.

Let G_i denote the graph consisting of the edges colored i. We claim that G_i contains no copy of $K_{2,t+1}$. The proof follows [18]. We show that for $(a, b), (a', b') \in (\mathbf{F}\setminus\{0\}) \times (\mathbf{F}\setminus\{0\})$, $(a, b) \not\sim (a', b')$ these two vertices have at most t common neighbors in G_i. Consider the equation system

$$\begin{aligned} ax + by &= u \\ a'x + b'x &= v \end{aligned} \tag{5}$$

We claim it has at most one solution (x, y) for every $u, v \in H_i$. Indeed, the solution is unique if the determinant of the system $\det\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$ is not 0. Otherwise, there exists an α such that $a = \alpha a'$, $b = \alpha b'$. If there exists a solution of (5) at all, then multiplying the second equation by α and subtracting it from the first one we get on the right hand side $u - \alpha v = 0$. We know that $(u/v) \in H$ hence $\alpha \in H$, contradicting the fact that (a, b) and (a', b') are not equivalent. Finally, there are t^2 possibilities for $u, v \in H_i$ in (5). The set of solutions form t-element equivalent classes, so there are at most t-classes $\langle x, y\rangle$ joint simultaneously to $\langle a, b\rangle$ and $\langle a', b'\rangle$.

Now turn to the still uncolored edges $(\langle a, b\rangle, \langle x, y\rangle)$ with $ax + by = 0$. Let G_0 be the graph formed by them. We are going to finish the proof of the Theorem by coloring the edges of G_0 by an additional $O(\sqrt{q}\log q)$ colors. Partition the underlying set of K_n into equivalence classes, $V_1, \ldots, V_{q-1}$, of size $(q-1)/t$ as follows: $\langle a, b\rangle$ and $\langle x, y\rangle$ are in the same class if $a/b = x/y$. If $\langle a, b\rangle \in V_i$ and $\langle x, y\rangle \in V_j$ $(i \neq j)$ and the edge $(\langle a, b\rangle, \langle x, y\rangle)$ is in G_0, then clearly every edge between V_i and V_j is also in G_0, and no edge in G_0 has only one endpoint in $V_i \cup V_j$. If some edge with both endpoints in V_i is in G_0, then all edges with both endpoints in either V_i or V_j are in G_0. Hence the graph G_0 consists of vertex disjoint unions of complete bipartite graphs $K_{(q-1)/t,(q-1)/t}$ joining a V_i to a V_j completely, and perhaps also some complete graphs. For these graphs we can use (4) together with the lower bound for

$\mathrm{ex}(K_n, K_{2,t+1})$ from [18] to color the edges of each of them simultaneously using the same set of at most $O(\sqrt{q}\log q)$ new colors such that each color class is $K_{2,t+1}$-free. □

The applications of symmetric block designs to construct $K_{2,t+1}$-free graphs is not new. To cite one example, Parsons [24] extended the 'Friendship Theorem' of Erdős, Rényi and Sós [17] and used symmetric (v, k, λ)-block designs admitting a polarity to obtain certain Ramsey numbers.

3. A General Upper Bound

Erdős and Gyárfás obtained an upper bound for $f(n,p,q)$ from the Local Lemma. Using the same method, we obtain an upper bound for $r(G,H,q)$. We always assume that G has n vertices, and that H has v vertices and e edges. Below we present the symmetric version of the Lemma. For a proof, see [2].

Theorem 3.1. (Lovász Local Lemma) *Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of at least $n - D$ other events, and suppose that* $\Pr(A_i) \le p$ *for all* $1 \le i \le n$. *If* $3pD \le 1$, *then* $\Pr(\cap_i \overline{A_i}) > 0$.

Theorem 3.2. *For graphs G, H with $n = |V(G)|$, $v = |V(H)|$, $e = |E(H)|$ and $1 \le q \le e$, there is a constant $c = c(H,q)$ such that*

$$r(G,H,q) < cn^{\frac{v-2}{e-q+1}}.$$

Proof. If $q = 1$, the result is trivial, so assume that $q \ge 2$. Color the edges of G independently with t colors, where the colors are assigned with equal probability. The probability that a given copy of H receives at most $q-1$ colors is bounded by

$$P = \binom{t}{q-1}\left(\frac{q-1}{t}\right)^e < t^{q-1}\left(\frac{q-1}{t}\right)^e.$$

Furthermore, the coloring of a fixed H is independent of the colorings of all other H's except those that intersect it in at least one edge. The number of these is at most

$$D = e\binom{n}{v-2} < en^{v-2}.$$

Solving $3PD < 1$ yields

$$t \ge (3(q-1)^e e)^{\frac{1}{e+1-q}} n^{\frac{v-2}{e-q+1}} = c(H,q)n^{\frac{v-2}{e-q+1}}. \tag{6}$$

The Local Lemma therefore implies that if t is at least this large, then an (H,q)-coloring with t colors exists. □

Corollary 3.3. $r(K_{n,n}, K_{p,p}, q) \le c(K_{p,p}, q)n^{\frac{2p-2}{p^2-q+1}}$.

4. Thresholds for Linear and Quadratic $r(K_{n,n}, K_{p,p}, q)$

For fixed p, we find the smallest q for which $r(K_{n,n}, K_{p,p}, q)$ is linear in n, and the smallest q for which $r(K_{n,n}, K_{p,p}, q)$ is quadratic in n. It turns out that these values are fairly close.

Theorem 4.1. *Suppose that $q = e - v + 3$ and that H is connected. Then*

$$\frac{n-1}{2v-4} \le r(K_n, H, q) < cn \ ,$$

and $r(K_n, H, q-1) \le c' n^{1-1/(v-1)}$ for some constants $c = c(H, q)$ and $c' = c(H, q-1)$.

Proof. The upper bounds follow from Theorem 3.2. For the lower bound, it is sufficient to show that in an (H, q)-coloring of K_n each color class contains at most $(v-2)n$ edges. Let S be a spanning tree of H. A monochromatic copy of S can be completed to a copy of H with a total of at most $e - (v-1) + 1 = q - 1$ colors. Thus each color class of $E(K_n)$ contains at most $\mathrm{ex}(K_n, S)$ edges. It is well-known (and easy, see [5]) that $\mathrm{ex}(K_n, S) \le (v-2)n$. □

Note that the Erdős-Sós Conjecture (i.e., $\mathrm{ex}(K_n, S) \le (v-2)n/2$; for latest developments, see Ajtai, Komlós and Szemerédi [1]) would yield a twice larger lower bound.

The above proof can easily be modified to give the following. Let S be a spanning tree of the bipartite graph H. Then $n^2/\mathrm{ex}(K_{n,n}, S) \le r(K_{n,n}, H, e-v+3)$. Let $S_{a,b}$ denote the double star, a spanning tree of $K_{a,b}$ with an adjacent pair of degrees a and b. By considering edges with an endpoint at a vertex of small degree, it is easy to see that $\mathrm{ex}(K_{n,n}, S_{a,b}) < 2n(b-1)$ for $b \ge a$. Thus we have the following.

Corollary 4.2. *Fix $p \ge 2$. If $q = p^2 - 2p + 3$, then $r(K_{n,n}, K_{p,p}, q)$ is linear in n, in particular, $\frac{n}{2p-2} < r(K_{n,n}, K_{p,p}, q) < c(K_{p,p}, q)n$. On the other hand, $r(K_{n,n}, K_{p,p}, q-1) \le c(K_{p,p}, q-1)n^{1-1/(2p-1)}$.* □

Remark: It can easily be shown from (6) in the proof of Theorem 3.2 that $c(K_{p,p}, q-1) < 3p^{p+2}$.

Next we compute the threshold for quadratic $r(K_{n,n}, K_{p,p}, q)$.

Theorem 4.3. *Let $q = p^2 - p + 2$, $p \ge 3$. Then $r(K_{n,n}, K_{p,p}, q) \ge C(n^2 - n)$, where $C = (\lfloor p/2 \rfloor^2 + \lfloor p/2 \rfloor + 1)/(\lfloor p/2 \rfloor^3 + \lfloor p/2 \rfloor^2 + \lfloor p/2 \rfloor + 1)$. If $p \ge 6$ and $n > p^{3/2}$, then $n^{4/3} - 2n^{2/3} + 1 \le r(K_{n,n}, K_{p,p}, q-1) \le c' n^{2-2/p}$ for $c' = c(K_{p,p}, q-1)$.*

Proof. Let $E(K_{n,n}) = C_1 \cup \dots \cup C_r$ be a $(K_{p,p}, q)$-coloring. Then every color class has at most $p-1$ edges, hence $r(K_{n,n}, K_{p,p}, q) \ge n^2/(p-1)$ is immediate. Next we improve the coefficient $1/(p-1)$ to C. Here C is slightly less then $1/\lfloor p/2 \rfloor$ and (as it will be shown in Theorem 7.1) gives the right coefficient of n^2 for $p = 3$.

Denote the partite sets of $K_{n,n}$ by X and Y. Let e_i denote the size of C_i, let V_i be the set of vertices incident to an edge of C_i, and let $E_i = K_{n,n}|V_i$ be the edges contained in V_i. Call a color class C_i *large* if $e_i \ge \lfloor p/2 \rfloor + 1$. Let $\ell + m$ be the number of large color classes. Suppose that ℓ of these, $C_1, \dots, C_\ell$ are matchings, but for each C_i with $\ell < i \le \ell + m$ one can find a vertex v_i incident to at least 2 edges of color i. For the rest of the colors $C_{\ell+m+1}, \dots, C_r$ we have $e_i \leq \lfloor p/2 \rfloor$.

We claim that $\sum_{\ell<i\leq\ell+m} e_i \leq n$. Indeed, if $\ell < i < j \leq \ell+m$, then the vertices of degree at least 2, v_i and v_j belong to the same partite set X or Y. Assume that $v_i, v_j \in Y$. Then $V_i \cap V_j \cap X = \emptyset$, implying that large color classes which are not matchings altogether span at most n edges. Considering the three types of colors we obtain

$$n^2 = \sum e_i = \sum (e_i - \lfloor p/2 \rfloor) + r\lfloor p/2 \rfloor \leq \sum_{1\leq i\leq\ell} (e_i - \lfloor p/2 \rfloor) + n + r\lfloor p/2 \rfloor \ . \tag{7}$$

We may suppose that $\ell \geq 2$, otherwise a slightly sharper version of (7) gives a better lower bound than $C(n^2-n)$. For each large color class C_i which is a matching observe that $|V_i| = 2e_i$, and $|E_i| = e_i^2$. It follows that for $1 \leq i < j \leq \ell$ we have $E_i \cap E_j = \emptyset$. Even more, if $e, e' \in E_i \cup E_j$, then e and e' have different colors unless they both belong to one of C_i or C_j. Thus, letting t denote the number of distinct colors in $\cup_{1\leq i\leq\ell} E_i$, we have

$$\sum_{1\leq i\leq\ell} (e_i^2 - e_i + 1) = t \leq r \ . \tag{8}$$

Let $\alpha = 1/(\lfloor p/2 \rfloor^2 + \lfloor p/2 \rfloor + 1)$. Multiplying (8) by α, adding the result to (7), and rearranging yields

$$n^2 - n \leq r(\lfloor p/2 \rfloor + \alpha) + \sum_{1\leq i\leq\ell} (e_i - \lfloor p/2 \rfloor - \alpha(e_i^2 - e_i + 1)) \ .$$

Since $x - \lfloor p/2 \rfloor - \alpha(x^2 - x + 1) \leq 0$ for $x \geq \lfloor p/2 \rfloor + 1$, the number of colors r is at least $(n^2-n)/(\lfloor p/2 \rfloor + \alpha)$, as claimed.

Now we are going to prove the polynomial bounds for $r(K_{n,n}, K_{p,p}, q-1)$. The upper bound follows from Theorem 3.2. For the lower bound, consider a $(K_{p,p}, q-1)$-coloring of $K_{n,n}$. If every color class has at most $n^{2/3}$ edges, then the total number of color classes is at least $n^{4/3}$. We may therefore suppose that there is a color class $C \subseteq E(K_{n,n})$ of size at least $n^{2/3} > p$. Let V_C be the set of vertices incident to an edge from C, let $V_X = V_C \cap X$, and let $V_Y = V_C \cap Y$. Let G be the graph formed by the edges in C, i.e., $V(G) = V_C$ and $E(G) = C$. If there exist $x \in V_X$ and $y \in V_Y$ with $\min\{d_G(x), d_G(y)\} \geq 2$, then there is a $(q-2)$-colored $K_{p,p}$, (containing $p+1$ edges from C) so we may assume by symmetry that $d_G(x) \leq 1$ for all $x \in V_X$. Thus $|V_X| \geq \max\{n^{2/3}, |V_Y|\}$. Let $H \subseteq K_{n,n}$ be the complete bipartite graph spanned by V_C. Observe that all edges other than the edges from C have distinct colors in H. If $|V_Y| \leq |V_X| \leq |V_Y| + 1$ then the number of colors on $E(H)$ is at least

$$|V_X||V_Y| - |V_X| + 1 \geq |V_X|(|V_Y| - 1) + 1 \geq n^{4/3} - 2n^{2/3} + 1.$$

If $|V_X| > |V_Y| + 1$, then either there are $u, v \in V_Y$ with $d_G(u) \geq 2$ and $d_G(v) \geq 2$, or there is a $w \in V_Y$ with $d_G(w) \geq 3$. In the first case, let $Y' = Y - \{u, v\}$, and in the second case, let $Y' = Y - \{w\}$. Since $p \geq 6$, there are no repeated colors on the edges between Y' and V_X except the color on edges of C. Thus the total number of colors is at least $(n-3)|V_X| + 1 \geq n^{4/3} - 2n^{2/3} + 1$. □

5. When is $r(K_{n,n}, K_{p,p}, q) = n^2 - O(1)$?

In this section we determine, for fixed p, the threshold for q beyond which all edges but a constant number must be colored with distinct colors. We also determine an infinite family of ramsey numbers.

Theorem 5.1. *If $q \geq p^2 - \lfloor p/2 \rfloor + 1$, then $r(K_{n,n}, K_{p,p}, q) = n^2 - (p^2 - q)$. However, $r(K_{n,n}, K_{p,p}, p^2 - \lfloor p/2 \rfloor) \leq n^2 - \lfloor n/2 \rfloor$, with equality for $p \geq 7$ and p odd, and $r(K_{n,n}, K_{5,5}, 23) = n^2 - 2\lfloor n/2 \rfloor + 2$. Moreover, $r(K_{n,n}, K_{p,p}, p^2 - \lfloor p/2 \rfloor) = n^2 - \lceil n/2 \rceil$ for $p \geq 14$ and p even.*

Proof. The upper bounds for $r(K_{n,n}, K_{p,p}, q)$ are provided by the following constructions. Suppose that the partite sets of $K_{n,n}$ are $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$.

When $q \geq p^2 - \lfloor p/2 \rfloor + 1$, color the edges $x_{2i-1}y_{2i-1}$ and $x_{2i}y_{2i}$ with color i, for $1 \leq i \leq p^2 - q$. When $q = p^2 - \lfloor p/2 \rfloor$, color in the same way, except let $1 \leq i \leq \lfloor n/2 \rfloor$. In both cases, color all the other edges with new distinct colors. The total number of colors used is $n^2 - (p^2 - q)$ in the first case, and $n^2 - \lfloor n/2 \rfloor$ in the second case. When n is odd and p is even, we can also color the pair x_1y_n and y_3x_n with the same color; this saves one color, giving only $n^2 - \lceil n/2 \rceil$ colors.

Our construction for $r(K_{n,n}, K_{5,5}, 23)$ is slightly different. For $2 \leq i \leq \lfloor n/2 \rfloor$, let x_1y_{2i-1} and x_2y_{2i} have the same color, with different pairs getting distinct colors. Similarly, let y_1x_{2i-1} and y_2x_{2i} have the same color (but distinct from the previous color set), with different pairs getting distinct colors. Give all other edges new distinct colors. This is a $(K_{5,5}, 23)$-coloring, since no $K_{5,5}$ contains three monochromatic matchings. The number of colors is $n^2 - 2\lfloor n/2 \rfloor + 2$.

To prove the lower bounds consider a $(K_{p,p}, q)$-coloring with r colors. Let $e_1^i, e_2^i, \ldots, e_s^i$ be the edges of color i with $s \geq 2$. Form the 4-element sets F_j^i $(1 \leq j < s)$ by taking the union $e_1^i \cup e_{j+1}^i$ and adding an arbitrary additional vertex of $X \cup Y$ if needed in such a way that $|F_j^i \cap X| = |F_j^i \cap Y| = 2$. Finally, let $\mathbf{F}$ be the edge-set of the (multi)hypergraph of the four-tuples obtained in this way. Since every color class with t edges gives rise to precisely $t-1$ four-tuples, $|\mathbf{F}| = n^2 - r$. For p-element sets $X' \subset X$, $Y' \subset Y$, we have

$$X' \cup Y' \text{ contains at most } p^2 - q \text{ members of } \mathbf{F}. \tag{9}$$

If $2(p^2 - q + 1) \leq p$ and $|\mathbf{F}| > p^2 - q$, then we can take $p^2 - q + 1$ four-tuples from $\mathbf{F}$ that are contained in a copy H of $K_{p,p}$; $E(H)$ will have fewer than q colors. Hence if $2(p^2 - q + 1) \leq p$, then $|\mathbf{F}| \leq p^2 - q$ and we are done.

Call a member $F \in \mathbf{F}$ of type X (type Y) if no other member of $\mathbf{F}$ contains any vertex from $F \cap X$ ($F \cap Y$, resp.). If each edge is of type X then $|\mathbf{F}| \leq |X|/2$ and we are done. The same is true for type Y.

Suppose that F_1 is not of type X and F_2 is not of type Y, for example $F_1 \cap F_3 \cap X \neq \emptyset$ and $F_2 \cap F_4 \cap Y \neq \emptyset$. Suppose first, that these 4 sets are distinct members of $\mathbf{F}$. In the case p odd, $p \geq 7$ adding $(p-7)/2$ arbitrary additional members to F_1, F_2, F_3, F_4 one gets a contradiction to (9) and we are done.

Suppose that $\mathbf{F}$ contains another 4 members $F_5, \ldots, F_8$ such that $F_5 \cap F_7 \cap X \neq \emptyset$ and $F_6 \cap F_8 \cap Y \neq \emptyset$. In the case p even, $p \geq 14$ adding $(p-14)/2$ arbitrary additional members of $\mathbf{F}$ to $F_1, \ldots, F_8$ one gets a contradiction to (9) and we are done.

In case of coincidencies among $F_1, \ldots, F_4$ one needs to add more members. The details are omitted.

It remains to consider the case $p = 5$. Suppose that F_1 is of neither type, i.e., $F_1 \cap F_2 \cap X \neq \emptyset$, and $F_1 \cap F_3 \cap Y \neq \emptyset$. Then $F_1 \cup F_2 \cup F_3$ can be covered by the vertex set of a $K_{5,5}$ (by a $K_{3,3}$ when F_2 coincides with F_3) a contradiction to (9). Thus every member is of type X or of type Y. If each member is of both types we obtain $|\mathbf{F}| \leq (2n)/4$, otherwise we have $|\mathbf{F}| \leq 2\lfloor (n-2)/2 \rfloor$. □

6. Densities of hypergraphs

In this section we determine, for fixed p, the threshold for q beyond which all edges but $\Theta(n)$ must be colored with distinct colors. Our main tool is an estimate of the size of a hypergraph with bounded densities of small subhypergraphs.

A k-uniform hypergraph with edge-set $\mathbf{F}$ is called $(u, v+1)$-free if every u vertices span at most v members of $\mathbf{F}$. Let $g_k(n, u, v)$ be the maximum number of edges of a $(u, v+1)$-free k-uniform hypergraph with n vertices. Turán's classical theorem determines $g_2(n, u, \binom{u}{2} - 1)$, for example, $g_2(n, 3, 2) = \lfloor n^2/4 \rfloor$. For a recent account on graph-density questions see Griggs, Simonovits and Thomas [20]. Brown, Erdős and Sós [7] proved that

$$g_k(n, u, v) > cn^{k - \frac{u-k}{v}},$$

by constructing a $(u, v+1)$-free k-uniform hypergraph on n vertices with $cn^{k-\frac{u-k}{v}}$ edges (here $c = c(k, u, v) > 0$ is independent of n). Consider the hypergraph $\mathbf{H}$ on $2n$ vertices obtained from their construction for $k = 4$ and $u = 2p$. Their proof also implies that for the case $v \leq p - 2$ one can also suppose that $|H \cap H'| \leq 1$ for all $H, H' \in \mathbf{H}$. Randomly partition the vertices of $\mathbf{H}$ into two equal sets X and Y. As the probability that a 4-element set is partitioned into two equal parts is $6/16$, this yields a family of 4-subsets $\mathbf{F} = \{F_1, \ldots, F_m\}$ of an underlying set $X \cup Y$ such that every $2p$-element subset contains at most v of the F_i's and
1) $|F_i \cap X| = |F_i \cap Y| = 2$ for every F_i,
2) $|F_i \cap F_j| \leq 1$ for $i \neq j$ (assuming that $v \leq p - 2$), and
3) $m > c_p n^{4-(2p-4)/v}$, where $c_p > 0$ depends only on p.
(Here c_p is smaller than the constant in the result of Brown, Erdős, and Sós.)
Now replace each 4-element set F_i by two disjoint pairs contained in it connecting X to Y, color these two edges by color i, and color the rest of the pairs between X and Y by distinct new colors. Since the total number of colors used is $n^2 - m$, we obtain

$$r(K_{n,n}, K_{p,p}, p^2 - v) < n^2 - c_p n^{4 - \frac{2p-4}{v}} \tag{10}$$

for $1 \leq v \leq p - 2$ and some constant $c_p > 0$.

Theorem 6.1. *If $p^2 - \lfloor\frac{2p-1}{3}\rfloor + 1 \le q \le p^2 - \lfloor\frac{p}{2}\rfloor$, then $n^2 - 2\lfloor(p-2)/3\rfloor(n-1) < r(K_{n,n}, K_{p,p}, q) \le n^2 - \lfloor n/2\rfloor$. However, $r(K_{n,n}, K_{p,p}, p^2 - \lfloor\frac{2p-1}{3}\rfloor) < n^2 - c_p n^{1+\varepsilon_p}$. Here c_p and ε_p are positive constants depending only on p.*

Proof. The upper bound in the last statement follows from (10) by letting $v = \lfloor(2p-1)/3\rfloor$.

For the case $\lfloor\frac{p}{2}\rfloor \le p^2 - q < \lfloor\frac{2p-1}{3}\rfloor$, a $(K_{p,p}, q)$-coloring with $n^2 - \lfloor n/2\rfloor$ colors was given in Section 5. We have to prove that all such colorings use more than $n^2 - 2\lfloor(p-2)/3\rfloor(n-1)$ colors. Let E be the set of edges whose color is used also on at least one other edge, and let $G \subseteq K_{n,n}$ be the subgraph spanned by E. Set $t = \lfloor(p+1)/3\rfloor$. We claim that if $u, v \in V(G)$ with $d_G(u), d_G(v) \ge t$, then $uv \notin E$, i.e., high degree vertices in G are nonadjacent in G. To prove this claim, suppose that $uv \in E$.

Case 1: $p \not\equiv 1 \pmod 3$. Then there is a $K_{p,p}$ containing a pair of edges of each color that appears on the edges incident with uv (and perhaps some more pairs e_i, f_i, with color i if there are colors adjacent to both u and v). The number of colors on this copy is at most $p^2 - (2t-1) < q$, a contradiction.

Case 2: $p \equiv 1 \pmod 3$. Then $3t - 1 = p - 2$, so in addition to the edges in the previous case, our copy of $K_{p,p}$ can be chosen to contain another 2 edges with the same color. The number of colors on this copy is $p^2 - (2t-1) - 1 < q$, a contradiction.

Counting the edges in E by their endpoint of lower degree gives $|E| \le 2(n-1)(t-1)$, which yields the required lower bound on the number of colors. □

The coefficient $2\lfloor(p-2)/3\rfloor$ in Theorem 6.1 can be improved by choosing t more carefully, noting its dependence on q. We could also include the colors from the nontrivial color classes. We do not attempt to find the optimal bound.

Note that substituting $v = p - 2$ into (10) we obtain a matching upper bound for $q = p^2 - p + 2$ (cf. Theorem 4.3).

$$r(K_{n,n}, K_{p,p}, p^2 - p + 2) < (1 - c_p)n^2. \tag{11}$$

7. The exact value of $r(K_{n,n}, K_{3,3}, 8)$.

When $p = 3$ and $2 \le q \le 8$, our upper bounds are those in Theorem 3.2. (See the chart in Section 9). We have nontrivial lower bounds only for $q \in \{6, 8\}$. Corollary 4.2 states that $n/4 < r(K_{n,n}, K_{3,3}, 6) \le cn$ for some constant c. Theorem 4.3 states that $(3/4)(n^2 - n) < r(K_{n,n}, K_{3,3}, 8)$. In this section we give the exact value of this Ramsey number.

Theorem 7.1. *$r(K_{n,n}, K_{3,3}, 8)$ is $(3/4)n^2$ if n is even, and $\lceil(3/4)n^2 + n/4\rceil$ if n is odd.*

Proof. First, we show the upper bound by constructing the colorings. Let the partite sets of $K_{n,n}$ be $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$. We color $E(K_{n,n})$ with ordered pairs as follows.

Case 1: $n = 2k$. For $i, j \in \{1, \ldots, \lfloor n/2 \rfloor\}$, let $x_{2i}y_{2j}$ and $x_{2i-1}y_{2j-1}$ both have color (i, j). Let all other edges have new distinct colors. Since every $K_{3,3}$ has at most one pair of edges of the form $x_{2i}y_{2j}, x_{2i-1}y_{2j-1}$, our construction is a $(K_{3,3}, 8)$-coloring.

Case 2: $n = 4k+1$. Let $c(x_{j-(2i-1)}y_j) = c(x_{j+2i}y_{j+1}) = (j, i)$ for $1 \le i \le (n-1)/4$, $1 \le j \le n$, where addition is taken modulo n. Color all other edges with distinct colors. Since the union of any two color classes of size 2 spans at least 4 vertices either in X or in Y, every $K_{3,3}$ has at most one color class of size two. Thus our construction is a $(K_{3,3}, 8)$-coloring.

Case 3: $n = 4k+3$. Let $c(x_{j-(2i-1)}y_j) = c(x_{j+2i}y_{j+1}) = (j, i)$ as before for $1 \le i \le (n-3)/4$, $1 \le j \le n$, and let $c(x_iy_i) = c(x_{i+(n-1)/2}y_{i+(n-1)/2}) = (i, 0)$ for $1 \le i \le (n-1)/2$. Color all other edges with distinct colors. It is easy to check that the color classes of size 2 induce edge-disjoint copies of C_4, so the obtained coloring is a $(K_{3,3}, 8)$-coloring. The total number of colors in the last case is $n^2 - n(n-3)/4 - (n-1)/2 = (3/4)n^2 + n/4 + 1/2$.

For the lower bound, consider a $(K_{3,3}, 8)$-coloring of $K_{n,n}$. Obviously, each color class contains at most 2 edges. Let $C_1, \ldots, C_t$ be the color classes of two edges, $C_i = \{\{x_1^i, y_1^i\}, \{x_2^i, y_2^i\}\}$. We are going to define t edge-disjoint cycles of length four $Q_1, \ldots, Q_t$, $C_i \subset Q_i$. Consider a color class C_i forming $2K_2$, i.e., $x_1^i \ne x_2^i$ and $y_1^i \ne y_2^i$. Then let Q_i be the 4 edges spanned by the vertices of C_i. Consider a color class C_i forming P_3, for example $x_1^i = x_2^i$ and $y_1^i \ne y_2^i$. Then choose a vertex x_3^i arbitrarily from $X \setminus \{x_1^i\}$, and let Q_i be spanned by $\{x_1^i, x_3^i, y_1^i, y_2^i\}$. It is easy to check that edges of Q_i and Q_j are disjoint for $i \ne j$.

Finally we need an upper bound for the number of edge-disjoint four-cycles. Each $x \in X$ is contained in at most $n/2$ of the Q_i's, thus $2t \le n\lfloor n/2 \rfloor$. □

It is easy to see that although our construction is not unique, every optimal $(K_{3,3}, 8)$-coloring contains no two adjacent edges of the same color, and the coloring can be obtained from a four-cycle packing of $K_{n,n}$. Note that in the same way one can show that if $m, n \ge 3$, then $r(K_{n,m}, K_{3,3}, 8) = nm - t$, where t is the maximum number of edge-disjoint four-cycles packed into $E(K_{n,m})$. On the other hand (denoting this maximum t by $t(m, n)$) one can easily extend the above constructions, or use a recurrence like $t(m, n) \ge t(m, n-2) + \lfloor m/2 \rfloor$ to determine the exact value of t. This yields

$$r(K_{n,m}, K_{3,3}, 8) = nm - \min\left\{ \left\lfloor \frac{n}{2} \left\lfloor \frac{m}{2} \right\rfloor \right\rfloor, \left\lfloor \frac{m}{2} \left\lfloor \frac{n}{2} \right\rfloor \right\rfloor \right\} . \tag{12}$$

8. Bounds for $r(K_{n,n}, C_4, 3)$.

The next case we consider is $r(K_{n,n}, C_4, 3)$. Since monochromatic P_4's are forbidden, each color class consists of disjoint stars. Using this observation, it is easy to prove that $r(K_{n,n}, C_4, 2) \ge n^2/(2n-2) \sim n/2$. Later we improve this lower bound but first we provide a simple construction.

Theorem 8.1. *If n is odd, then $r(K_{n,n}, C_4, 3) \le n$. If n is even, then $r(K_{n,n}, C_4, 3) \le n+1$.*

Proof. First suppose that n is odd. Let the partite sets of $K_{n,n}$ be $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$. Color $E(K_{n,n})$ with n colors by letting the j^{th} color class consist of the edges $x_i y_{i+j}$, $1 \le i \le n$, $0 \le j \le n-1$, where the subscripts are taken modulo n.

Since each color class is a matching, a 2-colored C_4 must consist of 2 monochromatic matchings of size 2. Assume without loss of generality that one of these matchings is in color 0 and that the four vertices of the 4-cycle are x_1, y_1, x_k, y_k. Since n is odd, $n+1-k \neq k-1$. Thus $x_1 y_k$ and $x_k y_1$ have distinct colors, and our construction is a $(C_4, 3)$-coloring.

When n is even we color the edges of $K_{n+1,n+1}$ as before and consider the coloring restricted to $K_{n,n}$. This gives an upper bound of $n+1$. □

Improving this upper bound seems to be very hard. D. Eichhorn [13] improved it by one when $n = 4, 12, 20, 36$, and 60 by exhibiting $(C_4, 3)$-colorings of $K_{n,n}$ with n colors. In the matrix below, the i, j^{th} entry represents the color of $x_i y_j$, where the partite sets of G are $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$. A construction for $n = 12$ is shown.

$$\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 12 & 9 & 10 & 11 \\
4 & 1 & 2 & 3 & 8 & 5 & 6 & 7 & 12 & 11 & 9 & 10 \\
4 & 3 & 1 & 2 & 6 & 8 & 7 & 5 & 9 & 10 & 11 & 12 \\
2 & 4 & 3 & 1 & 8 & 7 & 5 & 6 & 10 & 12 & 11 & 9 \\
5 & 6 & 7 & 8 & 12 & 9 & 10 & 11 & 1 & 2 & 3 & 4 \\
8 & 5 & 6 & 7 & 12 & 11 & 9 & 10 & 4 & 1 & 2 & 3 \\
6 & 8 & 7 & 5 & 9 & 10 & 11 & 12 & 4 & 3 & 1 & 2 \\
8 & 7 & 5 & 6 & 10 & 12 & 11 & 9 & 2 & 4 & 3 & 1 \\
12 & 9 & 10 & 11 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
12 & 11 & 9 & 10 & 4 & 1 & 2 & 3 & 8 & 5 & 6 & 7 \\
9 & 10 & 11 & 12 & 4 & 3 & 1 & 2 & 6 & 8 & 7 & 5 \\
10 & 12 & 11 & 9 & 2 & 4 & 3 & 1 & 8 & 7 & 5 & 6
\end{pmatrix}$$

$$r(K_{12,12}, C_4, 3) \le 12$$

We have already observed that $r(K_{n,n}, C_4, 2) \ge n/2$. Through a more careful examination of both the structure of each color class, and the interaction between color classes in a $(C_4, 3)$-coloring, we improve the lower bound to $2n/3$.

Theorem 8.2. $r(K_{n,n}, C_4, 3) > \left\lfloor \frac{2n}{3} \right\rfloor$.

Proof. Consider a $(C_4, 3)$-coloring of $K_{n,n}$ with color classes $D_1, D_2, \ldots, D_g$. Suppose that the i^{th} color class D_i consists of l_i disjoint stars $S_{i,j}$, where $1 \le j \le l_i$. Let $S_{i,j}$ have $d_{i,j}$ edges, and set $L = \sum_i l_i$.

Since every edge is covered once and $\sum_{j=1}^{l_i} d_{i,j} \le 2n - l_i$, we have

$$n^2 = \sum_{i=1}^{g} \sum_{j=1}^{l_i} d_{i,j} \le 2ng - L. \tag{13}$$

Two monochromatic paths of length two with common endpoints would yield a 2-colored C_4. Letting t denote the number of monochromatic paths of length two, we thus obtain

$$\sum_{i=1}^{g}\sum_{j=1}^{l_i}\binom{d_{i,j}}{2} = t \leq 2\binom{n}{2}. \tag{14}$$

From (14) we obtain

$$\frac{\sum_{i=1}^{g}\sum_{j=1}^{l_i}(d_{i,j})^2}{\sum_{i=1}^{g}\sum_{j=1}^{l_i}d_{i,j}} = \frac{2\sum_{i=1}^{g}\sum_{j=1}^{l_i}\binom{d_{i,j}}{2} + \sum_{i=1}^{g}\sum_{j=1}^{l_i}d_{i,j}}{\sum_{i=1}^{g}\sum_{j=1}^{l_i}d_{i,j}} \leq \frac{3n^2-2n}{n^2}. \tag{15}$$

Since the double sum in (14) has $\sum_i l_i = L$ terms, the Cauchy-Schwarz inequality yields

$$\left(\sum_{i=1}^{g}\sum_{j=1}^{l_i}d_{i,j}\right)^2 \leq L\left(\sum_{i=1}^{g}\sum_{j=1}^{l_i}(d_{i,j})^2\right). \tag{16}$$

By rearranging (16) and using (15), we obtain $L \geq n^3/(3n-2)$. Substituting back into (13) gives $2ng \geq n^2 + L \geq n^2 + n^3/(3n-2)$. Solving for g yields

$$g \geq \left\lceil \frac{n(2n-1)}{3n-2} \right\rceil > \left\lfloor \frac{2n}{3} \right\rfloor.$$

□

An *alternating* C_4 is a 2-colored C_4 whose edges alternate between its 2 colors when viewed cyclically. One might feel there is hope in improving the lower bound above because the proof allows alternating C_4's. Unfortunately, we have been unable to obtain any significant improvement from this observation. It is, however, interesting to define a function similar to $r(K_{n,n}, C_4, 3)$ with the exception that alternating C_4's are permitted.

Definition. A *weak* $(C_4, 3)$-coloring of $K_{n,n}$ is a coloring of the edges of $K_{n,n}$ in which every copy of C_4 has at least three colors or is alternately 2-colored. Let $r'(K_{n,n}, C_4, 3)$ denote the minimum number of colors in a weak $(C_4, 3)$-coloring of $K_{n,n}$.

Since the definition of $r'(K_{n,n}, C_4, 3)$ is a relaxation of that of $r(K_{n,n}, C_4, 3)$, we certainly have $r(K_{n,n}, C_4, 3) \geq r'(K_{n,n}, C_4, 3)$. Furthermore, the proof of Theorem 7.2 yields

$$r'(K_{n,n}, C_4, 3) > \left\lfloor \frac{2n}{3} \right\rfloor.$$

In the remaining part of this section we prove an upper bound on $r'(K_{n,n}, C_4, 3)$ that is asymptotic to $3n/4$. The proof requires a deep theorem about edge-coloring of hypergraphs. We describe this first.

Given a hypergraph $H = (V, E)$, the *degree* of a vertex $v \in V$, $d(v)$, is the number of edges containing v. For vertices v, w, the *codegree* of v and w, $cod(v, w)$, is the number of edges containing both v and w. Let

$$\Delta(H) = \max_{v \in V} d(v),$$
$$\delta(H) = \min_{v \in V} d(v),$$

$$C(G) = \max_{u,v \in V, u \neq v} cod(u,v).$$

A *matching* in H is a set of pairwise disjoint edges of H. A matching is *perfect* if every vertex of H is in exactly one of its edges. Let $\chi'(H)$, the *chromatic index* of H, denote the minimum number of matchings needed to partition the edges of H. A hypergraph H is k-uniform if each of its edges consists of exactly k elements.

Theorem 8.3. (Pippenger-Spencer [25]) *For every $k \geq 2$ and $\varepsilon > 0$, there exist $\varepsilon' > 0$ and n_0 such that if H is a k-uniform hypergraph on $n(H) \geq n_0$ vertices satisfying*

$$\delta(H) \geq (1-\varepsilon')\Delta(H) \tag{17}$$

and

$$C(H) \leq \varepsilon' \Delta(G), \tag{18}$$

then

$$\chi'(H) \leq (1+\varepsilon)\Delta(G). \tag{19}$$

We rephrase Theorem 8.3 in more convenient asymptotic notation.

Let $H_1, H_2, \ldots$ be hypergraphs, with $|V(H_i)| \to \infty$. If

$$\delta(H_n) \sim \Delta(H_n), \tag{20}$$

and

$$C(H_n) = o(\Delta(H_n)), \tag{21}$$

then

$$\chi'(H_n) \sim \Delta(H_n). \tag{22}$$

A *Steiner Triple System* (STS) is a 3-uniform hypergraph in which each pair of vertices has codegree one. It is well known that a STS on n points exists if and only if $n \equiv 1, 3$ $(mod\ 6)$.

We use Steiner Triple Systems and the following "large deviation" result in probability theory to prove an upper bound on $r'(K_{n,n}, C_4, 3)$.

Theorem 8.4. (Chernoff [12]) *Suppose that $p \in [0,1]$ and $X_1, \ldots, X_n$ are mutually independent random variables with $\Pr(X_i = 1) = p = 1 - \Pr(X_i = 0)$. If $X = X_1 + \ldots + X_n$ and $a > 0$, then $\Pr(|X - pn| > a) \leq 2e^{-2a^2/n}$.*

Theorem 8.5. *As $n \to \infty$, $r'(K_{n,n}, C_4, 3) \leq \frac{3n}{4}(1 + o(1))$.*

Proof. We first prove the result for a sufficiently dense set of positive integers. Later we use standard approximation arguments to obtain the result asymptotically for all n. Suppose that $2n + 1 \equiv 1, 3 \pmod 6$, and let S be a STS of $[2n + 1]$. Select a set $A \subseteq [2n + 1]$ by picking each point of $[2n + 1]$ with probability $1/2$, independently. Let $A \subseteq [2n + 1]$ be the (random) set of points thus picked, and let H be the 3-uniform hypergraph with vertex set $[2n + 1]$ and edges from the STS that intersect both A and $[2n + 1] - A = B$.

The calculations in the following paragraphs will show that, with high probability, the sizes of A and B differ by very little. Also, the degree of each vertex in H is close to $3n/4$.

Since H is 3-uniform and has codegree bounded by 1, the hypothesis for Theorem 8.3 will be satisfied and we therefore obtain a proper edge-coloring of H with about $3n/4$ colors. This coloring of $E(H)$ will yield a weak $(C_4, 3)$-coloring of the underlying bipartite graph with bipartition A, B.

Set $a = |A|$ and $b = |B|$. Let X be the event that $|a - n| \leq 2\sqrt{n}$, and let Y be the event that $|d_H(i) - 3n/4| \leq \sqrt{\frac{n \log(10n)}{2}}$ for all $i \in [2n+1]$. Since each edge of S is retained in H with probability $3/4$, and every vertex i has degree n in the STS, each vertex in H has expected degree $3n/4$. Since the expected size of A is n (actually $n + 1/2$, but this is insignificant in the following calculation) Theorem 8.4 gives

$$\Pr(\overline{X} \cup \overline{Y}) \leq \Pr(\overline{X}) + \Pr(\overline{Y}) \leq 2\exp\left\{-\frac{8n}{2n+1}\right\} + (2n+1)2\exp\left\{-\frac{n\log(10n)}{n}\right\} < 1.$$

Thus $\Pr(X \cap Y) > 0$, so there is a set A such that both X and Y hold. Choose such a set A. Since X holds, we may assume without loss of generality that $n - 2\sqrt{n} \leq a \leq b \leq n + 2\sqrt{n}$. Let G be the complete bipartite graph with partite sets A and B.

Using this random process, we obtain a hypergraph H satisfying (20) and (21). Theorem 8.3 implies that $\chi'(H) \sim \Delta(H) \sim 3n/4$; consider a decomposition of $E(H)$ into $\chi'(H)$ matchings. An edge in H contains either 2 vertices from A and one from B or vice versa. In G, this edge corresponds to the 3 vertex path with the same vertices. For each color class of edges in H, color all the edges of the corresponding P_3's in G with the same color.

Since each pair of vertices in a STS belongs to a unique edge, all edges in G are colored. Because a color class of edges in G arose from a matching in H, each color class in G consists of disjoint paths of length 2. Lastly, since every pair of vertices in a STS has codegree one, no two monochromatic P_3's in G share each of their two ends. These remarks together imply that the coloring of G is a weak $(C_4, 3)$-coloring with $(1 + o(1))3n/4$ colors.

For each m with $2m + 1 \equiv 1, 3 \pmod 6$, we have obtained a weak $(C_4, 3)$-coloring of $K_{a,b}$ with $(1 + o(1))3m/4$ colors, where $m - 2\sqrt{m} \leq a \leq m \leq b \leq m + 2\sqrt{m}$. Since weak $(C_4, 3)$-colorings are preserved under taking subgraphs, we have $r'(K_{a,a}, C_4, 3) \leq (1 + o(1))3m/4$ for some a with $m - 2\sqrt{m} \leq a \leq m$, by considering a copy of $K_{a,a} \subseteq K_{a,b}$. It remains to extend this to all n.

Given any n, choose m such that $m - 3\sqrt{m} \leq n \leq m - 2\sqrt{m}$ and $m \equiv 1, 3 \pmod 6$. Then certainly $n/m \sim 1$ as $n \to \infty$. Let a correspond to m as in the preceding paragraph. Since $r'(K_{n,n}, C_4, 3)$ is a nondecreasing function of n,

$$r'(K_{n,n}, C_4, 3) \leq r'(K_{a,a}, C_4, 3) \leq \frac{3m}{4}(1 + o(1)) \sim \frac{3n}{4}(1 + o(1)),$$

completing the proof. □

9. Chart of bounds on $r(K_{n,n}, K_{p,p}, q)$

In the charts below, "<< $f(n)$" means "$O(f(n))$", and ">> $g(n)$" means "$\Omega(g(n))$".

q	$r(K_{n,n}, C_4, q)$	$r(K_{n,n}, K_{3,3}, q)$
2	$\sqrt{n}(1+o(1))$ Thm. 2.3	$n^{1/3}(1+o(1))$
3	$> \lfloor (2/3)n \rfloor$ Thm. 8.1, $\leq n+1$ Thm. 8.2	$<< n^{4/7}$ Thm. 3.2
4	n^2	$<< n^{2/3}$ Thm. 3.2
5	—	$<< n^{4/5}$ Thm. 3.2
6	—	$> n/4$, $< cn$ Cor. 4.2
7	—	$>> n$, $<< n^{4/3}$ Thm. 3.2
8	—	$\lceil \frac{n}{2} \lceil \frac{3n}{2} \rceil \rceil$ Thm. 7.1
9	—	n^2

q	$r(K_{n,n}, K_{p,p}, q)$
2	$>> n^{1/p}$ (3)
p^2-2p+2	$<< n^{1-1/(2p-1)}$ Cor. 4.2
p^2-2p+3	$\Theta(n)$ Cor. 4.2
p^2-p+1	$<< n^{2-2/p}$ Thm. 4.3
p^2-p+2	$\geq C_p(n^2-n)$, $< (1-c_p)n^2$ Thm. 4.3, (11)
$p^2 - \lfloor (2p-1)/3 \rfloor$	$< n^2 - c_p n^{1+\varepsilon}$ Thm. 6.1
$p^2 - \lfloor (2p-1)/3 \rfloor + 1$	$> n^2 - 2\lfloor (p-2)/3 \rfloor (n-1)$ Thm. 6.1
$p^2 - \lfloor p/2 \rfloor$	$\leq n^2 - \lfloor n/2 \rfloor$, $= n^2 - \lfloor n/2 \rfloor$ if p odd and ≥ 7 Thm. 5.1
$p^2 - \lfloor p/2 \rfloor + 1$	$n^2 - \lfloor p/2 \rfloor + 1$ Thm. 5.1
p^2	n^2

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Department of Mathematics, University of Illinois, Urbana, IL 61801

E-mail address: axenovic@math.uiuc.edu, z-furedi@math.uiuc.edu

Mathematical Institute of Hungarian Academy, POB 127 Budapest 1364, Hungary

E-mail address: furedi@math-inst.hu

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160

E-mail address: mubayi@math.gatech.edu