Faces of maximal plane graphs without short cycles

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Abstract

For a positive integer g, we study a family of plane graphs G without cycles of length less than g that are maximal in a sense that adding any new edge to G either makes it non-plane or creates a cycle of length less than g. We show that the largest face length $\mathrm{f}_{\max}(g)$ of a 2-connected graph from this family satisfies $3g-12 \leq \mathrm{f}_{\max}(g) \leq 2(g-2)^2+1$.

1 Introduction

Turán-type problems play a substantial role in combinatorics since their introduction by Mantel [15] and Turán [20] in the first half of the 20th century. Perhaps the most extensively studied question of this type is the following. For a given family $\mathcal{F}$ of graphs, what is the largest possible number of edges $\mathrm{ex}(n, \mathcal{F})$ in an n-vertex $\mathcal{F}$*-free* graph, that is, a graph that does not contain any $F \in \mathcal{F}$ as a subgraph? Let $\mathcal{C}_{<g} = \{C_3, \ldots, C_{g-1}\}$ be a family of cycles of length less than g. It is known, for a fixed g, that $\mathrm{ex}(n, \mathcal{C}_{<g}) = O(n^{1+1/\lfloor (g-1)/2 \rfloor})$ and the bound is known to be asymptotically tight for some values of g, see [2,8,10].

Problems of this kind have a rich history of study and numerous variations, see the surveys [10,19,21]. Another variation was suggested by Dowden [7], who asked for the largest possible number of edges $\mathrm{ex}_{\mathcal{P}}(n, \mathcal{F})$ in an n-vertex *plane* $\mathcal{F}$-free graph. A direct application of Euler's formula yields that $\mathrm{ex}_{\mathcal{P}}(n, \mathcal{C}_{<g}) \leq \frac{g}{g-2}(n-2)$ for all $n \geq g \geq 3$ which is essentially tight. For more partial results on Dowden's problem, we refer the reader to [5,11,12,14,17,18] and the references therein.

In this note, we consider another fundamental parameter of a plane graph G, the largest face length of G, that we denote by $\mathrm{f}_{\max}(G)$. We say that a plane $\mathcal{C}_{<g}$-free graph G is *maximal plane* $\mathcal{C}_{<g}$*-free graph* if adding any new edge to G with both endpoints in G either makes it non-plane or creates a cycle of length less than g. The case when G is a star shows that $\mathrm{f}_{\max}(G)$ can be arbitrary large in terms of g even if our graph is maximal plane $\mathcal{C}_{<g}$-free. To avoid this rather degenerate situation, we consider only 2-connected graphs. More formally, let

$$\mathrm{f}_{\max}(g) = \max\{\mathrm{f}_{\max}(G) : \ G \text{ is a 2-connected maximal plane } \mathcal{C}_{<g}\text{-free graph}\}.$$

By taking $G = C_{2g-3}$, we immediately see that $\mathrm{f}_{\max}(g) \geq 2g-3$. Here, we show that this inequality is tight for $g = 3, 4, 5$, while the first author, Ueckerdt, and Weiner [3, Lemma 5] proved this for $g = 6$. Our main result provides general upper and lower bounds on $\mathrm{f}_{\max}(g)$. In particular, it implies that $\mathrm{f}_{\max}(g)$ is finite and strictly larger than $2g-3$ for all $g \geq 7$.

Theorem 1. *If $3 \leq g \leq 6$, then $\mathrm{f}_{\max}(g) = 2g-3$. If $7 \leq g \leq 9$, then $\mathrm{f}_{\max}(g) \geq 3g-9$. Moreover, for any integer $g \geq 7$, we have*

$$3g-12 \leq \mathrm{f}_{\max}(g) \leq 2(g-2)^2+1.$$

We prove the general bounds in Section 3. In Section 4, we deal with small values of g, while in Section 5, we provide a better upper bound for a special class of graphs. Some additional definitions are given in Section 2. Finally, we give concluding remarks in Section 6.

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2 Definitions and basic observations

When clear from the context, we shall identify a *plane* graph with the corresponding *planar* one. For all standard graph theoretic notions, we refer the reader to a book by Diestel [6]. We call a path S in a graph G an *ear* of a cycle C in G if the endpoints of S are vertices of C and no other vertex or edge of S belongs to C. We say that an ear S *splits* C into paths C' and C'' if $C = C' \cup C''$ and the endpoints of C' and C'' are those of S. Moreover, for every two vertices x and y of G, let us denote the length of a shortest *x, y-path*, i.e., a path between x and y, by $\mathrm{dist}_G(x, y)$. When the graph under consideration is clear from the context, we simply write $\mathrm{dist}(x, y)$. We denote the set of integers $\{1, \ldots, n\}$ by $[n]$.

We say that a graph is a *subdivided wheel* if it is a union of a cycle C, called the *outer cycle of the wheel*, and a tree T with exactly one vertex c, called the *center of the wheel*, of degree more than 2 such that a vertex of T belong to C if and only if it is a leaf of T. We say that a path in T connecting c to a leaf of T is a *spoke of the wheel* and a path in C connecting two consecutive leaves of T is a *segments of the wheel*. Note that a maximal plane $\mathcal{C}_{<g}$-free graph $W(g)$ in Figure 1 (a) is a subdivided wheel.

We shall repeatedly use the following observation.

Lemma 1. *Let $g \geq 4$, G be a 2-connected maximal plane $\mathcal{C}_{<g}$-free graph, and C be its facial cycle. If x and y are two non-consecutive vertices of C, then $2 \leq \mathrm{dist}(x, y) \leq g - 2$.*

Proof. Let x and y be two non-consecutive vertices on C.

If x and y are not adjacent, adding the edge xy to G doesn't break planarity, since x and y belong to the same face of G. Now the maximality of G implies that this new edge xy belongs to a cycle of length at most $g - 1$, and thus $\mathrm{dist}_G(x, y) \leq g - 2$, as desired.

Assume now that x and y are adjacent. This edge xy splits C into two paths C' and C'', each of length at least $g - 1$, since otherwise G contains a cycle of length less than g. Pick two vertices, x' on C' and y' on C'', such that $\mathrm{dist}_C(x, x') = \lfloor g/2 \rfloor = \mathrm{dist}_C(y, y')$. Note that $\mathrm{dist}(x, x') = \lfloor g/2 \rfloor$ otherwise the union of a shortest x, x'-path in G and the x, x'-path in C contains a cycle of length at most $g - 1$. Let P be a shortest x, x'-path. Assume that $\mathrm{dist}(x', y) < \lceil g/2 \rceil - 1$, i.e., that there is an x', y-path P' of length at most $\lceil g/2 \rceil - 2$. Then P' is shorter than P, and thus $P \cup P' \cup xy$ contains a cycle of length less than g. This is a contradiction implying that $\mathrm{dist}(x', y) \geq \lceil g/2 \rceil - 1$. Similarly, $\mathrm{dist}(y, y') = \lfloor g/2 \rfloor$ and $\mathrm{dist}(x, y') \geq \lceil g/2 \rceil - 1$. Since each x', y'-path contains either x or y by planarity, we conclude that $\mathrm{dist}(x', y') \geq \lfloor g/2 \rfloor + \lceil g/2 \rceil - 1 = g - 1$, which contradicts the first part of the lemma and thus completes the proof. □

3 Proof of Theorem 1 for $g \geq 7$

Let $g \geq 7$. For the general lower bound, consider the graph $W(g)$, see Figure 1 (a), that is a subdivided wheel with three spokes of length two each and each segment of length $g - 4$. Observe that any two non-adjacent vertices of $W(g)$ belong to at least one of the three cycles of length $2g - 4$, and thus adding an edge between them creates a cycle of length less than g. Hence, $W(g)$ is a 2-connected maximal plane $\mathcal{C}_{<g}$-free graph. Therefore, we have $\mathrm{f}_{\max}(g) \geq \mathrm{f}_{\max}(W(g)) = 3g - 12$, as claimed.

If $g = 7, 8$, or 9, consider a different construction $W'(g)$, that is an edge disjoint union of C_9 and C_{3g-9} that share three vertices equidistant on each of the cycles, see Figure 1 (b). As earlier, note that any two non-adjacent vertices of $W'(g)$ belong either to at least one of the three cycles of length $2g - 3$ or to at least one of the three cycles of length g. Hence, $W'(g)$ is a 2-connected maximal plane $\mathcal{C}_{<g}$-free graph, and so $\mathrm{f}_{\max}(g) \geq \mathrm{f}_{\max}(W'(g)) = 3g - 9$ for $g = 7, 8, 9$, as claimed.

Now we shall give a general upper bound. Let G be a 2-connected maximal plane $\mathcal{C}_{<g}$-free graph, and C be its largest facial cycle. The main idea of the proof is as follows. We argue that if C is sufficiently long, then there is an ear S that splits it into two sufficiently long paths. Then we show that there are two vertices, one on each path, each at a distance more

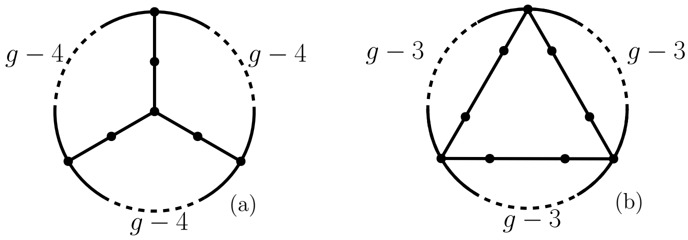

Figure 1: 2-connected maximal plane $\mathcal{C}_{<g}$-free graphs $W(g)$ and $W'(g)$.

than $g/2-1$ to S. As a result, these two vertices are at a distance more than $g-2$ in G, which contradicts Lemma 1.

Let $S = v_1 \dots v_{k+1}$ be an ear of C that splits it into $C' = v_1 u_1 \dots u_{m-1} v_{k+1}$ and $C'' = v_1 u'_1 \dots u'_{m'-1} v_{k+1}$. Our goal is to bound $\mathrm{dist}_C(v_1, v_{k+1}) = \min(m, m')$ in terms of g and k, that we shall state in Corollary 1. To do so, we order the vertices of S according to their indices, i.e., we say that $v_i < v_{i'}$ if $i < i'$. Let

$$d(j) = \mathrm{dist}(u_j, S) = \min_{i \in [k+1]} \mathrm{dist}(u_j, v_i) \qquad \text{and}$$
$$v(j) = v_i, \quad \text{where } i = \min\{i' : d(j) = \mathrm{dist}(u_j, v_{i'})\},$$

i.e., $v(j)$ is the smallest vertex from S such that $\mathrm{dist}(u_j, S) = \mathrm{dist}(u_j, v(j))$. Let $P(j)$ be some shortest $u_j, v(j)$-path. Note that each $P(j)$ is in the region bounded by C' and S, see Figure 2. For a vertex v of S, let

$$J(v) = \{j \in [m-1] : v(j) = v\}.$$

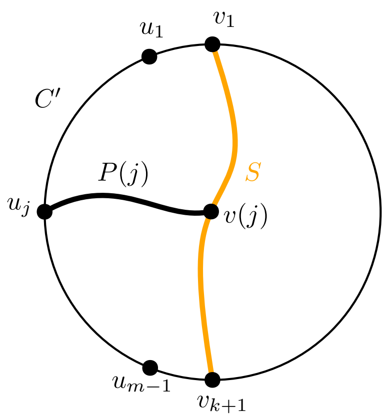

Figure 2: Setting for the proof of the general upper bound on $\mathrm{f}_{\max}(g)$.

Lemma 2. *If $d(j) \le g/2-1$ for each $j \in [m-1]$, then $m \le k(g-3)+2$.*

Proof. Our argument consists of several steps.

Claim 1. *For any $j \in [m-2]$, $v(j) \le v(j+1)$. In particular, for any vertex v of S, $J(v)$ is an interval of consecutive integers, that also could be empty.*

Indeed, assume that $v(j+1) < v(j)$ for some j. Since the edges of $P(j)$ and $P(j+1)$ are in the same region bounded by C' and S, by planarity, $P(j)$ and $P(j+1)$ share a vertex. Note that either u_j or u_{j+1} is a common vertex of $P(j)$ and $P(j+1)$, since otherwise $P(j) \cup P(j+1) \cup u_j u_{j+1}$ contains a cycle of length less than g.

If u_{j+1} belongs to $P(j)$, then $1 + \mathrm{dist}(u_{j+1}, v(j)) = \mathrm{dist}(u_j, v(j))$. Recall that $d(j+1) = \mathrm{dist}(u_{j+1}, v(j+1)) \le \mathrm{dist}(u_{j+1}, v(j))$ by the definition of $v(j+1)$. Thus $\mathrm{dist}(u_j, v(j+1)) \le$

$1+\mathrm{dist}(u_{j+1}, v(j+1)) \leq 1+\mathrm{dist}(u_{j+1}, v(j)) = \mathrm{dist}(u_j, v(j)) = d(j)$, which in turn contradicts the definition of $v(j)$, since $v(j+1) < v(j)$.

Similarly, if u_j belongs to $P(j+1)$, then $1+\mathrm{dist}(u_j, v(j+1)) = \mathrm{dist}(u_{j+1}, v(j+1))$. Recall that $d(j) = \mathrm{dist}(u_j, v(j)) < \mathrm{dist}(u_j, v(j+1))$ by the definition of $d(j)$, since $v(j+1) < v(j)$. Thus $\mathrm{dist}(u_{j+1}, v(j)) \leq 1 + \mathrm{dist}(u_j, v(j)) < 1 + \mathrm{dist}(u_j, v(j+1)) = \mathrm{dist}(u_{j+1}, v(j+1)) = d(j+1)$, which in turn contradicts the definition of $d(j+1)$. This proves the claim.

Claim 2. *For each vertex v of S, the function $d(\cdot)$ is strictly unimodal on $J(v)$. Moreover, $d(\cdot)$ is strictly monotone on $J(v_1)$ and on $J(v_{k+1})$.*

Fix a vertex v in S. If $j, j+1 \in J(v)$, then $P(j) \cup P(j+1) \cup u_j u_{j+1}$ is a closed walk of length less than g. Hence, this walk is trivial, i.e., either $P(j) = u_j u_{j+1} \cup P(j+1)$ (in which case $d(j) = d(j+1)+1$) or $P(j+1) = u_{j+1} u_j \cup P(j)$ (in which case $d(j+1) = d(j)+1$). Assume that there is a local maximum of $d(\cdot)$ at $j \in J(v)$, i.e., that $d(j-1) = d(j)-1 = d(j+1)$ for some $j-1, j, j+1 \in J(v)$. Then $P(j) = u_j u_{j-1} \cup P(j-1) = u_j u_{j+1} \cup P(j+1)$. This is a contradiction, since $P(j)$ has an endpoint u_j and two edges incident to it. Therefore, the function $d(\cdot)$ has a unique minimum on $J(v)$, it first strictly decreases and then strictly increases. Moreover, note that $\mathrm{dist}(u_1, v_1) = \mathrm{dist}(u_{m-1}, v_{k+1}) = 1$, and thus the corresponding minimums of $d(\cdot)$ on $J(v_1)$ and on $J(v_{k+1})$ are their endpoints $j = 1$ and $j = m-1$, respectively. For the illustration, see Figure 3. This proves the claim.

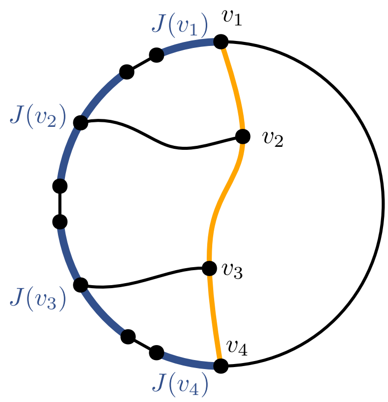

Figure 3: Special case in which the ear S (yellow) consists of 4 vertices; blue segments correspond to the sets $\{u_j : j \in J(v_i)\}$, $i \in [4]$; middle vertices on two blue segments correspond to the unique minimums of $d(\cdot)$ on these segments.

Claim 3. *For each vertex v of S, $|J(v)| \leq g-3$. Moreover $|J(v_1)|, |J(v_{k+1})| \leq g/2 - 1$.*

The second part is immediate from the fact that $d(\cdot)$ can have values only in $[g/2-1]$ and strictly monotone on $J(v_1)$ and on $J(v_{k+1})$. For the first part, note that the strict unimodality of $d(\cdot)$ on $J(v)$ implies that $|J(v)| \leq 2(g/2-1) - 1 = g-3$ for each $v \in V(S)$. This proves the claim.

Since each $j \in [m-1]$ belongs to $J(v)$, for some v and the sets $J(v)$ are pairwise disjoint, we have that $m - 1 = \sum_{i=1}^{k+1} |J(v_i)| \leq 2(g/2-1) + (k-1)(g-3) = k(g-3)+1$, which completes the proof of Lemma 2. □

Corollary 1. *Each ear $S = v_1 \dots v_{k+1}$ of C satisfies* $\mathrm{dist}_C(v_1, v_{k+1}) \leq k(g-3)+2$.

Proof. Indeed, if not, then some ear S splits C into two paths C' and C'', each of length more than $k(g-3)+2$. By Lemma 2, there are vertices x on C' and y on C'' such that $\mathrm{dist}(x, S) > g/2-1$ and $\mathrm{dist}(y, S) > g/2-1$. Since every x, y-path goes through S by planarity, we conclude that $\mathrm{dist}(x, y) > g-2$, which contradicts Lemma 1. □

Recall that C is a facial cycle of length $\mathrm{f}_{\max}(g)$, and take two "antipodal" vertices w and w' on it. Consider a w, w'-path P of length at most $g-2$, which exists by Lemma 1, and label the subsequence of its vertices that belong to C by $w = w_1, \dots, w_{t+1} = w'$. For $s \in [t]$, let $k_s - 1$ be the number of vertices of P strictly between w_s and w_{s+1}. Observe that $\sum_{s=1}^{t} k_s \le g - 2$ since this sum equals the length of P. Besides, if $k_s = 1$, then Lemma 1 implies that $\mathrm{dist}_C(w_s, w_{s+1}) = 1$. Moreover, note that if $k_s \ge 2$, then the w_s, w_{s+1}-subpath of P forms an ear of C, and thus $\mathrm{dist}_C(w_s, w_{s+1}) \le k_s(g-3)+2$ by Corollary 1, see Figure 4. Finally, the triangle inequality implies that

$$\begin{aligned}\mathrm{dist}_C(w, w') &\le \sum_{s=1}^{t} \mathrm{dist}_C(w_s, w_{s+1}) \le \sum_{s\colon k_s \ge 2} \big(k_s(g-3)+2\big) + \sum_{s\colon k_s = 1} 1 \\ &= (g-3) \sum_{s\colon k_s\ge 2} k_s + \Big(\sum_{s\colon k_s \ge 2} 2 + \sum_{s\colon k_s=1} 1 \Big) \\ &\le (g-3)\sum_{s=1}^{t} k_s + \sum_{s=1}^{t} k_s = (g-2)\sum_{s=1}^{t} k_s \le (g-2)^2.\end{aligned}$$

Recall that w and w' are antipodal vertices of the largest face of G, and thus

$$\mathrm{f}_{\max}(G) \le 2 \cdot \mathrm{dist}_C(w, w') + 1 \le 2(g-2)^2 + 1,$$

which completes the proof of Theorem 1 for $g \ge 7$.

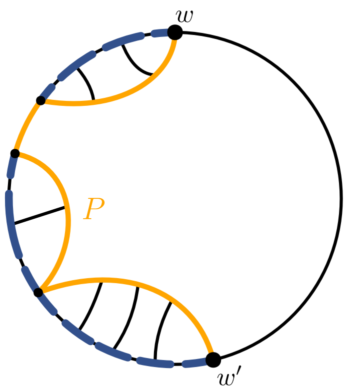

Figure 4: A path P between two antipodal vertices w and w' contains several ears of C; we apply Corollary 1 to bound the cyclic distance between the endpoints of each ear.

4 Proof of Theorem 1 for $3 \le g \le 6$

In this section, we argue that $\mathrm{f}_{\max}(g) = 2g - 3$ if $3 \le g \le 6$. Recall that the case $g = 6$ follows from Lemma 5 in [3]. Note that if $g = 3$, then the family $\mathcal{C}_{<g}$ of forbidden cycles is empty and any maximal plane graph is a triangulation, so $\mathrm{f}_{\max}(3) = 3$.

Let $g = 4$. Let $C = u_0u_1 \dots u_ku_0$ be a facial cycle of length at least 6 in a maximal plane $\mathcal{C}_{<g}$-free graph G. By Lemma 1 any two non-consecutive vertices of C are at distance exactly 2 in G. By planarity, u_0, u_3- and u_1, u_4-paths of length 2 must share a common midpoint, say w. Then u_0wu_1 is a cycle of length 3, a contradiction.

Let $g = 5$. Let $C = u_0u_1 \dots u_ku_0$ be a facial cycle of length at least 8 in a maximal plane $\mathcal{C}_{<g}$-free graph G. In this case, $k \ge 7$ and every two non-consecutive vertices of C are at distance either 2 or 3 in G by Lemma 1.

Assume first that $\mathrm{dist}(x, y) = 3$ for all vertices x, y of C such that $\mathrm{dist}_C(x, y) \ge 3$. This implies that the shortest u_0, u_4- and u_1, u_5-paths P and P' have no interior vertices on C. By planarity, they share a vertex. Thus $P \cup P' \cup u_0u_1 \cup u_4u_5$ is an edge-disjoint union of two nontrivial closed walks of total length 8. Thus there is a cycle of length at most 4, a contradiction.

Now assume that $\operatorname{dist}(x, y) = 2$ for some vertices x, y of C such that $\operatorname{dist}_C(x, y) \geq 3$. Assume without loss of generality that $\operatorname{dist}(u_2, u_j) = 2$ for some $5 \leq j \leq \lfloor k/2 \rfloor + 2$, and thus $u_2 w u_j$ is an ear of C for some vertex w. Note that $\operatorname{dist}(u_0, u_2) = 2$ and $\operatorname{dist}(u_0, w) \geq 2$, since otherwise $u_0 w$ is an edge and $u_0 u_1 u_2 w u_0$ is a cycle of length less than 5. Similarly, $\operatorname{dist}(u_4, u_2) = 2$ and $\operatorname{dist}(u_4, w) \geq 2$. In addition, u_j and u_0 are not consecutive on C since $j + 1 \leq \lfloor k/2 \rfloor + 3 < k$, and thus $\operatorname{dist}(u_0, u_j) \geq 2$. Since the shortest u_0, u_4-path of length at most 3 contains one of the vertices u_2, w, u_j by planarity, we conclude that this vertex is u_j and $\operatorname{dist}(u_0, u_j) = 2$, $\operatorname{dist}(u_4, u_j) = 1$. The latter equality implies that $j = 5$, and thus $\operatorname{dist}_C(u_0, u_j) > 2$. Now the equality $\operatorname{dist}(u_0, u_j) = 2$ implies that there exists a vertex w' such that $u_0 w' u_j$ is an ear of C.

Applying the argument from the previous paragraph to the vertices u_7, u_6, u_5, u_3, u_2 playing the roles of $u_0, u_1, u_2, u_4, u_5 = u_j$, respectively, we conclude that there exists a vertex w'' such that $u_7 w'' u_2$ is an ear of C. Moreover, it is easy to see that two ears $u_0 w' u_5$ and $u_7 w'' u_2$ must share their center vertex by planarity. In other words, $w' = w''$ and thus $u_2 w u_5 w' u_2$ is a cycle of length 4, a contradiction.

5 Subdivided wheels

Our next result shows that our general lower bound is tight up to an additive constant for a special class of graphs, the subdivided wheels.

Proposition 1. *Let G be a maximal plane $\mathcal{C}_{<g}$-free graph that is a subdivided wheel, then* $\mathrm{f}_{\max}(G) \leq 3g - 3$.

Proof. Let c the center of the wheel G and C be its outer cycle. Note that every face G is bounded either by C or by two consecutive spokes and a respective segment of C.

Consider a face F of the latter type, i.e., bounded by three path P_1, P_2, P_3, two of which are spokes and one is a cyclic segment. Let ℓ_1, ℓ_2, ℓ_3 be the lengths of these path and v_1, v_2, v_3 be their center vertices (chosen arbitrarily if needed), respectively. On the one hand, the distance from each v_i to either endpoint of P_i is at least $\lfloor \ell_i/2 \rfloor$, and thus $\operatorname{dist}(v_i, v_j) \geq \lfloor \ell_i/2 \rfloor + \lfloor \ell_j/2 \rfloor \geq (\ell_i + \ell_j)/2 - 1$. On the other hand, these distances do not exceed $g - 2$ by Lemma 1. Hence, the length of F satisfies $\ell_1 + \ell_2 + \ell_3 = (\ell_1 + \ell_2)/2 + (\ell_2 + \ell_3)/2 + (\ell_3 + \ell_1)/2 \leq 3g - 3$, as desired.

Now, we shall bound the length of C. Our argument relies on the following statement.

Claim 4. *Any set of $g - 2$ consecutive vertices of C contains a vertex u such that* $\operatorname{dist}(u, c) > g/2 - 1$.

Our argument is very similar to the proof of Claim 2. Take some m consecutive vertices $u_1, \ldots, u_m$ on C. For $j \in [m]$, let $P(j)$ be a shortest u_j, c-path and $d(j) = \operatorname{dist}(u_j, c)$ be its length. Assume that $d(j) \leq g/2 - 1$ for all $j \in [m]$. Note that $P(j) \cup P(j+1) \cup u_j u_{j+1}$ is a closed walk of length less than g. Hence, this walk is trivial, i.e., either $P(j) = u_j u_{j+1} \cup P(j+1)$ (in which case $d(j) = d(j+1) + 1$) or $P(j+1) = u_{j+1} u_j \cup P(j)$ (in which case $d(j+1) = d(j) + 1$). Assume that there is a local maximum of $d(\cdot)$ at $2 \leq j \leq m - 1$, i.e., that $d(j-1) = d(j) - 1 = d(j+1)$. Then $P(j) = u_j u_{j-1} \cup P(j-1) = u_j u_{j+1} \cup P(j+1)$. This is a contradiction, since $P(j)$ has an endpoint u_j and two edges incident to it. Therefore, the function $d(\cdot)$ has a unique minimum on $[m]$. In addition $d(\cdot)$ can have values only in $[g/2 - 1]$, and thus $m \leq 2(g/2 - 1) - 1 = g - 3$, as desired. This proves the claim.

Let w be a vertex of C such that $\operatorname{dist}(w, c) > g/2 - 1$. Assume that C has at least $3g - 2$ vertices. Then there is a set Q of at least $g - 2$ consecutive vertices on C, each at distance greater than $g - 2$ from w on C. By Claim 4 there is $w' \in Q$ such that $\operatorname{dist}(w', c) > g/2 - 1$. Any shortest w, w'-path either goes though c or is a subgraph of C. In the former case, $\operatorname{dist}(w, w') = \operatorname{dist}(w, c) + \operatorname{dist}(w', c) > g - 2$. In the latter case, $\operatorname{dist}(w, w') = \operatorname{dist}_C(w, w') > g - 2$. In either case, there is contradiction to Lemma 1. □

6 Concluding remarks

In this note we showed that the largest face length $\mathrm{f}_{\max}(g)$ of a 2-connected plane graph of girth at least g that is edge-maximal with respect to these two properties satisfies $\Omega(g) = \mathrm{f}_{\max}(g) = O(g^2)$. We would like to pose the following question.

Question 1. *Is it true that* $\mathrm{f}_{\max}(g) = \Theta(g)$*?*

While determining $\mathrm{f}_{\max}(g)$ remains most interesting question of this note, it was not originally obvious that $\mathrm{f}_{\max}(g)$ is bounded by any function of g. The shortest argument we know takes about a page and is based on multicolor Ramsey's theorem applied to the auxiliary complete graphs, where we color the edge xy according to $\mathrm{dist}_G(x, y)$ for all vertices x, y of G. It might be interesting to find a shorter argument.

Most Turán-type problems have their saturation counterparts, where the goal is to *minimize* the number of edges in a maximal $\mathcal{F}$-free graph. Note that if $\mathcal{F} = \mathcal{C}_{<g}$ with $g > 3$, then every maximal plane $\mathcal{C}_{<g}$-free contains at least $n - 1$ edges, which is tight as witnessed by stars. The following question is less trivial.

Question 2. *What is the minimum number* $\mathrm{sat}_{\mathcal{P}}^{2\text{-con}}(n, \mathcal{C}_{<g})$ *of edges in a 2-connected maximal plane* $\mathcal{C}_{<g}$*-free graph on* n *vertices?*

A direct application of Euler's formula yields that $\mathrm{sat}_{\mathcal{P}}^{2\text{-con}}(n, \mathcal{C}_{<g}) \geq (1 - 2/\mathrm{f}_{\max}(g))^{-1}(n - 2)$ for all $n \geq g \geq 3$. It would be interesting to improve this lower bound asymptotically.

Note that $\mathrm{f}_{\max}(g)$ is defined as the largest face length of a 2-connected maximal plane graph of girth *at least* g, while one could ask for a variant of this problem for graphs of girth *exactly* g. We claim that these two problems have the same answer. Indeed, consider a 2-connected plane graph G of girth at least g with a face F of length $\mathrm{f}_{\max}(g)$. Draw a cycle of length g in any face of G but F and add edges arbitrarily until the resulting graph G' is maximal plane $\mathcal{C}_{<g}$-free. It is easy to see that $\mathrm{f}_{\max}(G') \geq \mathrm{f}_{\max}(g)$ since F is still a face of G', while the girth of G' equals g, as desired.

Finally, let us note that the relations between face lengths in plane graphs and their other parameters including radius or diameter were also considered, see Ali, Dankelmann, Mukwembi [1] and Du Preez [16], respectively. See also a paper by Fernández, Sieger, and Tait [9] on planar subgraphs of given girth in planar graphs.

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