

EMBEDDING OF GRAPHS IN TWO-IRREGULAR GRAPHS

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ABSTRACT. A graph G on n vertices is called *two-irregular* if there are at most two vertices having the same degree for all possible degrees. We show that every graph with maximal degree at most $n/8 - O(n^{3/4})$ can be embedded into a two-irregular graph. We obtain it as a corollary of an algorithmic proof of a result about packing the graphs. This improves the bound of $O(n^{1/4})$ given by Faudree et al.

1. INTRODUCTION

We say that the graphs G_1, G_2, \dots, G_k on the same set of vertices V can be *packed* if there are permutations of vertices $\sigma_1, \sigma_2, \dots, \sigma_k$ such that for all $u, v \in V$ and for all $i = 1, \dots, k$ $\sigma_i(u)\sigma_i(v) \in E(G_i)$ then $\sigma_j(u)\sigma_j(v) \notin E(G_j)$ for all $j \neq i$. We say that graph G can be embedded in graph H on the same set of vertices V and write $G \subseteq H$, if there is a permutation of vertices σ such that for all $u, v \in V$ if $uv \in E(G)$ then $\sigma(u)\sigma(v) \in E(H)$. Sauer and Spencer [4] showed that if product of maximal degrees of two graphs is at most half the order of the graphs, then these two graphs can be packed. Wozniak [6] proved that 3 copies of every graph of order n and size at most $n - 2$ can be packed unless this graph is isomorphic to two special graphs, Caro and Yuster [?] solved the more general problem of packing several copies of a graph knowing average degree. There are many other related results on packing of several graphs or packing special classes of graphs, most of them are formulated in terms of the number of edges or the maximal degree. We obtain a new type of criterion for graphs to be packed (or embedded).

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We claim that if the maximal degree of a graph is not “too large” then it could be embedded in a graph with a certain prescribed degree sequence. One of the special classes of graphs we consider are *two-irregular graphs*, i.e., the graphs having at most two vertices of the same degree. The problem of embedding a graph into two-irregular graph has important applications. For instance, since the vertices of two-irregular graph can be recognized by their degree (up to at most one other vertex), then the vertices of a corresponding graph embedded into two-irregular graph could be recognized as well. This question was studied in the paper of Faudree et al. [2]. They show that every graph with maximal degree at most $O(n^{1/4})$ is embeddable into two-irregular graph of the same order. We improve their bound to $n/8 - O(n^{3/4})$. Moreover, we show that if a graph G has maximal degree at most $n/12$ and a graph H satisfies some special conditions, then for any permutation of vertices of G there is a graph F with degree sequence of H , such that G and F are packable with corresponding vertex order.

We define the graph $B_n = (V, E)$, a *half graph*, as follows: $V = A \cup B$, $A = \{a_1, \dots, a_{\lfloor n/2 \rfloor}\}$, $B = \{b_1, \dots, b_{\lfloor n/2 \rfloor}\}$, $E = \{\{a_i, a_j\} : i, j \in \{1, \dots, \lfloor n/2 \rfloor\}\} \cup \{\{b_i, b_j\} : i, j \in \{1, \dots, \lfloor n/2 \rfloor\}\} \cup \{\{a_i, b_j\} : i \leq j, i \in \{1, \dots, \lfloor n/2 \rfloor\}, j \in \{1, \dots, \lfloor n/2 \rfloor\}\}$. This graph has degrees $d(a_i) = n - i, i \in \{1, \dots, \lfloor n/2 \rfloor\}$, $d(b_j) = \lfloor n/2 \rfloor - 1 + j, j \in \{1, \dots, \lfloor n/2 \rfloor\}$. Thus B_n is a two-irregular graph.

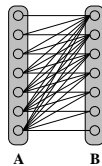


FIGURE 1. A half-graph on 14 vertices, A and B induce complete graphs

Given an underlying graph H and another graph H' on the same vertex set let the *excessive degree* of $x \in V$ with respect to H , $d_{\text{ex}}(H', x)$, be $d(H', x) - d(H, x)$. Let $\text{EX}(H') = \{x \in V : d_{\text{ex}}(H', x) > 0\}$ be the set of vertices with positive excessive degrees. If $A \subseteq V$ then $\sum_{v \in A} d_{\text{ex}}(H', v)$ is called a *total excess* of H' in A , or, if $A = V$ simply *total excess*. The graphs G and H have the *same degrees* if $V(G) = V(H)$ and $d(G, x) = d(H, x)$ for every $x \in V$. They have the *same degree sequence* if there is a bijection $f : V(G) \rightarrow V(H)$ such that $d(G, x) = d(H, f(x))$ for all $x \in V$.

2. EMBEDDING OF GRAPHS WITH BOUNDED MAXIMAL DEGREE

Let H be a complement of a bipartite graph on n vertices with almost equal parts A and B such that $A = \{a_1, \dots, a_{\lceil n/2 \rceil}\}$, $B = \{b_1, \dots, b_{\lfloor n/2 \rfloor}\}$ and for any $I \subset \{1, \dots, \lceil n/2 \rceil\}$ of size at least $n/8$, the set of vertices $\{a_i, b_i : i \in I\}$ induces at most $|I|^2/2$ edges in $H[A, B]$,

$$(1) \quad |E(H[A_I, B_I])| \leq |I|^2/2.$$

We call such graph a *generalized half-graph*.

Theorem 1. *Let G be a simple graph on n vertices with maximal degree $d \leq n/8 - O(n^{3/4})$ and let H be a generalized half-graph. Then G can be embedded into a graph F with the same degree sequence as H .*

Corollary 2. *If G is a simple graph on n vertices with maximal degree $d \leq n/8 - O(n^{3/4})$, then G is embeddable into a two-irregular graph having the same degree sequence as a half-graph B_n .*

Theorem 3. *Let H be generalized half-graph and G be the graph on the same vertex set as H having the maximal degree $d \leq n/12$. Then there is a graph F having the same degrees as H such that $G \subseteq F$.*

Moreover, for the Theorem 3 we give the algorithm of constructing a graph F with the required properties.

3. PROOF OF THE MAIN RESULT

Proof of Theorem 1. Let $d_0 = \frac{1}{2}d + 6\sqrt{n}$. By a result of Spencer [?] there exists a partition of $V(G) = A' \cup B'$, $A' = \{a'_1, \dots, a'_{\lceil n/2 \rceil}\}$, $B' = \{b'_1, \dots, b'_{\lfloor n/2 \rfloor}\}$ such that

$$(2) \quad |N(G, x) \cap A'| \leq d_0 \text{ and } |N(G, x) \cap B'| \leq d_0 \text{ for all } x \in V.$$

Place G into the vertex set of H by identifying a_i with a'_i and b_i with b'_i thus obtaining $G \cup H$.

Define a class of graphs, \mathbf{H}_1 , such that every member $Q \in \mathbf{H}_1$ has the same vertex set $V(Q) = V(H)$ and satisfies the following conditions:

- C1. Q contains G and $H[A, B]$.
- C2. Q is a subgraph of $G \cup H$.

C3. Q is obtained by deleting the same number of edges of $H \setminus G$ in A and in B , so it has the same excess with respect to H on A and B , i.e.,

$$\sum_{x \in A} d_{\text{ex}}(Q, x) = \sum_{x \in B} d_{\text{ex}}(Q, x).$$

C4. Q majorises the degrees of H ,

$$d_{\text{ex}}(Q, x) \geq 0 \text{ for every } x \in V.$$

Notice that \mathbf{H}_1 is not empty since $G \cup H$ satisfies C1–C4. Hence

$$\sum_{x \in V} d_{\text{ex}}(Q, x) \leq \sum_{x \in V} d_{\text{ex}}(G \cup H, x)$$

for every member Q of \mathbf{H}_1 . Let $H_1 \in \mathbf{H}_1$ minimize the total excess.

$$(3) \quad \sum_{x \in V} d_{\text{ex}}(H_1, x) \leq \sum_{x \in V} d_{\text{ex}}(Q, x) \quad \text{for all } Q \in \mathbf{H}_1.$$

We note that a minimal element of \mathbf{H}_1 (with respect to simultaneous edge deletion) is sufficient for our purposes. Finding such a minimal member of \mathbf{H}_1 is easy from algorithmic point of view, see later in Section 5.

Claim 1. The edges of G together with the deleted edges (i.e., the pairs from $H \setminus H_1$) form a complete graph on the vertices of positive excessive degrees either in A or in B . That is, one of the following holds:

- a) $xy \in E(G) \cup E(\overline{H}_1)$ for all $x, y \in \text{EX}(H_1) \cap A$,
- b) $xy \in E(G) \cup E(\overline{H}_1)$ for all $x, y \in \text{EX}(H_1) \cap B$.

Proof of Claim 1. Suppose that there are $a, a' \in \text{EX}(H_1) \cap A$ and $b, b' \in \text{EX}(H_1) \cap B$ such that $aa', bb' \notin E(G)$ and $aa', bb' \in E(H_1)$. Then by deleting the edges aa' and bb' from H_1 we obtain another member of \mathbf{H}_1 with smaller total excess (3), a contradiction. \square

From now on, we assume that a) holds. The case b) can be handled in the same way.

Claim 2. The number of vertices with positive excessive degree in A , $|\text{EX}(H_1) \cap A|$ is at most d . The total excess of H_1 in A is at most $(d+1)^2/4$

$$\sum_{x \in A} d_{\text{ex}}(H_1, x) \leq (d+1)^2/4.$$

Proof of Claim 2. It follows from C1 and C2 that H_1 is obtained from $G \cup H$ by deleting some edges of H in parts A and B which are not edges of G . In particular, $H_1[A, B] = (G \cup H)[A, B]$. Thus we could express the excessive degree of $x \in A \cap \text{EX}(H_1)$ in H_1 as the number of $G \setminus H$ edges joining x to B minus the number of deleted edges starting from x . We have

$$\begin{aligned} d_{\text{ex}}(H_1, x) &= d(H_1, x) - d(H, x) = \\ &= d(H_1[A], x) + d(H_1[A, B], x) - d(H[A], x) - d(H[A, B], x) = \\ &= -d(\overline{H}_1[A], x) + d((G \setminus H)[A, B], x) \leq \\ &= -d(\overline{H}_1[A], x) + d(G[A, B], x). \end{aligned}$$

From this we conclude that

$$d(\overline{H}_1[A], x) \leq d(G[A, B], x) - d_{\text{ex}}(H_1, x).$$

Adding $d(G[A], x)$ to both sides gives

$$(4) \quad d((G \cup \overline{H}_1)[A], x) \leq d(G, x) - d_{\text{ex}}(H_1, x).$$

Let us note that one can prove in the same way that for all $x \in B$

$$(5) \quad d((G \cup \overline{H}_1)[B], x) \leq d(G, x) - d_{\text{ex}}(H_1, x).$$

Let q be the maximal excessive degree in A ,

$$(6) \quad q = \max\{d_{\text{ex}}(H_1, x) : x \in \text{EX}(H_1) \cap A\}.$$

Claim 1 a) implies that x is joined to all vertices of $\text{EX}(H_1) \cap A$ by edges of $G \cup \overline{H}_1$, thus, together with (4) we have

$$(7) \quad |\text{EX}(H_1) \cap A| \leq 1 + d((G \cup \overline{H}_1)[A], x) \leq 1 + d - q.$$

Therefore

$$\sum_{x \in A} d_{\text{ex}}(H_1, x) = \sum_{x \in \text{EX}(H_1) \cap A} d_{\text{ex}}(H_1, x) \leq (d - q + 1)q \leq (d + 1)^2/4.$$

If $q = 0$, then there is nothing left to prove anymore ($\text{EX}(H_2) = \emptyset$), and for $q \geq 1$ the first half of the Claim follows from (7). \square

Now let's define the class of graphs \mathbf{H}_2 . Each member of $Q \in \mathbf{H}_2$ is constructed from H_1 by adding some edges between A and B and deleting the same number of edges in each of the parts A and B such that Q satisfies the conditions D1–D6 below.

- D1. $Q[A]$ and $Q[B]$ are subgraphs of $H_1[A]$ and $H_1[B]$ respectively and Q contains G and $H[A, B]$. (Recall that $(G \cup H)[A, B] = H_1[A, B]$.)
- D2. $d(H, x) \leq d(Q, x) \leq d(H_1, x)$, so
 $0 \leq d_{\text{ex}}(Q, x) \leq d_{\text{ex}}(H_1, x)$ for every $x \in V$.
- D3. $|E((H_1 \setminus Q)[A])| = |E((H_1 \setminus Q)[B])| = |E((Q \setminus H_1)[A, B])|$,
so $\sum_{x \in A} d_{\text{ex}}(Q, x) = \sum_{x \in B} d_{\text{ex}}(Q, x)$.
- D4. There are no new edges in Q incident to the vertices of positive excessive degree in A ,
 $Q[(\text{EX}(H_1) \cap A), B] = H_1[(\text{EX}(H_1) \cap A), B]$.
- D5. Define $d_1 = \lfloor d/2 \rfloor$. There are at most d_1 new edges at every vertex of B ,
 $d(Q \setminus H_1, x) \leq d_1$ for every $x \in B$.
- D6. Total excess of $Q[A]$ is the total excess of $H_1[A]$ minus the number of added (i.e., $Q \setminus H_1$) edges, hence
 $\sum_{x \in A} d(Q \setminus H_1, x) = \sum_{x \in A} (d_{\text{ex}}(H_1, x) - d_{\text{ex}}(Q, x))$.

Let $H_2 \in \mathbf{H}_2$ minimize the total excessive degree

$$\sum_{x \in V} d_{\text{ex}}(Q, x) \geq \sum_{x \in V} d_{\text{ex}}(H_2, x) \quad \text{for all } Q \in \mathbf{H}_2.$$

Claim 3. Total excessive degree of H_2 is zero, $\sum_{x \in V} d_{\text{ex}}(H_2, x) = 0$.

Proof of Claim 3. Suppose that there are two vertices $a \in A$ and $b \in B$ with positive excessive degrees. We shall find $a' \in A$ and $b' \in B$ such that deleting aa' and bb' (from $H_2 \setminus G$) and adding $a'b'$ (to $\overline{H_2}$) produces another member of \mathbf{H}_2 with smaller total excess, thus deriving a contradiction. We have $\text{EX}(H_2) \subseteq \text{EX}(H_1)$, so $a \in \text{EX}(H_1) \cap A$.

Define A' as a set of vertices $a' \in A$ with zero excessive degree such that aa' could be deleted,

$$A' = \{a' : aa' \in (E(H_2) \setminus G), a' \in A, a' \notin \text{EX}(H_2)\}.$$

The last condition for A' ($a' \notin \text{EX}(H_2)$) is implied by the first, since Claim 1 gives $\text{EX}(H_1) \cap A \subseteq \{a\} \cup N(G \cup \overline{H_1}, a)$. We shall show that

$$(8) \quad |A'| \geq \lceil n/2 \rceil - d - 1.$$

We have

$$A \setminus A' = \{a\} \cup N((G \cup \overline{H}_1)[A], a) \cup N((\overline{H}_2 \setminus \overline{H}_1)[A], a).$$

Obviously, $d((\overline{H}_2 \setminus \overline{H}_1)[A], a) = d(H_1 \setminus H_2, a)$. Then D4 implies that this is equal to $d_{\text{ex}}(H_1, a) - d_{\text{ex}}(H_2, a)$. Now by D2 we have

$$(9) \quad d((\overline{H}_2 \setminus \overline{H}_1)[A], a) \leq d_{\text{ex}}(H_1, a).$$

Adding (4) (with $x = a$) to (9) we have $d((G \cup \overline{H}_2)[A], a) \leq d$, implying $|A \setminus A'| \leq d + 1$ and (8) follows.

Define B' as a set of vertices $b' \in B$ satisfying *strict* inequality in D5 such that bb' can be deleted,

$$B' = \{b' : bb' \in (E(H_2) \setminus G), b' \in B, d(H_2 \setminus H_1, b') < d_1\}.$$

Let t be the number of vertices for which the equality of D5 is achieved,

$$(10) \quad t = |\{y' \in B : d(H_2 \setminus H_1, y') = d_1\}| \leq \frac{\sum_{x \in B} d_{\text{ex}}(H_1, x)}{d_1} \leq \frac{d}{2}.$$

We shall show that

$$(11) \quad |B'| \geq \lfloor n/2 \rfloor - d - d_1 - t - 1.$$

We have

$$\begin{aligned} B \setminus B' &= \{b\} \cup N((G \cup \overline{H}_1)[B], b) \cup N((\overline{H}_2 \setminus \overline{H}_1)[B], b) \\ &\quad \cup \{b' \in B : d(H_2 \setminus H_1, b') = d_1\}. \end{aligned}$$

As before we have $d((\overline{H}_2 \setminus \overline{H}_1)[B], b) = d(H_1 \setminus H_2, b)$. This is equal to $d_{\text{ex}}(H_1, b) - d_{\text{ex}}(H_2, b) + d((H_2 \setminus H_1)[B, A], b)$. Thus, D2 and D5 imply that this is at most

$$d((\overline{H}_2 \setminus \overline{H}_1)[B], b) \leq d_{\text{ex}}(H_1, b) + d_1.$$

Adding (5) to this inequality (with $x = b$) we obtain

$$d((G \cup \overline{H}_2)[B], b) \leq d + d_1.$$

Together with (10) we have

$$d((G \cup \overline{H}_2)[B], b) + |\{b' \in B : d(H_2 \setminus H_1, b') = d_1\}| \leq d + d_1 + t,$$

and (11) follows.

Define $I = \{i : a_i \in A', b_i \in B'\}$, $A_I = \{a_i : i \in I\}$, $B_I = \{b_i : i \in I\}$. The inequalities (8) and (11) give

$$(12) \quad |I| \geq |A'| + |B'| - \lfloor n/2 \rfloor \geq \lfloor n/2 \rfloor - 2d - d_1 - t - 2.$$

Notice that $|I| \geq n/8$. Now let's estimate the number of edges of H_2 between A_I and B_I . There are three types of edges – edges of H , edges of G and edges of $H_2 \setminus (G \cup H)$.

Using the density property (1) of H we have

$$(13) \quad |E(H[A_I, B_I])| \leq |I|^2/2.$$

Using (2) we have

$$(14) \quad |E(G[A_I, B_I])| \leq |I|d_0.$$

Concerning the $H_2 \setminus (G \cup H)$ edges between A and B , their number is at most $(d+1)^2/4$ by D2 and Claim 2. By definition, at least $d_1 t$ of these join A to $B \setminus B'$, so we have

$$(15) \quad |E((H_2 \setminus (G \cup H))[A_I, B_I])| \leq (d+1)^2/4 - d_1 t.$$

Now, summing up (13), (14) and (15) and applying (12), an easy calculation leads to

$$|E(H_2[A_I, B_I])| < |I|^2.$$

Thus, for $d \leq n/8 - O(n^{3/4})$ there exists a non-edge $a'b' \notin E(H_2)$, $a' \in A_I \subseteq A'$, $b' \in B_I \subseteq B'$. This concludes the proof of Claim 3. \square

Now we can take H_2 as the graph F in Theorem 1. \square

For the proof of Theorem 3 we mimic the proof of Theorem 1 starting with an arbitrary placement of the graph G and letting the constant d_0 be equal to d .

4. ALGORITHM

To construct a graph F for Theorem 3 we apply a two-step augmentation algorithm.
Step 1. We build graphs $\{H_i^*\}$ on the same set of vertices as H as follows. Let $H_0^* = G \cup H$. Suppose that H_i^* has been built. We say that H_i^* satisfies the condition of Step 1. if there is an edge aa' in A and edge bb' in B such that a, a', b, b' have positive excessive degrees in H_i^* and aa', bb' are not the edges of G . In this case, delete edges aa', bb' and set $E(H_{i+1}^*) = E(H_i^*) \setminus (\{aa'\} \cup \{bb'\})$. If H_{i+1}^* satisfies the condition of Step 1., repeat the above procedure. Assume that the last graph obtained is H_k^* . It is easy to see that this graph H_k^* satisfies all the properties of graph H_1 in the Theorem 3. Therefore Claim 1 (say a)) and Claim 2 hold.

Step 2. We build graphs $\{F_i^*\}$ on the same set of vertices as following. Let $F_0^* = H_k^*$. Suppose that F_i^* is built. We say that F_i^* satisfies the condition of Step 2. if if there is $a \in A$ and $b \in B$ with positive excessive degrees and there are $a' \in A$, $b' \in B$ such that aa' and bb' are the edges of F_i^* but not the edges of G and $a'b'$ is not the edge of F_i^* . In this case, delete aa' and bb' and add $a'b'$, i.e., set $E(F_{i+1}^*) = E(F_i^*) \setminus (\{aa'\} \cup \{bb'\}) \cup a'b'$. If F_{i+1}^* satisfies the condition of Step 2. repeat the above procedure. Denote the last graph obtained by F_l^* .

We can easily verify that F_l^* satisfies D1–D6, thus we obtained a desired graph F for Theorem 3.

Since each step of augmentation algorithm corresponds to a decrease of a total excess of G with respect to H , the number of steps in the algorithm is at most the total excess, which, in its turn, at most the number of edges in G . Therefore the algorithm runs in at most a quadratic time as a function of number of vertices.

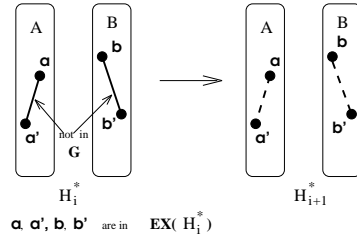


FIGURE 2. First step of augmentation algorithm

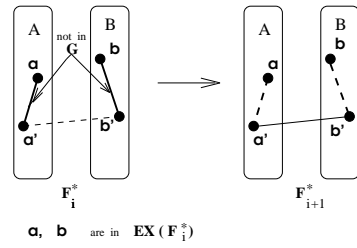


FIGURE 3. Second step of augmentation algorithm

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