A note on Ramsey numbers for Berge-G hypergraphs

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Abstract

For a graph $G = (V, E)$, a hypergraph H is called *Berge-G* if there is a bijection $\phi : E(G) \to E(H)$ such that for each $e \in E(G)$, $e \subseteq \phi(e)$. The set of all Berge-G hypergraphs is denoted $\mathcal{B}(G)$.

For integers $k \geq 2$, $r \geq 2$, and a graph G, let the Ramsey number $R_r(\mathcal{B}(G), k)$ be the smallest integer n such that no matter how the edges of a complete r-uniform n-vertex hypergraph are colored with k colors, there is a copy of a monochromatic Berge-G subhypergraph. Furthermore, let $R(\mathcal{B}(G), k)$ be the smallest integer n such that no matter how all subsets an n-element set are colored with k colors, there is a monochromatic copy of a Berge-G hypergraph.

We give an upper bound for $R_r(\mathcal{B}(G), k)$ in terms of graph Ramsey numbers. In particular, we prove that when G becomes acyclic after removing some vertex, $R_r(\mathcal{B}(G), k) \leq 4k|V(G)| + r - 2$, in contrast with classical multicolor Ramsey numbers.

When G is a triangle (or a K_4), we find sharper bounds and some exact results and determine some "small" Ramsey numbers:

- $k/2 - o(k) \leq R_3(\mathcal{B}(K_3)), k) \leq 3k/4 + o(k)$,
- For any odd integer t, $R(\mathcal{B}(K_3), 2^t) = t + 2$,
- $2^{ck} \leq R_3(\mathcal{B}(K_4), k) \leq e(1 + o(1))(k - 1)k!$,
- $R_3(\mathcal{B}(K_3), 2) = R_3(\mathcal{B}(K_3), 3) = 5, R_3(\mathcal{B}(K_3), 4) = 6, R_3(\mathcal{B}(K_3), 5) = 7, R_3(\mathcal{B}(K_3), 6) = 8, R_3(\mathcal{B}(K_3, 8) = 9, R_3(\mathcal{B}(K_4), 2) = 6$.

1 Introduction

For a graph G, a family $\mathcal{B}(G)$ consists of hypergraphs H each with $|E(G)|$ distinct hyperedges so that for each $xy \in E(G)$, there is a hyperedge e_{xy} of H such that $\phi(x), \phi(y) \in e_{xy}$ and $e_{xy} \neq e_{x'y'}$ if $xy \neq x'y'$, for an injective map $\phi : V(G) \to V(H)$. Here, we shall always denote the vertex set of F as $V(F)$ and the edge set of F as $E(F)$, for a graph or a hypergraph F. A copy of a graph F in a graph G is a subgraph of G isomorphic to F. When clear from context, we shall drop the word "copy" and just say that there is F in G.

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The members of $\mathcal{B}(G)$ are called *Berge-G hypergraphs*. We call a copy G' of G, where $G' = (\phi(V(G)), \{\phi(x)\phi(y) : xy \in E(G)\})$, the *underlying graph* of the Berge-G hypergraph.

The Ramsey number $R_r(\mathcal{F}, k)$ is the smallest integer n such that no matter how the edges of a complete r-uniform n-vertex hypergraph, that we denote K_n^r, are colored with k colors, there is a monochromatic subhypergraph from $\mathcal{F}$. We always assume that $k \geq 2$, $r \geq 2$, and simply write K_n for K_n^2. The classical k-color Ramsey number for a graph G (uniformity 2) is denoted $R(G, k)$, i.e., $R(G, k) = R_2(\{G\}, k)$. When we do not restrict our attention to uniform hypergraphs, we define the Ramsey number $R(\mathcal{B}(G), k)$ to be the smallest integer n such that no matter how all subsets an n-element set are colored with k colors, there is a monochromatic copy of a Berge-G hypergraph. It is convenient to define dual functions: $f(n, \mathcal{B}(G))$, the smallest number of colors in a coloring of $2^{[n]}$ such that there is no monochromatic Berge-G hypergraph and $f_r(n, \mathcal{B}(G))$, the smallest number of colors in a coloring of $\binom{[n]}{r}$ such that there is no monochromatic Berge-G hypergraph.

For $G = C_t$, the cycle on t vertices, and $r = 3$, this problem has been already investigated. It was proved by Gyárfás, Lehel, Sárközy, and Schelp [8] that $R_3(\mathcal{B}(C_t), 2) = t$ for $t \geq 5$. The fact that $R_3(\mathcal{B}(C_t), 3) \sim \frac{5t}{4}$ was the main result of the paper by Gyárfás and Sárközy, [9].

The Ramsey problem is closely related to Turán problems. For a family $\mathcal{F}$ of hypergraphs, the Turán number $\mathrm{ex}_r(n, \mathcal{F})$ is the largest number of edges in an r-uniform hypergraph on n vertices that does not contain any member from the family $\mathcal{F}$ as a subhypergraph. Indeed, let $N = R_r(\mathcal{F}, k) - 1$. Since in a coloring of K_N^r in k colors with no monochromatic member of $\mathcal{F}$ each color class has at most $\mathrm{ex}_r(N, \mathcal{F})$ edges, we have

$$\binom{N}{r} \leq k \cdot \mathrm{ex}_r(N, \mathcal{F}). \tag{1}$$

Note that this inequality gives easy upper bounds on N if $\mathrm{ex}_r(N, \mathcal{F}) = o(N^r)$. Győri [10] proved that $\mathrm{ex}_3(n, \mathcal{B}(K_3)) \leq n^2/8$ and the bound is tight. The non-uniform and multi-hypergraph case was addressed in that paper too. Other results on extremal numbers for Berge hypergraphs were provided by Gerbner and Palmer [5], Gerbner, Methuku, and Vizer, [3], Gerbner, Methuku, and Palmer, [4], as well as by Palmer, Tait, Timmons, and Wagner [12], and by Grósz, Methuku, and Tompkins [7].

We provide bounds on uniform and non-uniform Ramsey numbers for Berge-K_3 hypergraphs, including several exact results. We also give results for Ramsey numbers of Berge hypergraphs for general graphs.

Theorem 1. *Let $\mathcal{F} = \mathcal{B}(K_3)$. Then for $n \geq 3$*

1. $k/2 - o(k) \leq R_3(\mathcal{F}), k) \leq 3k/4 + o(k)$,
2. $R_3(\mathcal{F}, 2) = R_3(\mathcal{F}, 3) = 5$, $R_3(\mathcal{F}, 4) = 6$, $R_3(\mathcal{F}, 5) = 7$, $R_3(\mathcal{F}, 6) = 8$, $R_3(\mathcal{F}, 8) = 9$,
3. $2^{n-2}(1 - o(1)) \leq f(n, \mathcal{F}) \leq 2^{n-2}$,

4. $f(n, \mathcal{F})) = 2^{n-2}$, *for odd n.*

We make a connection between r-uniform Ramsey numbers of Berge-G hypergraphs, $r \geq 3$, and multicolor Ramsey numbers for auxiliary families of graphs. For a graph $G = (V, E)$ and $v \in V$, $\mathcal{G}^*(v)$ is defined as the class of all graphs obtained from G by the following procedure. Let $N(v) = \{q_1, \ldots, q_t\}$ denote the set of vertices adjacent to v in G. Let $G' = G - v$ be the graph obtained from G by deleting v and the edges of G incident to v. Then, for every $q_i \in N(v)$ add a *new edge* $q_i r_i$ (not in G') where r_i can be any vertex of G' or any new vertex (not in G'). These $|N(v)|$ new edges could be pendant, forming a matching, or could share endpoints. Thus $\mathcal{G}^*(v)$ includes G and many other graphs. The graph obtained this way is denoted by $G''(v; q_1r_1, \ldots, q_tr_t)$ and is called an *extension* of $G - v$. For example, if $G = K_4 - e$, the graph obtained from K_4 by deleting an edge, and v is a vertex of degree three in it, $\mathcal{G}^*(v)$ consists of G and four other graphs. When $G = K_4$, $\mathcal{G}^*(v)$ consists of G, $K_4 - e$ with a pendant edge incident to a vertex of degree 2 of $K_4 - e$, and a triangle with three pendant edges. See Figure 1.

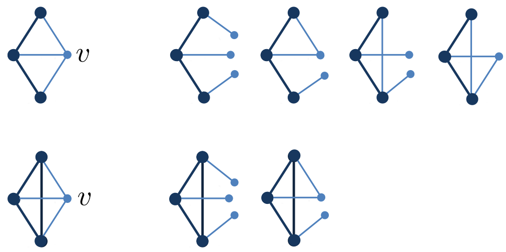

Figure 1: A family $\mathcal{G}^*(v)$ for $G = K_4 - e$ and for $G = K_4$

Theorem 2. *For any graph G, $v \in V(G)$, and any integer $r \geq 3$, we have $R_r(\mathcal{B}(G), k) \leq R(\mathcal{G}^*(v), k) + r - 2$.*

Proof. Consider a k-coloring c of $K = K_n^r$ on a vertex set V, $|V| = n \geq R(G^*(v), k) + r - 2$. Fix an arbitrary $(r-2)$-element subset $X \subset V$ and k-color the edges of the complete graph K' with vertex set $V \setminus X$ by the rule $c(xy) = c(X \cup \{x, y\})$. From the choice of n, we have a monochromatic, say red copy $G'' = G''(v; q_1r_1, \ldots, q_tr_t)$ of a member of $\mathcal{G}^*(v)$ in K'. We claim that K contains a red member of $\mathcal{B}(G)$.

To prove the claim, let $x \in X$ and define the graph F as follows. Let

$$V(F) = V(G'') \cup \{x\}, E(F) = (E(G'') \cup_{i=1}^{t} xq_i) \setminus (\cup_{i=1}^{t} q_i r_i).$$

We show that the edges of F can be covered by distinct red edges of K. Indeed, $xq_i \in (E(F) \setminus E(G''))$ is covered by $\{q_i, r_i\} \cup X$ and any edge $vw \in (E(F) \cap E(G''))$ is covered by $\{v, w\} \cup X$ and these sets are red edges of K. Thus the graph obtained from F upon removing those r_i-s that are not in $V(G')$ (they are isolated vertices of F) is isomorphic to G and its edges are covered by distinct red edges of K. This proves the claim and Theorem 2. □

Since $G \in \mathcal{G}^*(v)$ for every v, Theorem 2 implies

Corollary 3. *$R_r(\mathcal{B}(G), k) \leq R(G, k) + r - 2$.*

However, a much stronger bound follows from Theorem 2.

Corollary 4. *Set $\mathcal{G} = \cup_{v \in V(G)} \mathcal{G}^*(v)$. Then $R_r(\mathcal{B}(G), k) \leq R(\mathcal{G}, k) + r - 2$.*

Corollary 5. *If a graph G can be made acyclic by the removal of a vertex, then $R_r(\mathcal{B}(G), k) \leq 4k|V(G)| + r - 2$ for every $r \geq 3$.*

Proof. If $G - v$ is acyclic, $G' = G - v$ has an acyclic extension G'' obtained by adding a matching $q_i r_i$ from $N(v)$ to new vertices. Clearly $|V(G'')| < 2|V(G)|$ and since G'' is acyclic and $R(G'', k) \leq 2k|V(G'')|$ (see e.g. [11]), Corollary 5 follows from Theorem 2. □

For the non-uniform case we have the following bounds.

Theorem 6. *Let G be a graph with at least two edges. If $G \neq 2K_2$, then*

1.
$$\frac{2^{n-|V(G)|}}{|E(G)| - 1} \leq f(n, \mathcal{B}(G)) \leq 2^{n-1}.$$

2. *In addition, if G has maximum degree at most 2, then*
$$\frac{2^{n-1}}{|E(G)| - 1}(1 - o(1)) \leq f(n, \mathcal{B}(G)).$$

3.
$$f(n, \mathcal{B}(2K_2)) = 2^n - \binom{n}{2} - n - 1.$$

Moreover, we have some results for Ramsey numbers of $\mathcal{B}(K_4)$. Set $K_4^* = K_4^*(v)$, for a vertex v in K_4.

Theorem 7. *We have that for a positive constant c*

1. $2^{ck} \leq R_3(\mathcal{B}(K_4), k) \leq R(K_4^*, k) + 1 \leq e(1 + o(1))(k - 1)k!$,
2. $R_3(\mathcal{B}(K_4), 2) = 6$.

Note that part 1 in Theorem 7 shows that the upper bound of the multicolor Ramsey number for the family $K_4^*(v)$ differs from the best known upper bound of $R(K_3, k)$ only by a factor linear in k. It is also worth mentioning that part 2. in Theorem 7 shows that $R_3(\mathcal{B}(K_4), 2)$ is much smaller than its classical counterpart, $R_3(K_4^3, 2) = 13$, [13].

The rest of the paper is structured as follows. In Section 2 we treat the non-uniform case proving parts 3 and 4 of Theorem 1 and Theorem 6. In Section 3 we prove the remaining parts 1 and 2 of Theorem 1. Finally, in Section 4 we prove Theorem 7.

2 The non-uniform case

Proof of Theorem 6/1 - upper bound. If G is not a $2K_2$, make each color class consisting of a set and its complement. This gives a general upper bound.

Proof of Theorem 1/3 - upper bound. For the upper bound on $f(n, \mathcal{B}(K_3))$, consider the coloring of $2^{[n]}$ such that each color class consists of four sets: A, $[n-1]-A$, $[n]-A$, and $A \cup \{n\}$ for $A \subseteq [n-1]$. Then the total number of colors is 2^{n-2}. The four sets of each color class do not contain Berge-K_3.

Proof of Theorem 6/1 - lower bound. To prove the bound $f(n, \mathcal{B}(G)) \geq \frac{2^{n-|V(G)|}}{|E(G)|-1}$, consider a set S of $|V(G)|$ vertices and the set $\mathcal{S}$ of all subsets containing S. Note that any $|E(G)|$ sets from $\mathcal{S}$ form a Berge-G hypergraph. Thus there are at most $|E(G)|-1$ members of $\mathcal{S}$ of each color. Therefore the total number of colors is at least the number of colors used on $\mathcal{S}$, that in turn is at least

$$\frac{|\mathcal{S}|}{|E(G)|-1} = \frac{2^{n-|S|}}{|E(G)|-1} = \frac{2^{n-|V(G)|}}{|E(G)|-1}.$$

Proof of Theorem 6/2. Let G be a graph with maximum degree at most 2. Consider $\mathcal{X}$, the set of all subsets of $[n]$ of size at least $\frac{n+|V(G)|}{2}$. Then any two sets from $\mathcal{X}$ intersect in at least $|V(G)|$ elements. We claim that any $|E(G)|$ sets from $\mathcal{X}$ form a Berge-G graph. Assume that G is a cycle on k vertices, for other graphs of maximum degree at most 2, the argument is similar. Consider an arbitrary family $X_1, \dots, X_k$ of sets from $\mathcal{X}$. Pick vertices $x_1 \in X_1 \cap X_2$, $x_2 \in (X_2 \cap X_3) \setminus \{x_1\}$, and so on, $x_i \in (X_i \cap X_{i+1}) \setminus \{x_1, \dots, x_{i-1}\}$, $i = 2, \dots, k-1$. Finally pick $x_k \in (X_k \cap X_1) \setminus \{x_1, \dots, x_{k-1}\}$. Since pairwise intersections have size at least k, it is always possible to make such a choice of $x_1, \dots, x_k$. Then $\{x_1, \dots, x_k\}$ forms an underlying vertex set of a Berge-C_k.

Thus in a coloring of $\mathcal{X}$ with no monochromatic Berge-G, there are at most $|E(G)|-1$ sets of the same color. This implies that the number of colors is at least

$$\frac{|\mathcal{X}|}{|E(G)|-1} \geq \frac{2^{n-1} - c|V(G)|\frac{2^n}{\sqrt{n}}}{|E(G)|-1} \geq \frac{2^{n-1}}{|E(G)|-1} - o(2^n).$$

Proof of Theorem 1/3 - lower bound. The lower bound for Berge-K_3 follows from the previous proof.

Proof of Theorem 1/4. When n is odd, consider all sets of size at least $(n+1)/2$. Observe that any three of those form a Berge-K_3 hypergraph. Thus in any coloring of $2^{[n]}$ with no monochromatic copy of Berge-K_3 hypergraph, there are at most two subsets of size at least $(n+1)/2$ that have the same color. Thus the total number of colors in such a coloring is at least $|\{A : |A| \geq (n+1)/2, A \subseteq [n]\}|/2 = 2^{n-2}$. The upper bound follows from Theorem 6/1.

Proof of Theorem 6/3. To prove that $f(n, \mathcal{B}(2K_2)) = 2^n - \binom{n}{2} - n - 1$, note that in any coloring of subsets of size at least three in $[n]$ without monochromatic $\mathcal{B}(2K_2)$, all subsets must have distinct colors. Thus

$$f(n, \mathcal{B}(2K_2)) \geq |\{X : X \subseteq [n], |X| \geq 3\}| = 2^n - \binom{n}{2} - n - 1.$$

On the other hand, the following coloring has no monochromatic $\mathcal{B}(2K_2)$: color all sets of size at least 3 with distinct colors, color each set of at most two elements with the color of some 3-element set containing it. Then each color class of size at least two consists of subsets of some three element set, so it does not contain a copy of $\mathcal{B}(2K_2)$ and the number of colors is $2^n - \binom{n}{2} - n - 1$.

3 Ramsey number of the Berge triangle

In this subsection we set $\mathcal{F} = \mathcal{B}(K_3)$.

Proof of Theorem 1/1,2 - upper bounds. Consider a coloring of K_n^r, an r-uniform n-clique with k colors without monochromatic member of $\mathcal{F}$, $n = R_r(\mathcal{F}, k) - 1$. Then each color class has at most $\text{ex}_r(n, \mathcal{F})$ edges. A result of Győri [10] implies that $\text{ex}_r(n, \mathcal{F}) \leq \frac{n^2}{8(r-2)}$. Thus $\binom{n}{r}$, the total number of hyperedges, is at most $kn^2/(8(r-2))$. This provides the general upper bounds and all upper bounds for the Ramsey numbers for i colors, $i = 2, 3, 4, 5, 6, 7, 8$.

Proof of Theorem 1/1 - lower bound. We shall instead provide an upper bound on the number of colors needed to color the triples on n vertices so that no monochromatic Berge-K_3 is created. We split the vertex set in two almost equal parts, A and B. Let $A_1, A_2, \dots$ and $B_1, B_2 \dots$ be color classes of optimal proper edge-colorings of complete graphs (uniformity 2) on a vertex set A and on a vertex set B, respectively. Let $\mathcal{A}_i = \{\{x, y, z\} : \{x, y\} \in A_i, z \in B\}$, let $\mathcal{B}_i = \{\{x, y, z\} : \{x, y\} \in B_i, z \in A\}$. Then we see that each of $\mathcal{A}_i$'s and $\mathcal{B}_i$'s does not contain a member or $\mathcal{F}$. Moreover these classes of triples contain all hyperedges of K_n^3 with vertices in both A and B. Color all triples in $\mathcal{A}_i$ with color a_i, all tripes in $\mathcal{B}_i$ with color b_i, for distinct a_i's and b_i's. Further, color the triples with all elements in A or with all elements in B recursively using new colors such that the set of colors used on triples from A is the same as the set of colors used on triples from B. Let $f(n)$ be the number of colors given by this construction and $\chi'(G)$ denote the chromatic index of a graph G. Then

$$\begin{aligned} f(n) &\leq \chi'(K_{\lfloor n/2 \rfloor}) + \chi'(K_{\lceil n/2 \rceil}) + f(\lceil n/2 \rceil) \\ &\leq \lfloor n/2 \rfloor + \lceil n/2 \rceil + f(\lceil n/2 \rceil) \\ &= n + f(\lceil n/2 \rceil) \\ &\leq 2n + \log n. \end{aligned}$$

So, we have that the number of colors k is bounded as $k \leq 2n + \log n$, thus $n \geq k/2 - o(k)$.

Proof of Theorem 1/2 - lower bounds. The lower bound for 2 and 3 colors is obvious since two edges of K_4^3 can be colored red and the other two blue. An $\mathcal{F}$-free 4-coloring of K_5^3 on [5] can be given by splitting the edge set into color classes as follows:

$$123, 124, 125;\ 134, 234, 345;\ 135, 145;\ 235, 245.$$

Note that for each color class there is a pair of vertices that belongs to each hyperedge of this class, thus there is no monochromatic Berge-K_3 hypergraph. An $\mathcal{F}$-free 5-coloring of K_6^3 on $[5] \cup \infty$ can be given so that the first color class is

$$\infty 12, \infty 13, 345, 245$$

and each other color class is obtained from the first one by cyclically shifting the vertex labels that are not ∞ and keeping ∞ fixed. Finally, an $\mathcal{F}$-free 6-coloring of K_7^3 on $[5] \cup \infty_1 \cup \infty_2$ can given so that the first color class consists of 5 edges: $\infty_1\infty_2 1$, $\infty_1\infty_2 2$, $\infty_1\infty_2 3$, $\infty_1\infty_2 4$, $\infty_1\infty_2 5$. The second color class is

$$125, 134, \infty_1 25, \infty_2 25, \infty_1 34, \infty_2 34,$$

and third through sixth color classes are obtained from the second by keeping ∞_1, ∞_2 fixed and cyclically shifting other vertex labels. The lower bound for 8 colors comes from the general construction.

4 Ramsey results for Berge-K_4

In this section, set $\mathcal{F} = \mathcal{B}(K_4)$.

Proof of Theorem 7/1 - lower bound. A natural lower bound on $R_3(\mathcal{F}, k)$ comes from covering the edges of K_n^3 with the smallest possible number of 3-partite subhypergraphs. Indeed, a 3-partite 3-uniform hypergraph cannot contain any member of $\mathcal{F}$, thus $f(n)$, the minimum number of 3-partite hypergraphs needed to cover all edges of K_n^3 provides a coloring with $f(n)$ colors containing no monochromatic member from $\mathcal{F}$. This is a well studied problem in coding theory, a special perfect hash family. Apart from very small n, exact values of $f(n)$ are not known, only upper bound tables are available [1]. The order of magnitude of $f(n)$ is known, $c_1 \log(n) \leq f(n) \leq c_2 \log(n)$ [2]. The upper bound implies that $2^{ck} \leq R_3(\mathcal{F}, k)$ for a positive constant c.

Interestingly, the upper bound of $f(n)$ is easy from probabilistic constructions, however, simple explicit constructions are not known (for general n). It seems worthwhile to give a very simple construction leading to a $2\log_2^2(n)$ upper bound. Split the vertex set of K_n^3 into two almost equal parts A and B, consider colorings c_A, c_B on the pairs of A, B with no monochromatic K_3 and with disjoint color sets. Extend this coloring to edges of K_n^3 intersecting both A and B as follows: edges xyz with $x, y \in A, z \in B$ are colored with $c_A(xy)$, edges xyz with $x \in A, y, z \in B$ are colored with $c_B(yz)$. This can be easily done by using no more than $2\log_2(n)$ colors. The uncolored edges, i.e. edges inside A and inside B can be colored recursively, using the same set of new colors. This leads to the recursive bound $f(n) \leq 2\log_2(n) + f(\lceil n/2 \rceil)$.

Proof of Theorem 7/1 - upper bound. The inequality $R_3(\mathcal{F}, k) \leq R(K_4^*(v), k)$ follows from Theorem 2. Thus Theorem 7 follows from the following lemma.

Lemma 8. *For any ϵ, $0 < \epsilon < 1/4$, and any $k \geq 1$, $R(K_4^*(v), k) \leq (1+\epsilon)^k \epsilon^{-1} k!$. In particular, $R(K_4^*(v), k) \leq (k-1)e(1+o(1))k!$.*

Proof. We shall prove the statement by induction on k with a trivial basis for $k = 1$. Consider a coloring of $E(K_n)$, with k colors and no monochromatic copy of a member from $K_4^*(v)$. Note that from each monochromatic triangle T of a fixed color class i we can select *one* vertex $v_i(T)$ of degree two in color class i. Let $X_i = \cup v_i(T)$, where the union is taken over all monochromatic triangles of color i. Then X_i is an independent set in color i, i.e. cannot contain any edge of color i. By induction $|X_i| \leq (1+\epsilon)^{k-1}\epsilon^{-1}(k-1)!$.

Case 1. $|X_i| > n/((1+\epsilon)k)$ for some i.
Then $n \leq (1+\epsilon)k|X_i| \leq (1+\epsilon)k(1+\epsilon)^{k-1}\epsilon^{-1}(k-1)! = (1+\epsilon)^k\epsilon^{-1}k!$ by induction.

Case 2. $|X_i| \leq n/((1+\epsilon)k)$ for each i. Note that deleting all X_i's leaves vertex set X' of size at least $n - n/(1+\epsilon) = n\epsilon/(1+\epsilon)$ such that X' does not contain any monochromatic triangles. Then $n \leq (1+\epsilon)|X'|/\epsilon \leq (1+\epsilon)R(K_3,k)/\epsilon \leq k!(1+\epsilon)/\epsilon \leq (1+\epsilon)^k\epsilon^{-1}k!$.

Optimizing over ϵ, for large k we see that for $\epsilon = 1/(k-1)$, $n \leq (k-1)e(1+o(1))k!$. □

Proof of Theorem 7/2. Note that Theorem 2 gives

$$R_3(\mathcal{F},2) \leq R(K_4^*(v),2) + 1 \tag{2}$$

and the Ramsey number on the right can be determined easily.

Lemma 9. $R(K_4^*(v),2) = 7$.

Proof. Set $\mathcal{F} = \mathcal{B}(K_4^*(v))$. For the lower bound $R(\mathcal{F},2) \geq 7$, consider $K_{3,3}$ and its complement as a 2-coloring on K_6.

For the upper bound, suppose $K = K_7$ is colored with red and blue. A well-known result of Goodman [6] implies that any 2-colored K_7 contains at least four monochromatic triangles, among them two of the same color, say T_1, T_2 are vertex sets of red triangles.

Suppose for contradiction that we have no monochromatic member of $\mathcal{F}$. This implies that there exist $v_1 \in T_1, v_2 \in T_2$ with red degree two in K. Consequently the edge v_1v_2 and all edges incident to v_1, v_2 and not on T_1, T_2 are blue. Set

$$T = (T_1 \cup T_2) \setminus \{v_1, v_2\},\ S = V(K) \setminus (T_1 \cup T_2).$$

Easy inspection shows that either there is a blue member of $\mathcal{F}$ with base triangle v_1, v_2, s with $s \in S$ or all edges of $[S,T]$ are red. In the latter case any red triangle with a red edge in T^* and with any $s \in S$ is a base triangle of a red member of $\mathcal{F}$, apart from one case: when $|S| = 3, |T| = 2$. In this exceptional case a red edge s_1s_2 gives a red member of $\mathcal{F}$ (with base triangle $s_1 \cup T$) and a blue edge s_1s_2 gives a blue member of $\mathcal{F}$ (with base triangle $s_1s_2v_1$). □

Unfortunately, Lemma 9 implies through (2) only $R_3(\mathcal{F},2) \leq 8$, to decrease it by two seems to require more difficult argument. First we need the Turán number of $\mathcal{F}$ for $n = 5$.

Lemma 10. $\text{ex}_3(5,\mathcal{F}) = 5$.

Proof. Five edges clearly do not form $\mathcal{F}$ thus we have to show $\text{ex}_3(5,\mathcal{F}) < 6$. Assume H is a 3-uniform hypergraph with six edges on a vertex set $[5]$ without any member of $\mathcal{F}$. Observe that the maximum vertex degree of H is at least $\lceil \frac{6\times 3}{5} \rceil = 4$.

- Suppose that for some $1 \leq i < j \leq 5$, ij is not covered by any edge of H. By symmetry, let $i = 1, j = 2$. Then H either contains the six edges meeting $\{1,2\}$ in one vertex or one of them, say 234 is missing. In the first case the assignment

  $$e_{13} = 134, e_{14} = 145, e_{15} = 135, e_{34} = 234, e_{35} = 235, e_{45} = 245$$

  defines a $\mathcal{F}$, otherwise the assignment $e_{34} = 234$ is replaced by the assignment $e_{34} = 345$ to get a $\mathcal{F}$. In both cases we have a contradiction. Thus all pairs of vertices are covered by some edge of H. Assume that vertex 1 has maximum degree in H.

- If $d(1) = 6$ then the edges containing 1 can be obviously assigned to pairs of $\{2,3,4,5\}$ to form a member of $\mathcal{F}$. Similarly, if $d(1) = 5$ then the five edges containing 1 with the edge covering the yet uncovered pair of the link of 1 form a member of $\mathcal{F}$ on $\{2,3,4,5\}$. Both cases yield contradiction.

- If $d(1) = 4$ then the link of 1 is either the four cycle $2,3,4,5,2$ or the graph with edges $23,34,24,45$. In the first case we have a member of $\mathcal{F}$ on $\{2,3,4,5\}$ with the assignment

  $$e_{23} = 123, e_{34} = 134, e_{45} = 145, e_{25} = 125$$

  extended by the edges covering the uncovered pairs $24,35$. In the second case we make the assignments

  $$e_{23} = 123, e_{24} = 124, e_{34} = 134, e_{45} = 145.$$

  If there exist two distinct edges of H to cover the yet uncovered pairs $25,35$ we get a member of $\mathcal{F}$ on vertex set $\{2,3,4,5\}$. Otherwise $234,235$ are both edges of H and we have a member of $\mathcal{F}$ on $\{1,2,3,4\}$ by the assignments

  $$e_{12} = 123, e_{13} = 134, e_{14} = 145, e_{23} = 235, e_{24} = 124, e_{34} = 234,$$

  a contradiction, finishing the proof.

□

We are ready to prove that $R_3(\mathcal{F},2) = 6$. It is easy to see that $R_3(\mathcal{F},2) \geq 6$ since in any 2-coloring of K_5^3 with five edges in each color class there is no monochromatic member of $\mathcal{F}$.

For the upper bound consider a 2-coloring c of the edges of $K = K_6^3$ with no monochromatic member of $\mathcal{F}$. Let H be the hypergraph defined by the edges of the majority color, containing at least 10 hyperedges. By Lemma 10, any 5 vertices of H induce at most 5 hyperedges, thus the remaining at least $|E(H)|-5$ hyperedges contain the sixth vertex, i.e., $d(v) \geq |E(H)| - 5$ for every $v \in V(H)$ implying

$$6(|E(H)| - 5) \leq \sum_{v \in v(H)} d(v) = 3|E(H)|, \tag{3}$$

that in turn implies that $|E(H)| \leq 10$. On the other hand $|E(H)| \geq 10$, thus $|E(H)| = 10$ and so the inequality in (3) hold as equality. In particular, H is

5-regular, implying the same for the other color. Thus we may assume that K is partitioned into a red and a blue hypergraph H_r, H_b both 5-regular.

Let v be an arbitrary vertex of H and consider the hypergraphs induced by H_r, H_b on $V - v = [5]$. By Lemma 10 both contain $\mathcal{B}(K_4 - e)$, we show that some of them can be extended to $\mathcal{F}$ by adding a suitable edge of K containing v. It is convenient to represent H_r, H_b with the graphs G_r, G_b of their complements (with their inherited colorings). Apart from color changes we have four possible cases.

Case 1. G_r, G_b are complementary five cycles. Assume the edges of G_r are $i, i+1$, thus edges $i, i+1, i+2$ belong to H_r. Every four element subset form a $\mathcal{B}(K_4 - e)$ in H_r, for example the edges

$$e_{12} = 123, e_{45} = 145, e_{25} = 125, e_{24} = 234, e_{35} = 345$$

cover all pairs of $\{2, 3, 4, 5\}$ except $\{3, 4\}$. Thus we have a red $\mathcal{F}$ unless $\{v, i, i+1\}$ are all forming blue edges in K. The same argument gives that $\{v, i, i+2\}$ must be red edges of K. But then there are many monochromatic $\mathcal{F}$s, for example the assignment

$$e_{12} = 123, e_{45} = 145, e_{25} = 125, e_{24} = v24, e_{35} = v35, e_{45} = 345$$

gives one on $\{2, 3, 4, 5\}$.

Case 2. G_r, G_b are complementary bulls. Assume that the edges of G_r are $12, 23, 34, 24, 45$, implying that the edges $345, 123, 145, 125, 135$ belong to of H_r and the edges $234, 235, 134, 245, 124$ belong to H_b. Then we have two $\mathcal{B}(K_4 - e)$s in H_r on $\{1, 2, 3, 5\}$. One of them with

$$e_{12} = 123, e_{13} = 135, e_{15} = 145, e_{25} = 125, e_{35} = 135$$

implying that $v23$ must be blue. The other is

$$e_{13} = 135, e_{15} = 145, e_{23} = 123, e_{25} = 125, e_{35} = 135$$

implying that $v12$ must be blue. However, then vertex 2 has degree at least six in H_b, contradiction.

Case 3. $G_r = K_4 - e$, G_b is its complement. Assume that the edges of G_r are $12, 23, 34, 14, 24$, then edges $125, 135, 145, 235, 345$ belong to H_r and edges $123, 124, 134, 234, 245$ belong to H_b. We have three $\mathcal{B}(K_4 - e)$s in H_b on vertices $\{1, 2, 3, 4\}$. All use assignments

$$e_{23} = 234, e_{24} = 245, e_{34} = 134$$

and they can be extended to $\mathcal{B}(K_4 - e)$ with $e_{13} = 123, e_{14} = 124$, or $e_{12} = 123, e_{14} = 124$ or $e_{12} = 124, e_{13} = 123$, respectively. Since we have no $\mathcal{B}(K_4)$ in H_b, the edges $12v, 13v, 14v$ are all red and vertex 1 has degree at least six in H_r, contradiction.

Case 4. G_r is a four-cycle with a pendant edge, G_b is its complement. Assume that the edges of G_r are $12, 23, 34, 45, 25$, then edges $123, 125, 145, 134, 345$ belong to H_r and edges $124, 135, 234, 235, 245$ belong to H_b. Then we have two $\mathcal{B}(K_4 - e)$s in H_b on $\{1, 3, 4, 5\}$. We can assign in both

$$e_{14} = 124, e_{34} = 234, e_{35} = 235, e_{45} = 245$$

and complete it with either $e_{13} = 135$ or $e_{15} = 135$. Since we have no $\mathcal{B}(K_4)$ in H_b, $13v, 15v$ are both red edges, consequently vertex 1 has degree at least six in H_r, contradiction.

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