Faces in girth-saturated graphs on surfaces

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Abstract

What is the maximum length $\mathrm{f}_{\max}(\ell, \Sigma)$ of a facial cycle of an inclusion-maximal graph with girth at least ℓ embedded on a given surface Σ? If $\Sigma = \mathcal{P}$ is a plane, we show that $3\ell - 11 \le \mathrm{f}_{\max}(\ell, \mathcal{P}) \le 8\ell - 13$. We also prove that $\mathrm{f}_{\max}(\ell, \Sigma)$ is bounded for any integer ℓ and any closed surface Σ. For a fixed ℓ, the bound is polynomial in genus of Σ.

1 Introduction

Turán-type problems play a substantial role in combinatorics since their introduction by Mantel [22] and Turán [31] in the first half of the 20th century. Perhaps the most extensively studied question of this type is the following. For a given family $\mathcal{F}$ of graphs, what is the largest possible number of edges $\mathrm{ex}(n, \mathcal{F})$ in an n-vertex $\mathcal{F}$*-free* graph, that is, a graph that does not contain any $F \in \mathcal{F}$ as a subgraph? Let $\mathcal{C}_{<\ell}$ be a family of cycles of length less than ℓ. It is known that, for a fixed ℓ, $\mathrm{ex}(n, \mathcal{C}_{<\ell}) = O(n^{1+1/\lfloor(\ell-1)/2\rfloor})$, and that the bound is asymptotically tight for some small values of ℓ, see [2, 11, 15].

Problems of this kind have a rich history of study and numerous variations, see surveys [15, 30, 32]. Another variation was suggested by Dowden [10], who asked for the largest possible number of edges $\mathrm{ex}_{\mathcal{P}}(n, \mathcal{F})$ in an n-vertex *plane* $\mathcal{F}$-free graph, where $\mathcal{P}$ stands for the plane. A direct application of Euler's formula yields that $\mathrm{ex}_{\mathcal{P}}(n, \mathcal{C}_{<\ell}) \le \frac{\ell}{\ell-2}(n-2)$ for all $n \ge \ell \ge 3$ which is essentially tight. For more partial results on Dowden's problem, we refer the reader to [7, 16, 18, 21, 28, 29] and the references therein.

Here we consider another natural parameter of graphs on surfaces: the length of a longest facial cycle. Let Σ be a connected surface and G be a graph embedded on Σ. Denote by $\mathrm{f}_{\max}(G)$ the length of a longest facial cycle of G. If G has no facial cycles, e.g. if the boundary of every face contains pendant edges, then we define $\mathrm{f}_{\max}(G) = 0$. We refer the reader to a classical book [24] by Mohar and Thomassen for basic definitions. We say that G is *maximal* $\mathcal{C}_{<\ell}$*-free graph embedded on* Σ if G is $\mathcal{C}_{<\ell}$-free, but adding any new edge to G with both endpoints in G creates either a crossing on Σ or a cycle of length less than ℓ. Further, let

$$\mathrm{f}_{\max}(\ell, \Sigma) = \max\{\mathrm{f}_{\max}(G) : \ G \text{ is a maximal } \mathcal{C}_{<\ell}\text{-free graph embedded on } \Sigma\}.$$

By considering a cycle of length $2\ell - 3$ bounding a disc on a surface Σ, we immediately see that $\mathrm{f}_{\max}(\ell, \Sigma) \ge 2\ell - 3$ for each surface Σ and $\ell \ge 3$. Here, we show that $\mathrm{f}_{\max}(\ell, \mathcal{P}) = 2\ell - 3$ for $\ell = 3, 4, 5$, while the first author, Ueckerdt, and Weiner [3, Lemma 5] proved this for $\ell = 6$. Our first main result provides general upper and lower bounds on $\mathrm{f}_{\max}(\ell, \mathcal{P})$. In particular, it implies that $\mathrm{f}_{\max}(\ell, \mathcal{P})$ is finite and strictly larger than $2\ell - 3$ for all $\ell \ge 7$.

Theorem 1. *If* $3 \le \ell \le 6$*, then* $\mathrm{f}_{\max}(\ell, \mathcal{P}) = 2\ell - 3$*. For any* $\ell \ge 7$*, we have*

$$3\ell - 11 \le \mathrm{f}_{\max}(\ell, \mathcal{P}) \le 8\ell - 13.$$

Moreover, if $7 \le \ell \le 9$*, then* $\mathrm{f}_{\max}(\ell, \mathcal{P}) \ge 3\ell - 9$*.*

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Each connected *closed* surface, namely a compact surface without a boundary, is homeomorphic to either a sphere with g handles $\mathbb{S}_g$ or to a sphere with g crosscaps $\mathbb{N}_g$ for some non-negative integer g, see e.g. [24, Theorem 3.1.3]. Our second main result provides a general upper bound on $\mathrm{f}_{\max}(\ell, \Sigma)$ for these surfaces as well as a lower bound which is better than $\mathrm{f}_{\max}(\ell, \Sigma) \geq 2\ell - 3$ for most of the values of g and ℓ.

Theorem 2. *Let $g \geq 1$ and $\ell \geq 3$ be integers and Σ be a surface, $\Sigma \in \{\mathbb{S}_g, \mathbb{N}_g\}$. Then*

$$\mathrm{f}_{\max}(\ell, \Sigma) \leq \left((4g+4)^2\ell\right)^\ell.$$

Moreover, if $\ell \geq 6$, then

$$\mathrm{f}_{\max}(\ell, \Sigma) \geq g(\ell - 4).$$

We remark that if a connected surface Σ' is homeomorphic to a surface obtained from some other surface Σ by removing a finite number of points or discs, then any finite graph G can be embedded on Σ' if and only if it can be embedded on Σ, and so $\mathrm{f}_{\max}(\ell, \Sigma') = \mathrm{f}_{\max}(\ell, \Sigma)$ for all ℓ. In particular, for a sphere $\mathbb{S}_0$, we have $\mathrm{f}_{\max}(\ell, \mathcal{P}) = \mathrm{f}_{\max}(\ell, \mathbb{S}_0)$. Hence, the assumptions of Theorem 2 do no reduce the generality.

Paper outline. In Section 2, we introduce our notations and make some basic observations. We prove Theorem 1 for $\ell \geq 7$ in Section 3, while in Section 4, we deal with the remaining small values of ℓ. In Section 5, we prove Theorem 2. Finally, we give concluding remarks and state open problems in Section 6.

2 Definitions and basic observations

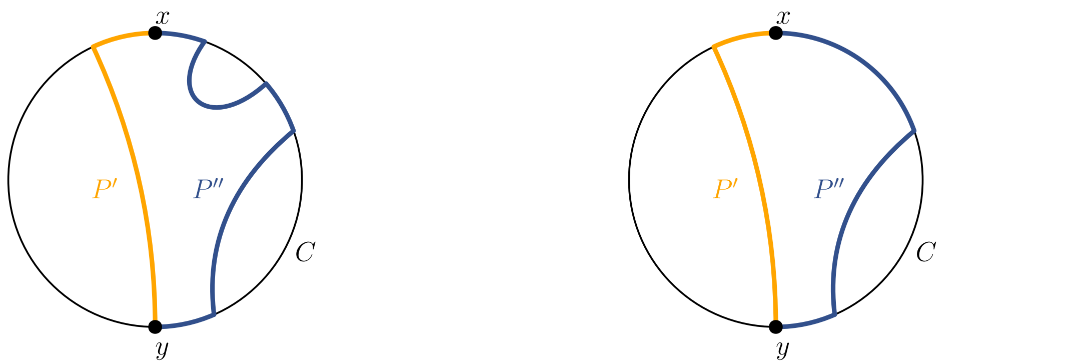
We denote the set of integers $\{1, \ldots, n\}$ by $[n]$. All graphs in this paper are finite. For a graph $G = (V, E)$, we denote the number of its edges by $\|G\|$. We denote a cycle of length n by C_n. When clear from the context, we shall identify a graph embedded on a surface with its embedding. In particular, we identify a *planar* graph with the corresponding *plane* one. Let G be a graph embedded on a connected closed surface Σ. We call the connected components of $\Sigma \setminus G$ *faces of* G, see [17, Section 3.1.4] If a cycle in G forms a boundary of a face, we call the cycle *facial*. For all standard graph theoretic notions, we refer the reader to a book by Diestel [9].

For a walk Q, we denote its length, i.e., the number of its edges counting repetitions, by $\|Q\|$. We say that a walk is *non-trivial* if the underlying graph contains a cycle. Equivalently, a walk is non-trivial if it contains at least one edge of odd multiplicity.

Let us denote the length of a shortest x, y-path by $\mathrm{dist}_G(x, y)$ or simply $\mathrm{dist}(x, y)$ when G is clear from the context. For a path P and two of its vertices x and y, we write xPy to denote the subpath of P with the endpoints x and y. We also concatenate these objects in a straightforward manner, e.g. $xPyQz$ stands for the walk that consists of the path xPy followed by the path yQz.

We call proper connected subgraphs of a cycle its *segments*. Two vertices of a cycle C are called *antipodal* if the distance between them on the cycle is $\lfloor \|C\|/2 \rfloor$. We call a path P with at least two vertices an *ear* of a cycle C if the endpoints of P are vertices of C and no other vertex or edge of P belongs to C. Observe that every path P with the endpoints on a cycle C is an edge-disjoint union of segments of the cycle C and its ears. We say that P is C*-convex* if there is at most one such ear. More formally, for a cycle C and two of its vertices x and y, we say that an x, y-path P is C*-convex* if either P is a segment of C or for some vertices x' and y' on P, both xPx' and $y'Py$ are segments of C and $x'Py'$ is an ear of C. Note that x' and y' may coincide with x and y, respectively, see Figure 1a.

We say that vertices x and y *split* a cycle C' into two edge-disjoint x, y-paths P' and P'' if $C' = P' \cup P''$. For a cycle C' of length at most $2\ell - 2$ and two of its vertices x and y, we call C' an $(x, y; \ell)$*-lens*, or simply x, y*-lens* when the value of ℓ is clear from the context. If x and y split an x, y-lens C' into two x, y-paths that are both C-convex for a cycle C, we say that C' is C*-convex* as well, Figure 1b.



(a) An x, y-path P' is C-convex, while P'' is not. (b) Both P' and P'' are C-convex. The cycle $C' = P' \cup P''$ is a C-convex $(x, y; \ell)$-lens if $\|C'\| \le 2\ell - 2$.

Figure 1: An illustration to the definitions of C-convex path and lens.

A *spider* is a tree that is a subdivision of a star, i.e., a tree with exactly one vertex of degree at least three, called the *head* of the spider. A *leg* of a spider is a path with endpoints that are the head and a leaf of the spider. A *t-subspider* of a spider S is a spider obtained by taking the union of some t legs of S. A *pseudo-spider* with *head* u and *leaf-set* U is a union of some not necessarily edge-disjoint u, u'-paths, $u' \in U$, that are called its *legs*. Note that as a graph, a pseudo-spider may not be a tree, since the vertices of its leaf-set may be of degree larger than 1.

We say that a graph is a *subdivided wheel* if it is a union of a cycle C, called the *outer cycle of the wheel*, and a spider S, whose head c is called the *center* of the wheel, such that $V(S) \cap V(C)$ is the set of leaves of S. We say that a path in S connecting c to a leaf of S is a *spoke of the wheel* and a path in C connecting two consecutive leaves of S is a *segment of the wheel*.

We shall repeatedly use the following observations.

Lemma 1. *Let ℓ be an integer, Σ be a connected closed surface, G be a maximal $\mathcal{C}_{<\ell}$-free graph embedded on Σ, and C be its facial cycle. If x and y are two vertices of C, then $\mathrm{dist}_G(x, y) \le \ell - 2$.*

Proof. Let x and y be two vertices of C. If x and y are not adjacent, adding the edge xy to G inside a face bounded by C does not create a crossing. Now the maximality of G implies that this new edge xy belongs to a cycle of length at most $\ell - 1$, and thus $\mathrm{dist}_G(x, y) \le \ell - 2$, as desired. □

Lemma 2. *If z is a common vertex of a shortest x, y-path P and a shortest x', y-path Q in a graph G, then $Q' = x'QzPy$ gives a shortest x', y-path as well.*

Proof. If $\|Q\| < \|Q'\|$, then $\|zQy\| < \|zPy\|$. Hence, an x, y-walk $P' = xPzQy$ is shorter than the shortest x, y-path P, a contradiction. Therefore, the x', y-walk Q' is no longer than the shortest x', y-path Q, and thus Q' gives a shortest x', y-path as well. □

3 Proof of Theorem 1 for $\ell \ge 7$

Throughout this section, let $\ell \ge 7$ be an integer[1], G be a maximal $\mathcal{C}_{<\ell}$-free plane graph, and C be its facial cycle. We shall need several preliminary lemmas about shortest paths and lenses in G.

Lemma 3. *Let $xx', yy' \in E(C)$, Q be a shortest x', y-path and Q' be a shortest x, y'-path in G. If $xx', yy' \notin E(Q) \cup E(Q')$, then Q and Q' are vertex disjoint.*

[1] All the arguments in this section are valid for all $\ell \ge 3$, but they give only weak bounds when $3 \le \ell \le 6$. For the exact result for these values of ℓ, see Section 4.

Proof. Assume for the contrary that Q and Q' share a vertex z. Note that a closed walk $C' = xx'QzQ'x$ is non-trivial, since both Q and Q' do not use the edge xx'. By a similar argument, a closed walk $C'' = yy'Q'zQy$ is non-trivial as well. Moreover, their total length is $\|C'\| + \|C''\| = \|x'Qz\| + \|zQ'x\| + \|y'Q'z\| + \|zQy\| + 2 = \|Q\| + \|Q'\| + 2 \leq 2\ell - 2$, where the latter inequality is due to Lemma 1. Hence, either C' or C'' contains a cycle of length less than ℓ, a contradiction. □

Lemma 4. *If $x, y \in V(C)$ and P is a shortest x, y-path in G, then P is C-convex.*

Proof. Assume that P contains at least two ears of C. Since every subpath of P is a shortest path between its endpoints, we can assume without loss of generality that our counterexample is minimal with respect to inclusion, i.e., that it contains exactly two ears of C as subpaths each of which shares an endpoint with P. Namely, for some vertices x' and y' on P, both xPx' and $y'Py$ are ears of C and $x'Py'$ is a segment of C, see Figure 2. Note that x' and y' may coincide.

Let R be the x, y-path in C that contains x' and y'. Consider the family $\mathcal{F}$ of shortest w, w'-paths of the form $wRxPyRw'$, over all possible $w, w' \in V(R)$. This family $\mathcal{F}$ is non-empty since it contains P. Let $P' = w_xRxPyRw_y$ be a maximal element of $\mathcal{F}$ ordered by inclusion. Further, let w'_x be a vertex adjacent to w_x on C such that $w_xw'_x$ is not an edge of P'. Observe that $w'_x \notin V(P')$ since otherwise the w_x, w_y-path $w_xw'_xP'w_y$ is shorter than P', a contradiction. Similarly, let w'_y be a vertex adjacent to w_y on C such that $w_yw'_y$ is not an edge of P', and observe that $w'_y \notin V(P')$.

Consider a shortest w'_x, w_y-path Q. We see that Q intersects $w_xP'x'$ by planarity. Denote the first point of $w_xP'x'$, counting from w_x, that lies on Q by z. Lemma 2 implies that $\widehat{Q} = w'_xQzP'w_y$ is also a shortest w'_x, w_y-path. In addition, the maximality of P' implies that $z \neq w_x$ and thus $w'_xw_x \notin E(\widehat{Q})$. Similarly, a shortest w'_y, w_x-path Q' intersects $w_yP'y'$, and we denote the first point of $w_yP'y'$, counting from w_y, that lies on Q' by z'. Then $\widehat{Q}' = w'_yQ'z'P'w_x$ is also a shortest w'_y, w_x-path and $w'_yw_y \notin E(\widehat{Q}')$.

On the one hand, note that both $\widehat{Q}$ and $\widehat{Q}'$ contain $x'Py'$ as a subpath. On the other hand, Lemma 3 implies that $\widehat{Q}$ and $\widehat{Q}'$ are vertex disjoint, a contradiction. □

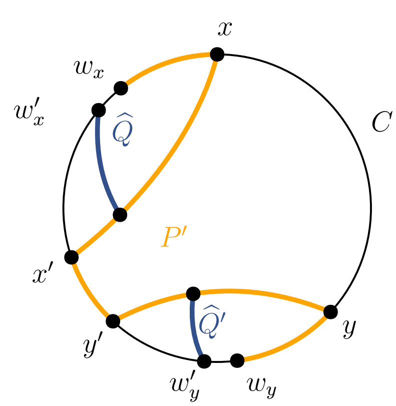


Figure 2: An illustration to the proof of Lemma 4.

Lemma 5. *For every $x, y \in V(C)$ such that $\mathrm{dist}_C(x, y) > \ell - 2$, there exists a C-convex $(x, y; \ell)$-lens.*

Proof. Let P be a shortest x, y-path. By Lemma 1, $\|P\| \leq \ell - 2$, and thus P is not a segment of C. Lemma 4 implies that P is C-convex, and thus for some vertices x' and y' on P, both xPx' and $y'Py$ are segments of C and $x'Py'$ is an ear of C. Note that x' and y' may coincide with x and y, respectively. There are two possible cases depending on whether the set $\{x', y'\}$ separates x and y in C or not, see Figure 3. However, for our argument, they are treated the same way.

Consider a family of shortest w, w'-paths in G, over all possible $w, w' \in V(C)$. Let $\mathcal{F}$ be its subfamily consisting of those paths that contain P as a subpath. In particular, for any $P' \in \mathcal{F}$ with endpoints w, w', $\|P'\| = \mathrm{dist}_G(w, w')$. This family $\mathcal{F}$ is non-empty since it contains P. Let P' be a maximal element from $\mathcal{F}$ ordered by inclusion. By Lemma 4, P' is C-convex. Since $x'P'y' = x'Py'$ is an ear of C, all the other vertices and edges of P' are in C. Let w_x and w_y be the endpoints of P', where w_x is closer to x than to y on P', see Figure 3. Note that w_x and w_y may coincide with x and y, respectively.

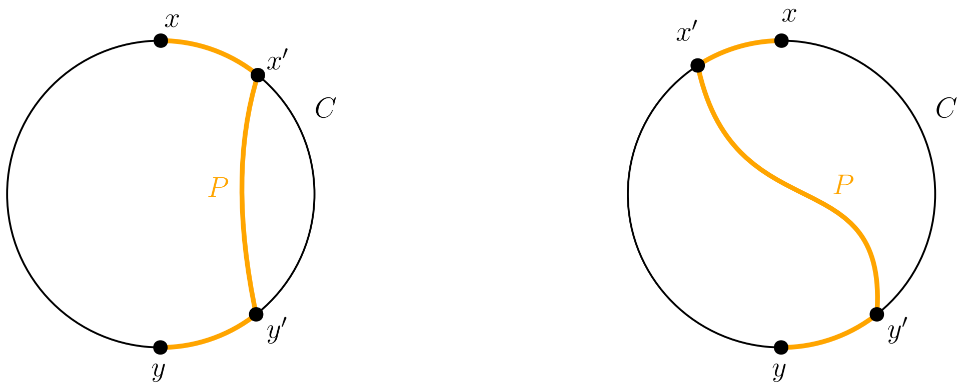
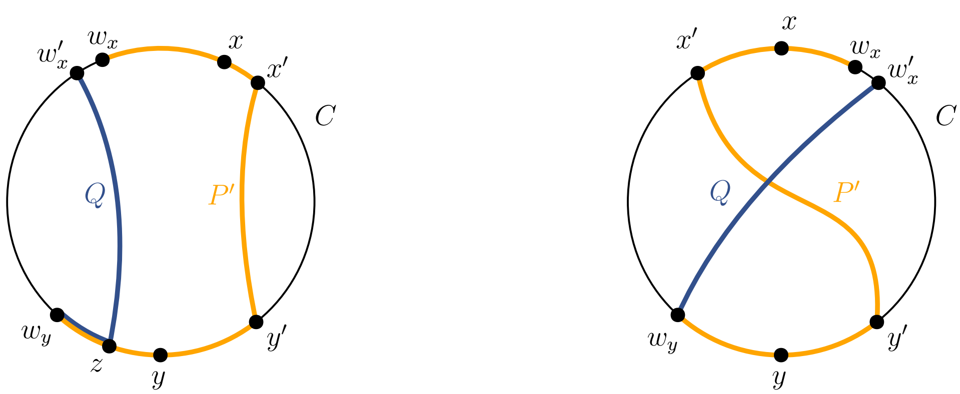


Figure 3: The set $\{x', y'\}$ can separate x and y in C (left) or not (right).

Let w'_x be a vertex adjacent to w_x on C such that $w_x w'_x$ is not an edge of P'. Observe that $w'_x \notin V(P')$ since otherwise the w_x, w_y-path $w_x w'_x P' w_y$ is shorter than P', a contradiction. Similarly, let w'_y be a vertex adjacent to w_y on C such that $w_y w'_y$ is not an edge of P', and observe that $w'_y \notin V(P')$. Further, let Q be a shortest w'_x, w_y-path and Q' be a shortest w'_y, w_x-path. By Lemma 4, both Q and Q' are C-convex.

Case 1. $V(Q) \cap V(w_x P' y) \subseteq \{y\}$ *or* $V(Q') \cap V(w_y P' x) \subseteq \{x\}$.

Note that this case could happen only if $\{x', y'\}$ does not separate x and y in C, see Figure 4b. If $V(Q) \cap V(w_x P' y) \subseteq \{y\}$, denote the first point of $yP'w_y$, counting from y, that lies on Q by z, see Figure 4a. Observe that $z = y$ if and only if $V(Q) \cap V(w_x P' y) = \{y\}$. One can see that $C' = w_x P' z Q w'_x w_x$ is a cycle of length $\|C'\| \le \|P'\| + \|Q\| + 1 \le 2\ell - 3$, where the last inequality is by Lemma 1. Hence, C' is a C-convex x, y-lens, as desired. The situation when $V(Q') \cap V(w_y P' x) \subseteq \{x\}$ is symmetric.



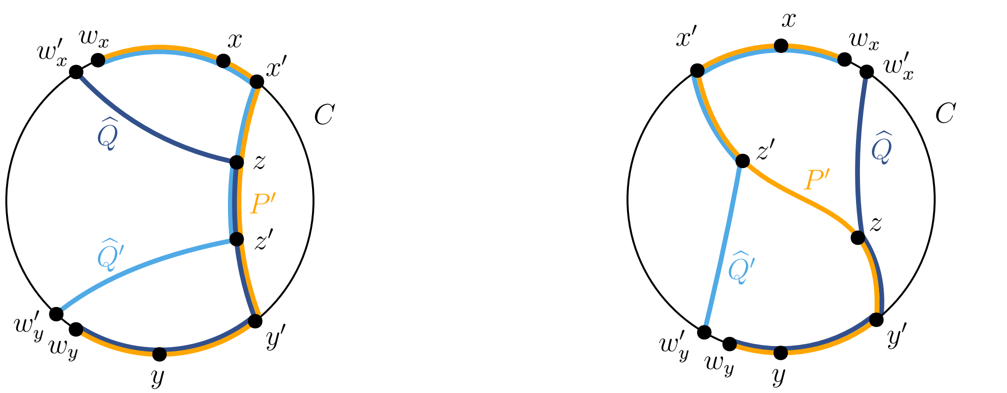
(a) If $V(Q) \cap V(w_x P' y) \subseteq \{y\}$, then $w_x P' z Q w'_x w_x$ is a C-convex x, y-lens.

(b) If $\{x', y'\}$ separates x and y on C, then $V(Q) \cap V(w_x P' y) \not\subseteq \{y\}$.

Figure 4: An illustration to Case 1.

Case 2. $V(Q) \cap V(w_x P' y) \not\subseteq \{y\}$ *and* $V(Q') \cap V(w_y P' x) \not\subseteq \{x\}$.

Let z be the first vertex of $w_x P' y$, counting from w_x, that lies on Q. Observe that $z \neq$ Let $\widehat{Q} = w'_x Q z P' w_y$. Lemma 2 implies that $\widehat{Q}$ is a shortest w'_x, w_y-path. Besides, $w_x w'_x \notin E(\widehat{Q})$ by maximality of P'. Recall that $z \in V(w_x P' y) \setminus \{y\}$ by our assumption, and thus $z \neq w_y$. Therefore, the path $\widehat{Q}$ contains the edge of P' adjacent to w_y, and thus $w_y w'_y \notin E(\widehat{Q})$. Similarly, let z' be the first vertex of $w_y P' x$, counting from w_y, that lies on Q'. Then $\widehat{Q}' = w'_y Q' z' P' w_x$ is a shortest w'_y, w_x-path and $w_x w'_x, w_y w'_y \notin E(\widehat{Q}')$. Now Lemma 3 applied to $\widehat{Q}$ and $\widehat{Q}'$ implies that these two paths are vertex-disjoint. Note that this can happen only if $\{x', y'\}$ separates w'_x and w'_y on C, see Figure 5a. Thus $C' = w_x w'_x \widehat{Q} w_y w'_y \widehat{Q}' w_x$ is a cycle of length $\|C'\| = \|\widehat{Q}\| + \|\widehat{Q}'\| + 2 \leq 2\ell - 2$, where the last inequality is by Lemma 1, see Figure 5b. In addition, both $\widehat{Q}$ and $\widehat{Q}'$ are C-convex by Lemma 4, and thus C' is a desired C-convex x, y-lens. □



(a) If $\{x', y'\}$ does not separate w'_x and w'_y on C, then $\widehat{Q}$ and $\widehat{Q}'$ cannot be vertex-disjoint.

(b) If $\widehat{Q}$ and $\widehat{Q}'$ are vertex-disjoint, then $w_x w'_x \widehat{Q} w_y w'_y \widehat{Q}' w_x$ is a C-convex x, y-lens.

Figure 5: An illustration to Case 2.

Lemma 6. *Let x, y be antipodal vertices of C and z be a center of a segment of C with endpoints x and y. Let C' be a C-convex $(x, y; \ell)$-lens. If $z \in V(C')$, then $\|C\| \leq 8\ell - 13$.*

Proof. Let x and y split C' into two C-convex x, y-paths P' and P'', so that $z \in V(P')$. Note that either $xP'z$ or $yP'z$ is a segment of C since P' is C-convex. Assume without loss of generality that $xP'z$ is a segment of C. Then $\|C\| \leq 4\mathrm{dist}_C(x, z) + 3 \leq 4(\|P'\| - 1) + 3 \leq 4(\|C'\| - 2) + 3 \leq 4(2\ell - 2 - 2) + 3 = 8\ell - 13$. □

Proof of Theorem 1 for $\ell \geq 7$.

Upper bound. Note that if $\|C\| < 2\ell$, then there is nothing to prove. So we assume that $\|C\| \geq 2\ell$. Let w_1, w_3 and w_2, w_4 be two pairs of antipodal vertices of C that split C into four segments of almost equal lengths. Let C' be a C-convex $(w_1, w_3; \ell)$-lens, and C'' be a C-convex $(w_2, w_4; \ell)$-lens, which exist by Lemma 5. Let Q_2, Q_4 be w_1, w_3-paths such that $C' = Q_2 \cup Q_4$ and Q_1, Q_3 be w_2, w_4-paths such that $C'' = Q_1 \cup Q_3$. Moreover, assume that w_i is closer in G to Q_i than to Q_{i+2}, $i \in [4]$, index addition modulo 4, see Figure 6.

If $w_i \in V(Q_i)$ for some $i \in [4]$, then $\|C\| \leq 8\ell - 13$ by Lemma 6, as desired. So from now one, we assume that $w_i \notin V(Q_i)$ for all $i \in [4]$.

By planarity, Q_i intersects Q_{i+1}, $i \in [4]$. When $|V(Q_i) \cap V(Q_{i+1})| = 1$ for each $i \in [4]$, we see that $Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ is a union of four edge-disjoint cycles of total length $\|C'\| + \|C''\| \leq 4\ell - 4$, see Figure 6, left. Then one of the cycles has length less than ℓ, a contradiction.

In general, recalling that $w_i \notin V(Q_i)$ for all $i \in [4]$, let u_i be the first vertex in Q_i, counting from w_{i-1}, that is in Q_{i-1}, see Figure 6, right. Consider non-trivial closed walks $U_i = w_i Q_{i+1} u_{i+1} Q_i u_i Q_{i-1} w_i$, $i \in [4]$.

Note that each Q_i is split into three paths by u_i and u_{i+1}, and each of these three paths is contained in exactly one of the U_i's. Therefore, $\sum_{i=1}^{4} \|U_i\| = \sum_{i=1}^{4} \|Q_i\| = \|C'\| + \|C''\| \leq 4\ell - 4$, and thus one of the non-trivial closed walks U_i contains a cycle of length less than ℓ, a contradiction again.

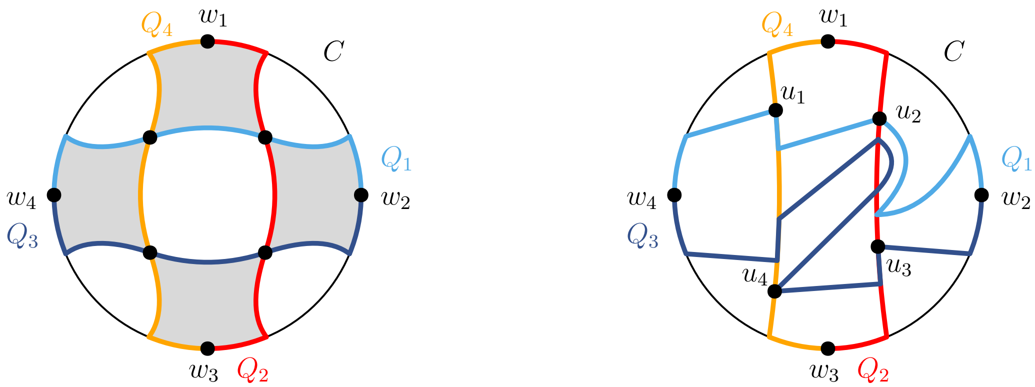


Figure 6: An illustration to the proof of Theorem 1.

Lower bound. Consider the graph $W(\ell)$ that is a subdivided wheel with three spokes of length 2 each, two segment of length $\ell - 4$, and the third segment of length $\ell - 3$, see Figure 7, left. Observe that any two non-adjacent vertices of $W(\ell)$ belong to some cycle of length either $2\ell - 4$ or $2\ell - 3$, and thus adding an edge between them creates a cycle of length less than ℓ. Hence, $W(\ell)$ is a maximal $\mathcal{C}_{<\ell}$-free plane graph. Therefore, we have $\mathrm{f}_{\max}(\ell) \geq \mathrm{f}_{\max}(W(\ell)) = 3\ell - 11$, as claimed.

If $\ell = 7, 8$, or 9, consider a different construction $W'(\ell)$, that is an edge-disjoint union of C_9 and $C_{3\ell-9}$ that share three vertices equidistant on each of the cycles, see Figure 7, right. Any two non-adjacent vertices of $W'(\ell)$ belong to a cycle of length $2\ell - 3$, ℓ, or 9. Hence, $W'(\ell)$ is a maximal $\mathcal{C}_{<\ell}$-free plane graph, and so $\mathrm{f}_{\max}(\ell) \geq \mathrm{f}_{\max}(W'(\ell)) = 3\ell - 9$ for $\ell = 7, 8, 9$, as claimed. □

Figure 7: Maximal $\mathcal{C}_{<\ell}$-free plane graphs $W(\ell)$ and $W'(\ell)$.

4 Proof of Theorem 1 for $3 \leq \ell \leq 6$

In this section, we argue that $\mathrm{f}_{\max}(\ell) = 2\ell - 3$ if $3 \leq \ell \leq 6$. Our proof relies on the following observation.

Lemma 7. *Let $\ell \geq 4$, G be a maximal $\mathcal{C}_{<\ell}$-free plane graph, and C be its facial cycle. If x and y are two non-consecutive vertices of C, then xy is not an edge of G.*

Proof. Assume that x and y are adjacent. They split C into two x,y-paths P' and P'', each of length at least $\ell - 1$, since otherwise G contains a cycle of length less than ℓ. Pick two vertices, x' on P' and y' on P'', such that $\mathrm{dist}_C(x, x') = \lfloor \ell/2 \rfloor = \mathrm{dist}_C(y, y')$. Note that $\mathrm{dist}(x, x') = \lfloor \ell/2 \rfloor$, since otherwise the union of

$xP'x'$ and a shortest x,x'-path in G contains a cycle of length less than ℓ. Assume that there is an x',y-path Q of length at most $\lceil \ell/2 \rceil - 2$. Then Q is shorter than $xP'x'$, and thus $xP'x'Qyx$ contains a cycle of length less than ℓ. This is a contradiction implying that $\mathrm{dist}(x',y) \geq \lceil \ell/2 \rceil - 1$. Similarly, $\mathrm{dist}(y,y') = \lfloor \ell/2 \rfloor$ and $\mathrm{dist}(x,y') \geq \lceil \ell/2 \rceil - 1$. Since each x',y'-path contains either x or y by planarity, we conclude that $\mathrm{dist}(x',y') \geq \lfloor \ell/2 \rfloor + \lceil \ell/2 \rceil - 1 = \ell - 1$, which contradicts Lemma 1 and thus completes the proof. □

Proof of Theorem 1 for $3 \leq \ell \leq 6$.

Recall that the lower bound is immediate by considering $C_{2\ell-3}$, so we proceed with the upper bound.

If $\ell = 3$, then the family $\mathcal{C}_{<\ell}$ of forbidden cycles is empty and any maximal plane graph is a triangulation, so $\mathrm{f}_{\max}(3) = 3$.

Let $\ell = 4$. Let $C = u_0u_1 \cdots u_ku_0$ be a facial cycle of length at least 6 in a maximal $\mathcal{C}_{<\ell}$-free plane graph G. By Lemmas 1 and 7, any two non-consecutive vertices of C are at distance exactly 2 in G. By planarity, u_0,u_3- and u_1,u_4-paths of length 2 must share their center vertex, say w. Then u_0wu_1 is a cycle of length 3, a contradiction. Hence, $\mathrm{f}_{\max}(4) \leq 5 = 2\ell - 3$.

Let $\ell = 5$. Let $C = u_0u_1 \cdots u_ku_0$ be a facial cycle of length at least 8 in a maximal $\mathcal{C}_{<\ell}$-free plane graph G. In this case, $k \geq 7$ and every two non-consecutive vertices of C are at distance either 2 or 3 in G by Lemmas 1 and 7.

Assume first that $\mathrm{dist}(x,y) = 3$ for all vertices x,y of C such that $\mathrm{dist}_C(x,y) \geq 3$. This implies that the shortest u_0,u_4- and u_1,u_5-paths P and P' have no inner vertices on C. By planarity, they share a vertex. Thus $u_0Pu_4u_5P'u_1u_0$ is an edge-disjoint union of two nontrivial closed walks of total length 8. Thus there is a cycle of length at most 4, a contradiction.

Now assume that $\mathrm{dist}(x,y) = 2$ for some vertices x,y of C such that $\mathrm{dist}_C(x,y) \geq 3$. Assume without loss of generality that $\mathrm{dist}(u_2,u_j) = 2$ for some $5 \leq j \leq \lceil k/2 \rceil + 2$, and thus u_2wu_j is an ear of C for some vertex w by Lemma 7, see Figure 8a. Note that $\mathrm{dist}(u_0,u_2) = 2$ and $\mathrm{dist}(u_0,w) \geq 2$, since otherwise u_0w is an edge and $u_0u_1u_2wu_0$ is a cycle of length 4. Similarly, $\mathrm{dist}(u_4,u_2) = 2$ and $\mathrm{dist}(u_4,w) \geq 2$. In addition, u_j and u_0 are not consecutive on C since $j + 1 \leq \lceil k/2 \rceil + 3 \leq k$, and thus $\mathrm{dist}(u_0,u_j) \geq 2$. Since the shortest u_0,u_4-path of length at most 3 contains one of the vertices u_2,w,u_j by planarity, we conclude that this vertex is u_j and $\mathrm{dist}(u_0,u_j) = 2$, $\mathrm{dist}(u_4,u_j) = 1$. The latter equality implies that $j = 5$, and thus $\mathrm{dist}_C(u_0,u_j) > 2$. Now the equality $\mathrm{dist}(u_0,u_j) = 2$ and Lemma 7 imply that there exists a vertex w' such that $u_0w'u_j$ is an ear of C. Moreover, $w' \neq w$ since $\mathrm{dist}(u_0,w) \geq 2$.

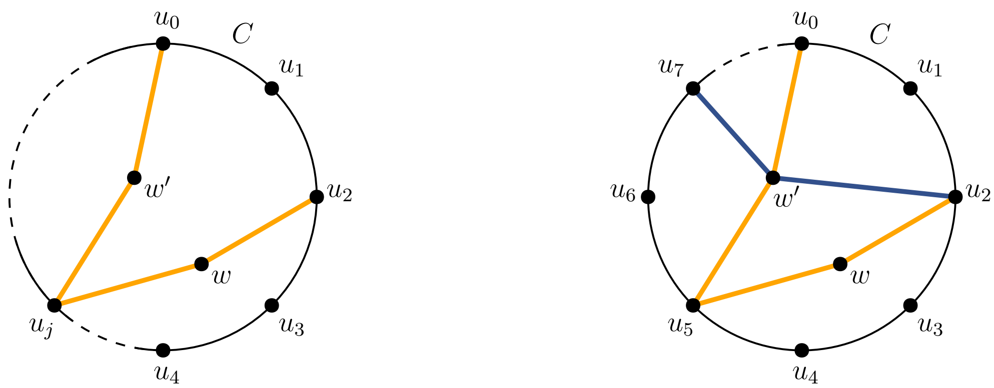
Repeating the argument from the previous paragraph verbatim replacing the vertices u_7,u_6,u_5,u_3,u_2 by $u_0,u_1,u_2,u_4,u_5 = u_j$, respectively, we conclude that there exists a vertex w'' such that $u_7w''u_2$ is an ear of C. Moreover, one can see that two ears $u_0w'u_5$ and $u_7w''u_2$ must share their center vertex by planarity. In other words, $w' = w''$ and thus $u_2wu_5w'u_2$ is a cycle of length 4, a contradiction, see Figure 8b. Hence, $\mathrm{f}_{\max}(5) \leq 7 = 2\ell - 3$, as desired.

The case $\ell = 6$ follows from Lemma 5 in [3]. □

5 Proof of Theorem 2

Let $\Sigma = \mathbb{S}_g$ or $\Sigma = \mathbb{N}_g$. The celebrated Ringel's theorem, see e.g. [24, Theorem 4.4.7], gives a necessary and sufficient condition for the existence of an embedding of a complete bipartite graph in these surfaces.

Theorem 3 (Ringel [26, 27]). *Let* $s,t \geq 3$ *and* $g \geq 0$ *be intergers. Then* $K_{s,t}$ *can be embedded on* $\mathbb{S}_g$ *if and only if* $g \geq \left\lceil \frac{(s-2)(t-2)}{4} \right\rceil$. *Moreover,* $K_{s,t}$ *can be embedded on* $\mathbb{N}_g$ *if and only if* $g \geq \left\lceil \frac{(s-2)(t-2)}{2} \right\rceil$.



(a) u_2wu_j and $u_0w'u_j$ are ears of C.

(b) $u_2wu_5w'u_2$ is a cycle of length 4.

Figure 8: An illustration to the case $\ell = 5$.

Recall that the class of graphs that can be embedded on Σ is closed under taking minors, see e.g. [24, Section 5.9]. Hence, Theorem 3 also implies that a graph which has a 'large' bipartite clique as a minor cannot be embedded on Σ. Here we only need the following simple corollary of this general result.

Proposition 1. *Let $g \geq 0$, $t = 4g + 3$ be integers, $\Sigma = \mathbb{S}_g$ or $\Sigma = \mathbb{N}_g$, and H be a graph that has $K_{3,t}$ as a minor. Then H cannot be embedded on Σ.*

Throughout this section, let $\ell \geq 3$ and $g \geq 1$ be integers, $t = 4g + 3$, $\Sigma = \mathbb{S}_g$ or $\Sigma = \mathbb{N}_g$, G be a maximal $\mathcal{C}_{<\ell}$-free graph embedded on Σ, and C be its facial cycle. We shall need the following general lemmas about graphs embedded on surfaces, in particular their spiders and pseudo-spiders, see Section 2 for definitions.

Lemma 8. *Let S_1 be a spider and S_2 be a pseudo-spider in G with t leaves on C each such that their leaves alternate on C. Then $V(S_1) \cap V(S_2) \neq \varnothing$.*

Proof. Assume that $V(S_1) \cap V(S_2) = \varnothing$. Let $u_1, v_1, \ldots, u_t, v_t$ be the leaves of S_1 and S_2 that appear alternatingly in this order in C. Consider a star S with center w and t leaves $w_1, \ldots, w_t$, where all the vertices are not the vertices of G. Let T be a tree obtained from S by adding edges $w_j u_j$ and $w_j v_j$, $j = 1, \ldots, t$. We can embed T on the face bounded by C. Then we see that the graph $S_1 \cup S_2 \cup T$ embedded on Σ has $K_{3,t}$ as a minor. This contradicts Proposition 1. $\square$

Lemma 9. *For any segment S of C of length ℓ and any $z \in V(G)$ there is $x \in V(S)$ such that* $\mathrm{dist}(x, z) > \frac{\ell}{2} - 1$.

Proof. Let $S = x_1 \cdots x_\ell$. Assume for the contrary that for all $i \in [\ell]$, $d(i) := \mathrm{dist}_G(x_i, z) \leq \frac{\ell}{2} - 1$. For all $i \in [\ell]$, let Q_i be a shortest x_i, z-path in G. Note that if $i < \ell$, then $x_i Q_i z Q_{i+1} x_{i+1} x_i$ is a closed walk of length at most $2(\frac{\ell}{2} - 1) + 1 = \ell - 1$. Hence, this walk is trivial, i.e., either $Q_i = x_i x_{i+1} Q_{i+1} z$ (in which case $d(i) = d(i+1) + 1$) or $Q_{i+1} = x_{i+1} x_i Q_i z$ (in which case $d(i+1) = d(i) + 1$). Assume that there is a local maximum of $d(\cdot)$ at $i \in [\ell]$, i.e., that $d(i-1) = d(i) - 1 = d(i+1)$ for some $i-1, i, i+1 \in [\ell]$. Then $Q_i = x_i x_{i-1} Q_{i-1} z = x_i x_{i+1} Q_{i+1} z$. This is a contradiction, since Q_i has an endpoint x_i and two edges incident to it. Therefore, the function $d(\cdot)$ has a unique minimum on $[\ell]$, it first strictly decreases and then strictly increases. However, this implies that $\ell = |[\ell]| \leq d(1) + d(\ell) + 1 \leq 2(\frac{\ell}{2} - 1) + 1 = \ell - 1$, a contradiction. $\square$

Proof of Theorem 2.

Upper bound. Recall that $\ell \geq 3$ and $g \geq 1$ are integers, $t = 4g + 3$, $\Sigma = \mathbb{S}_g$ or $\Sigma = \mathbb{N}_g$, G is a maximal $\mathcal{C}_{<\ell}$-free graph embedded on Σ, and C it its facial cycle. Assume for the contrary that $\|C\| > \big((t+1)^2\ell\big)^\ell$. We shall find a spider S_1 with head z and a pseudo-spider S_2 with t leaves on C and legs of length less than

ℓ such that there are $t\ell$ leaves of S_1 on C between any two consecutive leaves of S_2 and $z \notin V(S_2)$. We shall argue that there is a leg of S_2 with at least ℓ distinct vertices from S_1, i.e., more than the total number of vertices in that leg. This will result in a final contradiction.

Building S_1. Consider a subset $W \subset V(C)$ such that the distance between its vertices on C is at least $\ell + 1$ and such that $|W| \geq \left((t+1)^2\ell\right)^{\ell-1}$. Fix a vertex $w \in W$ and construct a rooted tree T with the root w and the leaves in W such that every root-to-leaf path in T is a shortest path in G as follows.

Initially, the tree T contains only one vertex, its root w. At each step, we take a new vertex $w' \in W \backslash \{w\}$ and consider a shortest w, w'-path P in G. Let v be the last point of P, counting from w, that is also a vertex of T. Note that the union of vPw' with the unique w, v-path in T is also a shortest w, w'-path in G by Lemma 2. We add the path vPw' to the tree T.

On the one hand, note that $W \subset V(T)$ by construction. On the other hand, recall that the pairwise distance between the vertices of $V(C)$ is at most $\ell - 2$ by Lemma 1. Thus the height of T is at most $\ell - 1$. Hence, if the maximum degree of T is at most $d := (t+1)^2\ell$, then $|V(T)| < d^{\ell-1} \leq |W|$, a contradiction. Therefore, there exists a vertex $z \in V(T)$ of degree more than $(t+1)^2\ell$ in T. By following the edges incident to z to the respective leaves of T, we see that T contains a spider S_1 with head z and with $(t+1)^2\ell$ legs such that each of its leaves is also a leaf of T. Denote the set of leaves of S_1 by W' and observe that $W' \subset W$ by construction.

Building S_2. By Lemma 9, there is a set of vertices U obtained by picking one vertex between every two consecutive on C vertices of W' such that their distance to z is greater than $\ell/2 - 1$. Let $U' \subset U$ consists of every $t\ell^{\text{th}}$ vertex of U in their order on C. Recall that $|U| = |W'| = (t+1)^2\ell$, and thus $|U'| \geq t + 1$. Pick a vertex $u \in U'$ and a pseudo-spider S_2 with head u and t legs that are shortest paths in G from u to some t vertices in $U' \backslash \{u\}$. Observe that $z \notin V(S_2)$ because otherwise z is on some leg of S_2, say with a leaf u', and thus $\text{dist}_G(u, u') = \text{dist}_G(u, z) + \text{dist}_G(u', z) > 2(\frac{\ell}{2} - 1)$ which contradicts Lemma 1.

Crossings between S_1 and S_2. Recall that there are at least $t\ell$ leaves of S_1 between any two consecutive leaves of S_2 by construction. Therefore, there are $t\ell$ pairwise edge-disjoint t-subspiders of S_1 that are leaf-alternating with S_2. Each of these subspiders share a common vertex with S_2 by Lemma 8. Moreover, these vertices are different for different t-subspiders because $z \notin V(S_2)$. By the pigeonhole principle, there is a leg of S_2 that contains at least $(t\ell)/t = \ell$ different vertices. However, the legs of S_2 have lengths less than ℓ by Lemma 1. This contradiction yields that $\|C\| \leq \left((t+1)^2\ell\right)^{\ell}$, as desired.

Lower bound.

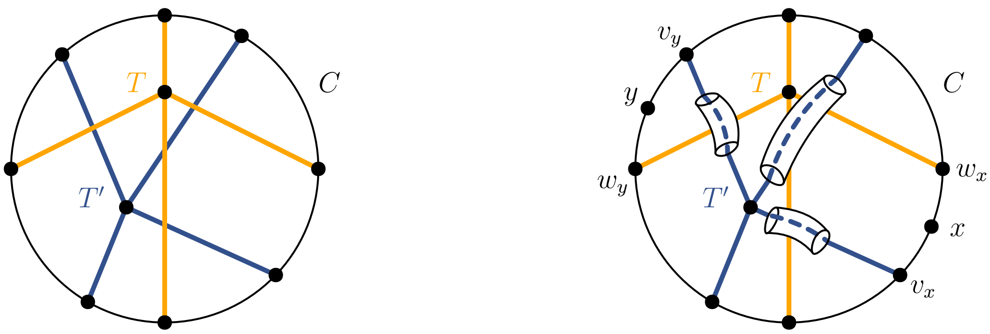
Assume that $\ell \geq 6$ and let $m = \ell - 4$. Consider a subdivided wheel with $g + 1$ spokes of length 1 and $g + 1$ segments of length $2m$ formed by a cycle C and a star T. Let T' be a star with $g + 1$ leaves such that its center is not a vertex of the wheel and its leaves are central vertices of the $g + 1$ segments of the wheel. Call the union of these graphs G.

Sketch G on the plane such that C is a circle, the centers of T and T' lie inside C and their edges are straight line segments, as shown in Figure 9a. Note that in this sketch of G, precisely g edges of T' cross T. For each of these edges, we take sufficiently small circles around two of its inner points, one before the first crossing and another one after the last crossing, replaces two discs bounded by these circles with a handle, and redirect the inner part of the edge along this handle. The resulting drawing of G on $\mathbb{S}_g$ is crossing-free, see [23, Section 2]. Moreover, the cycle C of length $2(g + 1)m$ is still a facial cycle of our drawing.

First, we observe that every cycle in G must contain at least two edge-disjoint segments of C of length m and at least two additional edges. In other words, G contains no cycles of length less than $2m + 2 \geq \ell$.

Second, we verify the maximality. Let x, y be two vertices of C. Note that if $\text{dist}_C(x, y) \leq m$, then adding the edge xy to G creates a cycle of length at most $m + 1 < \ell$, and there is nothing to prove. Hence, we assume without loss of generality that $\text{dist}_C(x, y) > m$. Let v_x and w_x be the vertices of T' and T, respectively, closest to x on C. Define v_y and w_y in a similar way. Note that v_x and w_x may coincide with v_y and w_y, respectively, but not simultaneously because $\text{dist}_C(x, y) > m$. Let Q_1 be an x, y-path formed by the union of the shortest x, v_x- and y, v_y-paths on C with the v_x, v_y-path on T'. Similarly, let Q_2 be an x, y-path formed by the union of the shortest x, w_x- and y, w_y-paths on C with the w_x, w_y-path on T, see Figure 9b. Observe that the union of Q_1 and Q_2 is a cycle and its length is at most $2m+4 \leq 2\ell-3$. Hence, adding the edge xy to G creates a cycle of length less than ℓ. Therefore, G is a maximal $\mathcal{C}_{<\ell}$-free graph embedded on $\mathbb{S}_g$. Now we conclude that $\mathrm{f}_{\max}(\ell, \mathbb{S}_g) \geq \|C\| = (2g+2)m$, which is even stronger than the desired lower bound.

For nonorientable surfaces, we cut off one or two small discs inside the face F and replace them with crosscaps. Since the resulting surface is homeomorphic to $\mathbb{N}_{2g+1}$ or $\mathbb{N}_{2g+2}$, respectively, see e.g. [24, Theorem 3.1.3], we conclude that $\mathrm{f}_{\max}(\ell, \mathbb{N}_{2g+1})$, $\mathrm{f}_{\max}(\ell, \mathbb{N}_{2g+2}) \geq \|C\| = (2g+2)m$. It remains only to note that for $g = 1, 2$, the desired inequality $\mathrm{f}_{\max}(\ell, \mathbb{N}_g) \geq g(\ell-4)$ follows from the lower bound $\mathrm{f}_{\max}(\ell, \mathbb{N}_g) \geq 2\ell-3$ discussed in the introduction. □



(a) A drawing of G on the plane in case $g = 3$. (b) A drawing of G on $\mathbb{S}_3$ without crossings.

Figure 9: An illustration to the lower bound.

6 Concluding remarks

We showed that the maximum length $\mathrm{f}_{\max}(\ell, \mathcal{P})$ of a facial cycle of an inclusion-maximal plane graph with girth at least ℓ satisfies $3\ell - 11 \leq \mathrm{f}_{\max}(\ell, \mathcal{P}) \leq 8\ell - 13$. We would like to pose the following question.

Question 1. *Is it true that* $\mathrm{f}_{\max}(\ell, \mathcal{P}) = 3\ell + o(\ell)$*?*

When $\ell \geq 6$ and the plane is replaced with an arbitrary closed surface Σ of genus $g \geq 1$, we showed that $g(\ell-4) \leq \mathrm{f}_{\max}(\ell, \Sigma) \leq \left((4g+4)^2\ell\right)^{\ell}$. In fact, our proof gives slightly better bounds depending on the orientability of Σ: $\mathrm{f}_{\max}(\ell, \mathbb{N}_g) \leq \left((2g+4)^2\ell\right)^{\ell}$ and $\mathrm{f}_{\max}(\ell, \mathbb{S}_g) \geq (2g+2)(\ell-4)$. By replacing one of the trees with a path in our construction, we can also show that $\mathrm{f}_{\max}(\ell, \mathbb{N}_g) \geq (2g+2)(\ell-g-2)$ for $\ell \geq 2g+3$.

Note that if $\ell \geq 6$ is fixed while g tends to infinity, then $\Omega(g) = \mathrm{f}_{\max}(\ell, \Sigma) = O(g^{2\ell})$ for every closed surface Σ of genus g. It would be interesting to reduce the polynomial gap between these bounds. In the regime when Σ is fixed while ℓ tends to infinity, we only know that $\Omega(\ell) = \mathrm{f}_{\max}(\ell, \Sigma) \leq \ell^{\ell+o(\ell)}$. We believe that in this regime, $\mathrm{f}_{\max}(\ell, \Sigma)$ is upper bounded by some polynomial function of ℓ.

We remark that it was not originally obvious for us that $\mathrm{f}_{\max}(\ell, \Sigma)$ is bounded by any function of ℓ even in the simplest case when Σ is a plane. Our argument in Section 5 is basically the shortest proof of the inequality $\mathrm{f}_{\max}(\ell, \Sigma) < \infty$ we have. It might be interesting to find a shorter argument.

Most Turán-type problems have their saturation counterparts, where the goal is to *minimize* the number of edges in an inclusion-maximal $\mathcal{F}$-free graph, see the survey [12] by Faudree, Faudree, and Schmitt. For the special case when $\mathcal{F}$ is a family of cycles, see [8, 14, 20]. The study of planar saturation numbers has been recently initiated by Clifton and Salia [6], see also [4]. Note that if $\mathcal{F} = \mathcal{C}_{<\ell}$ with $\ell > 3$, then every maximal $\mathcal{C}_{<\ell}$-free plane contains at least $n-1$ edges, which is tight as witnessed by stars. However, if we consider only 2-connected plane graphs, i.e. such graphs that all their faces are bounded by cycles, then the problems becomes less trivial.

Question 2. *What is the minimum number* $\mathrm{sat}_{\mathcal{P}}^{2\text{-con}}(n, \mathcal{C}_{<\ell})$ *of edges in a 2-connected maximal* $\mathcal{C}_{<\ell}$*-free plane graph on* n *vertices?*

A direct application of Euler's formula yields that $\mathrm{sat}_{\mathcal{P}}^{2\text{-con}}(n, \mathcal{C}_{<\ell}) \geq (1 - 2/\mathrm{f}_{\max}(\ell, \mathcal{P}))^{-1}(n-2)$ for all $n \geq \ell \geq 3$. It would be interesting to improve this lower bound asymptotically.

Note that $\mathrm{f}_{\max}(\ell, \Sigma)$ is defined as the maximum length of a facial cycle of an inclusion-maximal graph with girth *at least* ℓ embedded on Σ, while one could ask for a variant of this problem for graphs with girth *exactly* ℓ. We claim that these two problems have the same answer. Indeed, consider a graph G with girth at least ℓ embedded on Σ with a face F bounded by a cycle of length $\mathrm{f}_{\max}(\ell, \Sigma)$. Draw a cycle of length ℓ in any face of G but F and add edges arbitrarily until the resulting graph G' is an inclusion-maximal $\mathcal{C}_{<\ell}$-free graph. It is easy to see that $\mathrm{f}_{\max}(G') \geq \mathrm{f}_{\max}(G)$ since F is still a face of G', while the girth of G' equals ℓ, as desired.

Finally, let us note that the relations between lengths of facial cycles in plane graphs and their other parameters including radius or diameter were also considered, see Ali, Dankelmann, and Mukwembi [1] and Du Preez [25], respectively. See also a paper by Fernández, Sieger, and Tait [13] on planar subgraphs of given girth in planar graphs.

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