Bipartite independence number in graphs with bounded maximum degree

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Abstract

We consider a natural, yet seemingly not much studied, extremal problem in bipartite graphs. A *bi-hole* of size t in a bipartite graph G is a copy of $K_{t,t}$ in the bipartite complement of G. Let $f(n,\Delta)$ be the largest k for which every $n \times n$ bipartite graph with maximum degree Δ in one of the parts has a bi-hole of size k. Determining $f(n,\Delta)$ is thus the bipartite analogue of finding the largest independent set in graphs with a given number of vertices and bounded maximum degree. Our main result determines the asymptotic behavior of $f(n,\Delta)$. More precisely, we show that for large but fixed Δ and n sufficiently large, $f(n,\Delta) = \Theta(\frac{\log \Delta}{\Delta} n)$. We further address more specific regimes of Δ, especially when Δ is a small fixed constant. In particular, we determine $f(n,2)$ exactly and obtain bounds for $f(n,3)$, though determining the precise value of $f(n,3)$ is still open.

1 Introduction

The problem of finding $g(n,\Delta)$, the smallest possible size of a largest independent set in an n-vertex graph with given maximum degree Δ is not very difficult. Indeed, one can consider the graph that is the disjoint union of $\lfloor n/(\Delta+1) \rfloor$ complete graphs on $\Delta+1$ vertices each and a complete graph on the remaining vertices. This shows that $g(n,\Delta) \le \lceil n/(\Delta+1) \rceil$. On the other hand, every n-vertex graph of maximum degree Δ contains an independent set of size $\lceil n/(\Delta+1) \rceil$, obtained for example by the greedy algorithm. Consequently, $g(n,\Delta) = \lceil n/(\Delta+1) \rceil$. The situation is more interesting for regular graphs, see Rosenfeld [24] for a more detailed analysis. The analogous problem in the bipartite setting is more complex: determining the smallest possible 'bipartite independence number' of a bipartite graph with maximum degree Δ is still unresolved, even for $\Delta = 3$. To make this precise, we shall start with a few definitions.

A subgraph of the complete bipartite graph $K_{n,n}$ with n vertices in each part is called an $n \times n$ bipartite graph. A *bi-hole* of size k in a bipartite graph $G = (A \cup B, E)$ with a given bipartition A, B, is a pair (X, Y) with $X \subseteq A$, $Y \subseteq B$ such that $|X| = |Y| = k$, and such that there are no edges of G with one endpoint in X and the other endpoint in Y. Thus, the size of the largest bi-hole can be viewed as a bipartite version of the usual independence number. This work is devoted to studying the behavior of this function.

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For a graph G, we denote the degree of a vertex x by $\deg_G(x)$ or $\deg(x)$, the number of edges by $e(G)$, the number of vertices by $|G|$, and the maximum degree by $\Delta(G)$. We write log for the natural logarithm.

Definition 1.1. *Let $f(n, \Delta)$ be the largest integer k such that any $n \times n$ bipartite graph $G = (A \cup B, E)$ with $\deg(a) \leq \Delta$ for all $a \in A$ contains a bi-hole of size k. Let $f^*(n, \Delta)$ be the largest k such that any $n \times n$ bipartite graph G with $\Delta(G) \leq \Delta$ contains a bi-hole of size k.*

While $f(n, \Delta)$ is defined by restricting the maximum degree in one part of the graph, $f^*(n, \Delta)$ is its 'symmetric' version. Observe that $f(n, \Delta) \leq f^*(n, \Delta)$ for any natural numbers n and Δ for which these functions are defined.

Theorem 1.1. *There exists an integer Δ_0 and a positive constant c such that the following holds. For any $\Delta \geq \Delta_0$ there is $N_0 = N_0(\Delta) \geq 5\Delta \log \Delta$ such that for any $n > N_0$,*

$$f(n, \Delta) \geq \frac{1}{2} \cdot \frac{\log \Delta}{\Delta} n.$$

In addition, $f^*(n, \Delta) \geq c \frac{\log \Delta}{\Delta} n$.

Theorem 1.2. *Let Δ be an integer, $\Delta \geq 27$. If $n \geq \frac{\Delta}{\log \Delta}$, then*

$$f(n, \Delta) \leq 8 \cdot \frac{\log \Delta}{\Delta} n.$$

Note that the expression in the upper bound provided by Theorem 1.2 is trivial if $\Delta < 27$ since then $8 \cdot \frac{\log \Delta}{\Delta} \geq 1$.

Theorems 1.2 and 1.1 thus determine $f(n, \Delta)$ asymptotically for sufficiently large, but fixed Δ and growing n. We state this concisely in the following corollary.

Corollary 1.1. *There exists an integer Δ_0 such that the following holds. For any $\Delta \geq \Delta_0$ there is $N_0 = N_0(\Delta)$ such that for any $n > N_0$,*

$$\frac{1}{2} \cdot \frac{\log \Delta}{\Delta} n \leq f(n, \Delta) \leq 8 \cdot \frac{\log \Delta}{\Delta} n.$$

Corollary 1.2. *There is a Δ_0 such that if $\Delta > \Delta_0$, then $f^*(n, \Delta) = \Theta(\frac{\log \Delta}{\Delta} n)$.*

Note that Theorem 1.1 does not cover the entire range of Theorem 1.2: one wonders if $f(n, \Delta)$ is also $\Theta(\frac{\log \Delta}{\Delta} n)$ when, for example, n is in the interval $(\Delta/\log \Delta,\ \Delta \log \Delta)$. We leave this as an open problem, see Section 4.

Given Corollary 1.1, it is natural to consider the behavior of $f(n, \Delta)$ when Δ is small. In this case, we have only the following modest results. The bounds are obtained as corollaries of general bounds that become less and less precise as Δ grows; see Table 1 and Section 3.2 for more explicit values.

Theorem 1.3. *For any $\Delta \geq 2$ and any $n \in \mathbb{N}$ we have $f(n, \Delta) \geq \lfloor \frac{n-2}{\Delta} \rfloor$. Moreover, for any $n \in \mathbb{N}$ we have $f(n, 2) = \lceil n/2 \rceil - 1$, and there exists n_0 such that if $n > n_0$, then $0.3411n < f(n, 3) \leq f^*(n, 3) < 0.4591n$.*

Δ	Lower bound	Upper bound
3	0.34116	0.4591
4	0.24716	0.4212
5	0.18657	0.3887
6	0.14516	0.3621
7	0.11562	0.3395
8	0.09384	0.3201
9	0.07735	0.3031
10	0.06459	0.2882

Table 1: Explicit asymptotic lower and upper bounds on $f(n,\Delta)$, divided by n, obtained for small values of Δ, for n large enough.

We also consider the other end of the regime for Δ, when Δ is close to n. In particular, when Δ is linear in n, say $\Delta = n - cn$, it follows from Theorem 1.2 that $f(n, n-cn) = O(\log n)$. Furthermore, it is not too difficult to show that $f(n, n-cn) = \Omega(\log n)$ (see Proposition 3.2 for details). When Δ is much larger (i.e., $\Delta = n - o(n)$), bounding $f(n,\Delta)$ bears a strong connection to the Zarankiewicz problem, and we are able to obtain the following result. We formulate it in terms of a bound on the degrees guaranteeing a bi-hole of a constant size t. Let

$$\Delta(t) \coloneqq \max\{q \,:\, f(n,q) = t\}.$$

Theorem 1.4. *Let $t \geq 4$ be an integer. There is a positive constant C and an integer N_0 such that if $n > N_0$, then $n - Cn^{1-1/t} \leq \Delta(t) \leq n - Cn^{1-\frac{2}{t+1}}$. In addition there is an integer N_0, such that if $n > N_0$, then $\Delta(2) = n - n^{1/2}(1+o(1))$ and $\Delta(3) = n - n^{2/3}(1+o(1))$.*

The rest of the paper is structured as follows. We describe connections between the function $f(n,\Delta)$, classical bipartite Ramsey numbers, and the Erdős-Hajnal conjecture in Section 2. We prove Theorems 1.1 and 1.2 in Section 3.1 and prove Theorem 1.3 and establish the values for Table 1 in Section 3.2. We prove Theorem 1.4 in Section 3.3. Section 4 provides concluding remarks and open questions.

2 Related problems

The function $f(n,\Delta)$ is closely related to the bipartite version of the Erdős-Hajnal conjecture, bipartite Ramsey numbers, and the Zarankiewicz function.

A conjecture of Erdős and Hajnal [11] asserts that for any graph H there is a constant $\epsilon > 0$ such that any n-vertex graph that does not contain H as an induced subgraph has either a clique or a coclique on at least n^ϵ vertices. While this conjecture remains open, (see, for example, the survey by Chudnovsky [9]) the bipartite version of the problem has been considered. For a bipartite graph G, let $\tilde{h}(G)$ be the size of a largest *homogeneous set* in G respecting sides, i.e. the largest integer t such that G either contains a bi-hole of size t or a complete bipartite subgraph with t vertices in each part. Given a bipartite graph H, let $\tilde{h}(n,H)$ be the smallest value of $\tilde{h}(G)$ over all $n \times n$ bipartite graphs G that do not contain H as an induced subgraph respecting sides. It is implicit from a result of Erdős, Hajnal, and Pach [12] that if H is a bipartite graph with the smallest part of size k, then $\tilde{h}(n, H) = \Omega(n^{1/k})$. A standard probabilistic argument shows that if H or its bipartite complement contains a cycle, then $\tilde{h}(n, H) = O(n^{1-\epsilon})$ for a positive ϵ. Axenovich, Tompkins, and Weber [2] proved that $\tilde{h}(n, H)$ is linear in n for all but four strongly acyclic bipartite graphs H (a bipartite graph H being *strongly acyclic* if neither H nor its bipartite complement contains a cycle). Korándi, Pach, and Tomon [21] independently obtained similar results in the context of matrices. Note that $f(n, \Delta)$ for Δ sublinear in n corresponds to $\tilde{h}(n, K_{1,\Delta+1})$. Indeed, a bipartite graph not having a star with $\Delta + 1$ leaves (respecting sizes) is a graph with vertices in one part having degrees at most Δ. In such a graph there are clearly no complete bipartite graphs with $\Delta + 1$ vertices in each part, so the largest homogeneous set is a bi-hole, as Δ is sublinear in n. Its size is thus determined by $f(n, \Delta)$.

Furthermore, the parameter $f(n, \Delta)$ bears a connection to bipartite Ramsey numbers. If H_1 and H_2 are bipartite graphs, then the *bipartite Ramsey number* $\mathrm{br}(H_1, H_2)$ is the smallest integer n such that any red-blue coloring of the edges of $K_{n,n}$ produces a red copy of H_1 or a blue copy of H_2 respecting sides. Thus, if $f(n, \Delta) = k$, then $\mathrm{br}(K_{1,\Delta+1}, K_{k,k}) = n$. For results on bipartite Ramsey numbers, see Caro and Rousseau [8], Thomason [25], Hattingh and Henning [18], Irving [19], and Beineke and Schwenk [5].

Finally, considering the bipartite complement, determining $f(n, \Delta)$ is related to the Zarankiewicz problem in bipartite graphs. We let $z(n, t)$ denote the maximum number of edges in a subgraph of $K_{n,n}$ with no copy of $K_{t,t}$. Finding a large bi-hole in a bipartite graph is the same as finding a large copy of $K_{t,t}$ in the bipartite complement, where the bipartite complement has large minimum degree on one side (this is spelled out more carefully in Section 3.3). There is some literature on the Zarankiewicz problem for t large (see, for example, Balbuena et al. [3, 4], Čulík [10], Füredi and Simonovits [15], Griggs and Ouyang [16], and Griggs, Simonovits, and Thomas [17]). However, most of these results address the case when t is close to $n/2$, or when the results do not lead to improvements on our bounds.

3 Bounds on $f(n, \Delta)$

In this section we establish upper and lower bounds on $f(n, \Delta)$ for various ranges of Δ. First, we establish the exact value of $f(n, 2)$ that is used in other results. Then, we treat the case when Δ is fixed but large. We then move on to consider when Δ is a small fixed constant, and finally, when Δ is large, close to n.

Lemma 3.1. *For every positive integer n, we have $f(n, 2) = \lceil n/2 \rceil - 1$.*

Proof. To see that $f(n, 2) < n/2$, simply consider an even cycle C_{2n} on $2n$ vertices. It remains to establish the lower bound. Let $H = (A \cup B, E)$ be a bipartite graph with n vertices in each part and the degree of each vertex in A is at most 2. Note that we may assume without loss of generality that the degree of each vertex in A is exactly 2. Consider the auxiliary graph G with vertex set B, in which two vertices are adjacent if and only if they have a common neighbor in H. Consequently, there is a natural bijection between the edges of G and A, and thus G has n vertices and n edges. We assert that G contains a set E' of edges and a set V' of vertices each of size $\lceil n/2 \rceil - 1$, and such that no edge in E' has a vertex in V'. Note, then, that this pair of sets corresponds to parts of a bi-hole in H of size $\lceil n/2 \rceil - 1$, thus proving that $f(n, 2) \geq \lceil n/2 \rceil - 1$. The rest of the proof is devoted to proving the above assertion.

To this end, we consider the components of G: a component C is *dense* if $e(G[C]) \geq |C|$. Let $S_1, \ldots, S_k$ be the components of G, enumerated such that $S_1, \ldots, S_m$ are dense and the others are not. Note that we must have at least one dense component, so $m \geq 1$, and it could be that all components are dense. Let x be the number of components of G that are not dense, that is, $x \coloneqq k - m \in \{0, \ldots, k-1\}$. Let v and e be the number of vertices and edges, respectively, in the union of all dense components of G. Then the total number of edges in non-dense components of G is $n - e$ and the total number of vertices in these components is $n - v$. In addition, the number of vertices in non-dense components is at least the number of edges plus the number of components. Thus, $n - v \geq n - e + x$, so $x \leq e - v$.

Let G' be a subgraph of G with precisely $\lceil n/2 \rceil - 1$ edges and consisting of $S_1, \ldots, S_q$ and a connected subgraph of S_{q+1}, for some $q \in \{0, \ldots, k-1\}$. In particular, if S_1 has at least $\lceil n/2 \rceil - 1$ edges, then G' is a connected subgraph of S_1. It suffices to show that G' has at most $\lfloor n/2 \rfloor + 1$ vertices, since we can then choose a set V' of $\lceil n/2 \rceil - 1$ vertices in $V(G) \setminus V(G')$, which along with $E' \coloneqq E(G')$ will form the sought pair (V', E'). To this end, first notice that if G' has at most one non-dense component, then the number of vertices of G' is at most $|E'| + 1$, which is at most $\lceil n/2 \rceil \leq \lfloor n/2 \rfloor + 1$, as desired. Suppose now that G' has more than one non-dense component. It follows that G' contains all dense components of G. Let x' be the number of non-dense components of G'. Then $x' \leq x$. The number of edges in dense components of G' is e, thus the number of edges in non-dense components of G' is $\lceil n/2 \rceil - 1 - e$. This implies that the number of vertices in non-dense components of G' is at most $(\lceil n/2 \rceil - 1 - e) + x' \leq (\lceil n/2 \rceil - 1 - e) + x$. Adding the number v of vertices in dense components of G and the number of vertices in non-dense components of G', we see that the total number of vertices in G' is at most $v + ((\lceil n/2 \rceil - 1 - e) + x) \leq \lceil n/2 \rceil - 1 \leq \lfloor n/2 \rfloor + 1$. This concludes the proof. $\square$

3.1 Proof of Theorems 1.1 and 1.2

The upper bound given in Theorem 1.2 comes from suitably modifying the random bipartite graph $G\left(n, n, \frac{\Delta}{n}\right)$. The idea of the proof of the lower bound given in Theorem 1.1 is as follows. Let $G = (A \cup B, E)$ be an $n \times n$ bipartite graph with $\deg(x) \leq \Delta$ for every $x \in A$. We choose an appropriate parameter s and choose a subset S of B uniformly at random from the set of all s-element subsets of B and consider the set T of vertices in A that have at least $\Delta - 2$ neighbors in S. Lemma 3.1 can then be applied to the bipartite graph induced on parts $(T, B \setminus S)$, as in this bipartite graph every vertex in T has degree at most 2. Intuitively, the set T should be "large enough" to guarantee a large bi-hole in G. Floors and ceilings, when not relevant, are ignored in what follows. We start by establishing the lower bound, that is Theorem 1.1.

Proof of Theorem 1.1. Consider an arbitrary bipartite graph with parts A and B each of size n so that the degrees of vertices in A are at most Δ. Choose a subset S of B of size $(1-2x)n-2$ randomly and uniformly among all such subsets, where $x \coloneqq \frac{1}{2}\frac{\log \Delta}{\Delta}$. We assume that $n \geq N_0 \geq 5\Delta \log \Delta$ and $\Delta \geq \Delta_0$ chosen large enough to satisfy the last inequality in the proof. Let X be the random variable counting the number of vertices in A with at least $\Delta - 2$ neighbors in S. Then

$$\mathbb{E}X \geq n \cdot h(x, n, \Delta),$$

where

$$h(x,n,\Delta) = \binom{\Delta}{\Delta-2}\binom{n-\Delta}{(1-2x)n-\Delta}\binom{n}{(1-2x)n-2}^{-1}.$$

Observe that if $\mathbb{E}X \geq 2xn+2$, then there is a set A' of at least $2xn+2$ vertices in A, each sending at most 2 edges to $B \setminus S$. Since $|B \setminus S| = 2xn+2$, Lemma 3.1 implies that there is a bi-hole between A' and $B \setminus S$ of size at least xn. So, it is sufficient to prove that $h(x,n,\Delta) \geq 2x + 2/n$. Let us now verify this inequality.

Recall that $x = \frac{1}{2}\frac{\log\Delta}{\Delta}$. Let $\alpha = 1-2x$, so $\alpha = 1 - \frac{\log\Delta}{\Delta} = \frac{\Delta-\log\Delta}{\Delta} \in (0,1)$. Note that $\alpha n \geq \Delta$ since $n \geq 5\Delta\log\Delta$. Let $\beta = \frac{1}{\alpha} - 1$. Then $\beta = \frac{\log\Delta}{\Delta-\log\Delta}$. We have

$$\begin{aligned}
h(x,n,\Delta) &= \frac{\binom{\Delta}{\Delta-2}\binom{n-\Delta}{(1-2x)n-\Delta}}{\binom{n}{(1-2x)n-2}} \\
&= \binom{\Delta}{2}\prod_{j=2}^{\Delta-1}\left(\frac{\alpha n-j}{n-j}\right)\cdot\left[\frac{(2xn+2)(2xn+1)}{n(n-1)}\right] \\
&> \binom{\Delta}{2}\prod_{j=2}^{\Delta-1}\left(\frac{\alpha n-j}{n-j}\right)\cdot(2x)^2 \\
&= \binom{\Delta}{2}(2x)^2\alpha^{\Delta-2}\prod_{j=2}^{\Delta-1}\left(1-\frac{\beta j}{n-j}\right) \\
&\geq \binom{\Delta}{2}(2x)^2\alpha^{\Delta-2}\left(1-\frac{\beta\Delta}{n-\Delta}\right)^{\Delta-2} && (1)\\
&\geq \binom{\Delta}{2}(2x)^2\alpha^{\Delta}\left(1-\frac{\beta\Delta}{n-\Delta}\right)^{\Delta}, && (2)
\end{aligned}$$

where (1) holds because the function $j \mapsto \frac{\beta j}{n-j}$ is increasing, as $\beta > 0$. Now, expressing β in terms of Δ, we note that

$$\frac{\beta\Delta}{n-\Delta} \leq \frac{\Delta\log\Delta/(\Delta-\log\Delta)}{5\Delta\log\Delta-\Delta} = \frac{\log\Delta}{(5\log\Delta-1)(\Delta-\log\Delta)} \leq 1,$$

and therefore Bernoulli's inequality can be applied to (2). It follows that

$$\begin{aligned}
h(x,n,\Delta) &> \binom{\Delta}{2}(2x)^2(1-2x)^{\Delta}\left(1-\frac{\Delta^2\log\Delta}{(\Delta-\log\Delta)(n-\Delta)}\right) \\
&\geq \binom{\Delta}{2}(2x)^2(1-2x)^{\Delta}\left(1-\frac{4\Delta\log\Delta}{n}\right) && (3)\\
&\geq \binom{\Delta}{2}(2x)^2(1-2x)^{\Delta}\frac{1}{5}, && (4)
\end{aligned}$$

where (3) follows since $\frac{1}{n-\Delta} < \frac{2}{n}$ and $\log\Delta < \Delta/2$, and (4) holds since $n > 5\Delta\log\Delta$. Now, note that $(1-2x)^{\Delta} = \left(1 - \frac{\log\Delta}{\Delta}\right)^{\Delta} \geq \frac{1}{2} e^{-\frac{\log\Delta}{\Delta}\cdot\Delta} = \frac{1}{2\Delta}$. Thus, from (4) we obtain

$$h(x,n,\Delta) > \binom{\Delta}{2}(2x)^2 \frac{1}{10\Delta} = (2x)\frac{(\Delta-1)\log\Delta}{20\Delta}.$$

Finally, to bound the right-hand side of the above inequality from below, observe that

$$(2x)\frac{(\Delta-1)\log\Delta}{20\Delta} \geq (2x)\left(1 + \frac{1}{40}\log\Delta\right) = 2x + \frac{\log^2\Delta}{40\Delta} \geq 2x + \frac{2}{5\Delta\log\Delta} \geq 2x + \frac{2}{n},$$

where these inequalities hold for sufficiently large Δ. Accordingly, $h(x,n,\Delta) > 2x + \frac{2}{n}$, which concludes the proof of Theorem 1.1. $\square$

To prove Theorem 1.2 we shall need to use Chernoff's bound. Specifically, we use the following version (see [20, Corollary 21.7, p.401], for example).

Lemma 3.2 (Chernoff's bounds)**.** *Let X be a random variable with distribution* $\mathrm{Bin}(N,p)$ *and* $\varepsilon \in (0,1)$. *Then*

$$\mathbb{P}[X \geq (1+\varepsilon)\,\mathbb{E}\,X] \leq \exp\left(-\frac{\varepsilon^2}{3}\mathbb{E}X\right) \quad \textit{and} \tag{5}$$

$$\mathbb{P}[X \leq (1-\varepsilon)\,\mathbb{E}\,X] \leq \exp\left(-\frac{\varepsilon^2}{2}\,\mathbb{E}\,X\right). \tag{6}$$

Proof of Theorem 1.2. Let $\Delta \geq 27$. Suppose that $n \geq \frac{3\Delta}{20\log(\Delta/2)}$. Set $N := 2n$ and $\Delta' := \Delta/2$, so in particular $\Delta' \geq 13.5$. We consider first $H := G\left(N, N, \frac{\Delta'}{N}\right)$, that is, H is a random bipartite graph with parts A and B each of size N, where each edge ab with $a \in A$ and $b \in B$ is chosen independently and uniformly at random with probability Δ'/N. We first establish that the random graph H contains no "large" bi-holes with fairly large probability. In the following, for subsets $X \subseteq A$ and $Y \subseteq B$, let $e(X,Y)$ denote the number of edges with one endpoint in X and the other in Y.

(A). *With probability at least* 0.75*, any two subsets $X \subset A$ and $Y \subset B$ with $|X| = |Y| = \frac{2N\log\Delta'}{\Delta'}$ satisfy $e(X,Y) > 0$.*

Proof. Set $m := \frac{2N\log\Delta'}{\Delta'}$ and note that m is therefore at least $\frac{6}{5}$. Suppose that $X \subset A$ and $Y \subset B$ both have size m. Then

$$\mathbb{P}(e(X,Y) = 0) = \left(1 - \frac{\Delta'}{N}\right)^{m^2}.$$

Let p be the probability that there is a pair (X,Y), with $X \subset A$ and $Y \subset B$, $|X| = |Y| = m$, such that $e(X,Y) = 0$. Forming a union bound over all possible pairs of sets of size m, we have

$$p \leq \binom{N}{m}^2 \left(1 - \frac{\Delta'}{N}\right)^{m^2} \leq \left(\frac{Ne}{m}\right)^{2m} e^{-\frac{\Delta' m^2}{N}} = \left(\frac{\Delta' e}{2\log\Delta'} e^{-\log\Delta'}\right)^{2m} = \left(\frac{e}{2\log\Delta'}\right)^{2m} \leq 0.25.$$

Here, we used the standard estimates $\binom{t}{k} \leq \left(\frac{t\cdot e}{k}\right)^k$, $1 - x \leq e^{-x}$, the fact that $(e/2\log\Delta') < 0.53$ because $\Delta' \geq 13.5$, as well as inequality $2m \geq \frac{12}{5}$. This establishes (A). ⌟

Next we show that, with probability sufficiently large for our purposes, at least half of the vertices of H have degree at most $\Delta' + \sqrt{3\Delta'}$.

(B). *With probability greater than* 0.25*, the number of vertices* $v \in A$ *with more than* $\Delta' + \sqrt{3\Delta'}$ *neighbors in* B *is at most* $N/2$.

Proof. We use standard concentration inequalities to show that the degree of every vertex in A is approximately Δ'. For each vertex $v \in A$, let X_v be the degree of v in H. Noting that $\mathbb{E}X_v = \Delta'$, we apply (5) from Lemma 3.2 with $\varepsilon \coloneqq \sqrt{3/\Delta'} < 1$ to obtain

$$\mathbb{P}[X_v \geq (1+\varepsilon)\Delta'] \leq \exp\left(-\frac{\left(\sqrt{3/\Delta'}\right)^2}{3}\Delta'\right) = e^{-1}.$$

Letting X be the random variable counting those vertices $v \in A$ with $X_v \geq (1+\varepsilon)\Delta'$, we observe that $\mathbb{E}X \leq e^{-1}N$, and therefore, by Markov's inequality, we deduce that $\mathbb{P}[X \geq N/2] \leq \frac{2}{e} < 0.75$, thereby establishing (B). ⌟

It follows from (A) and (B) that with positive probability, H has no large bi-holes, and at least half of the vertices in A have degree at most $\Delta' + \sqrt{3\Delta'} \leq 2\Delta' \leq \Delta$. We now fix such a graph H. We can thus choose a subset A' of A of size $N/2 = n$ such that every vertex in A' has degree at most Δ in H. Now, arbitrarily choosing a subset B' of B of size n, we know that the subgraph H' of H induced by $A' \cup B'$ is an $n \times n$ bipartite graph with maximum degree Δ on one side and without bi-hole of size larger than

$$2(2n)\left(\frac{\log(\Delta/2)}{(\Delta/2)}\right) < 8n\left(\frac{\log \Delta}{\Delta}\right).$$

In order to obtain a lower bound on $f^*(n,\Delta)$, all that is required is to make the example obtained above have bounded maximum degree. Thus, it suffices to apply Chernoff's inequality to all vertices (instead of just the vertices in A). We may have to remove more vertices after doing this, but the loss will only be reflected in the constant. This completes the proof of Theorem 1.2. □

3.2 Bounding $f(n,\Delta)$ for small Δ

We have already established a part of Theorem 1.3 via Lemma 3.1, namely, we showed that $f(n,2) = \lceil n/2 \rceil - 1$. Our aim in this section is to investigate the behavior of $f(n,3)$ more closely, and to complete the proof of Theorem 1.3. First, let us note the following lower bound on $f(n,\Delta)$, valid for all integers n and Δ greater than 1.

Proposition 3.1. *If* n *and* Δ *are two integers greater than* 1*, then* $f(n,\Delta) \geq \left\lfloor \frac{n-2}{\Delta} \right\rfloor$.

Proof. We shall prove this by induction on Δ with the base case $\Delta = 2$ following from Lemma 3.1. Let $H = (A \cup B, E)$ be a bipartite graph with n vertices in each part and such that the degree of each vertex in part A is equal to Δ, $\Delta \geq 3$.

Consider a set X of $\lfloor (n-2)/\Delta \rfloor$ vertices in B. If $|N(X)| \leq n - \lfloor (n-2)/\Delta \rfloor$, then X and $A \setminus N(X)$ form a bi-hole with at least $\lfloor (n-2)/\Delta \rfloor$ vertices in each part. Otherwise, $|N(X)| > n - \lfloor (n-2)/\Delta \rfloor$. Let $G' \coloneqq G[N(X) \cup (B \setminus X)]$. Then each of the parts of G' has size at least $n - \lfloor (n-2)/\Delta \rfloor \geq n - (n-2)/\Delta$ and the maximum degree of vertices of $N(X)$ in G' is at most $\Delta - 1$. Thus, by induction G' has a bi-hole of size at least $\left\lfloor \frac{1}{\Delta-1}(n-(n-2)/\Delta - 2) \right\rfloor = \left\lfloor \frac{n-2}{\Delta} \right\rfloor$. □

It follows from the above proposition that $f(n, 3) \geq \lfloor (n-2)/3 \rfloor$. However, this lower bound can be improved slightly by choosing a random subset of B and considering the neighborhood of this set in A, similarly as in the proof of the lower bound in Theorem 1.1.

Lemma 3.3. *If n and Δ are two integers greater than 1, then $f(n, \Delta) \geq f(\lfloor \xi n \rfloor, \Delta - 1)$, where $\xi = \xi(\Delta)$ is a solution to the inequality $1 - \xi^{\Delta} \geq \xi$.*

Proof. For simplicity we omit floors in the following. Let G be a bipartite graph with parts A and B each of size n such that the vertices in A have degrees at most Δ. We shall show that there is a set $S \subset B$, such that $|S| = (1-\xi)n$ and such that $|N(S)| \geq \xi n$. To do this, we shall choose S randomly and uniformly out of all subsets of B of size $(1-\xi)n$ and show that the expected number X of vertices from A with at least one neighbor in S is at least ξn. Indeed, if p is the probability for a fixed vertex in A not to have a neighbor in S, then

$$p = \frac{\binom{n-\Delta}{(1-\xi)n}}{\binom{n}{(1-\xi)n}}.$$

Using the identity $\binom{n-l}{k}/\binom{n}{k} = \binom{n-k}{l}/\binom{n}{l}$, we see that $p = \binom{\xi n}{\Delta}/\binom{n}{\Delta}$. Now, using the inequality $\binom{\delta n}{r} \leq \delta^r \binom{n}{r}$, which is valid for every $\delta \in (0,1)$, we find that

$$p = \frac{\binom{\xi n}{\Delta}}{\binom{n}{\Delta}} \leq \xi^{\Delta}.$$

Thus, $\mathbb{E}X = n(1-p) \geq n(1-\xi^{\Delta}) \geq n\xi$ by our choice of ξ. Consequently, with positive probability $|N(S)| \geq \xi n$. We also have $|B \setminus S| = \xi n$. Since each vertex of $N(S)$ sends at most $\Delta - 1$ edges to $B \setminus S$, it follows that there is a bi-hole between $N(S)$ and $B \setminus S$ of size $f(\xi n, \Delta - 1)$. This completes the proof. □

We make explicit some lower bounds obtained using Lemma 3.3 (and Lemma 3.1).

Corollary 3.1. *There exists N_0 such that if $n \geq N_0$, then $f(n,3) > 0.34116n$, $f(n,4) > 0.24716n$, $f(n,5) > 0.18657n$, and $f(n,6) > 0.14516n$.*

The next natural step regarding small values of Δ is to evaluate how good the bounds written in Corollary 3.1 are. The following upper bounds are obtained by analysing the pairing model (also known as the *configuration model*) to build random regular graphs, tailored to the bipartite setting.

Lemma 3.4. *Let Δ be an integer greater than 2, and assume that $\beta \in (0,1)$ is such that*

$$\frac{(1-\beta)^{2\Delta(1-\beta)}}{\beta^{2\beta}(1-\beta)^{2(1-\beta)}(1-2\beta)^{\Delta(1-2\beta)}} < 1.$$

Then there exists $N_0 = N_0(\beta)$ such that for every $n > N_0$ we have $f(n,\Delta) \leq f^*(n,\Delta) < \beta n$. In particular, for n sufficiently large there exists a Δ-regular $n \times n$ bipartite graph with no bi-hole of size at least βn.

Proof. We shall work with the configuration model of Bollobás [6] suitably altered to produce a bipartite graph. Fix an integer n and consider two sets of Δn (labelled) vertices each: $X =$

$\{x_1^1, \dots, x_1^\Delta, \dots, x_n^1, \dots, x_n^\Delta\}$ and $Y = \{y_1^1, \dots, y_1^\Delta, \dots, y_n^1, \dots, y_n^\Delta\}$. Choose a perfect matching F between X and Y uniformly at random. We call F a *pairing*.

Given a pairing F, for each $i \in \{1, \dots, n\}$ the vertices $x_i^1, \dots, x_i^\Delta$ are identified into a new vertex x_i, and similarly the vertices $y_i^1, \dots, y_i^\Delta$ are identified into a new vertex y_i. This yields a multi-graph G'. We prove that with positive probability G' is a simple graph. To see why this holds, first notice that the total number of different pairings is $(\Delta \cdot n)!$. Second, each fixed (labelled) Δ-regular $n \times n$ bipartite graph arises from precisely $(\Delta!)^{2n}$ different pairings (because for each vertex x_i we can freely permute the vertices $\{x_i^1, \dots, x_i^\Delta\}$). Third, McKay, Wormald and Wysocka [23] proved that the number of different labelled Δ-regular $n \times n$ bipartite graphs is

$$(1 + o(1)) \exp\left(-\frac{(\Delta-1)^2}{2} \cdot ((\Delta-1)^2 + 1)\right) \frac{(\Delta \cdot n)!}{(\Delta!)^{2n}},$$

which is at least $c\frac{(\Delta \cdot n)!}{(\Delta!)^{2n}}$ for some $c > 0$. Combining these three facts, we find that $\mathbb{P}(G' \text{ is a graph}) \geq c > 0$, as announced.

Now, fix $k = k(n) = \beta \cdot n$ for some $\beta \in (0, 1)$. For each $i \in \{1, \dots, n\}$, set $X_i \coloneqq \{x_i^1, \dots, x_i^\Delta\}$ and $Y_i \coloneqq \{y_i^1, \dots, y_i^\Delta\}$. Fixing a family of k sets $\mathcal{X} = \{X_{i_1}, \dots, X_{i_k}\}$ and also $\mathcal{Y} = \{Y_{i_1}, \dots, Y_{i_k}\}$, let $W(\mathcal{X}, \mathcal{Y})$ be the event that the union U of the sets in $\mathcal{X} \cup \mathcal{Y}$ spans no edge in F (and thus corresponds to a bi-hole in G'). Let us now find the probability of $W(\mathcal{X}, \mathcal{Y})$. There are $2\Delta k$ edges incident to vertices in U. The number of ways to choose these $2\Delta k$ edges is as follows: if the edge is incident to a vertex in $\bigcup_{j=1}^k X_{i_j}$, then the other end-vertex must belong to $Y \setminus \bigcup_{j=1}^k Y_{i_j}$, and hence there are

$$(\Delta n - \Delta k) \dots (\Delta n - 2\Delta k + 1)$$

different ways of choosing the edges incident to a vertex in $\bigcup_{j=1}^k X_{i_j}$. The situation is analogous for edges incident to $\bigcup_{j=1}^k Y_{i_j}$, yielding a total of $(\Delta n - \Delta k)^2 \dots (\Delta n - 2\Delta k + 1)^2$ ways to choose the $2\Delta k$ edges incident to a vertex in U. For each such choice there are $(\Delta n - 2\Delta k)!$ ways to choose the remaining edges, for a total of $(\Delta n - 2\Delta k)! \cdot (\Delta n - \Delta k)^2 \dots (\Delta n - 2\Delta k + 1)^2$ different pairings in which U spans no edge. It follows that

$$\mathbb{P}(W(\mathcal{X}, \mathcal{Y})) = \frac{(\Delta n - 2\Delta k)! \cdot (\Delta n - \Delta k)^2 \dots (\Delta n - 2\Delta k + 1)^2}{(\Delta n)!} = \frac{((\Delta n - \Delta k)!)^2}{(\Delta n)!(\Delta n - 2\Delta k)!}.$$

Let $W = \bigcup_{\mathcal{X}, \mathcal{Y}} W(\mathcal{X}, \mathcal{Y})$ be the event that F contains a bi-hole of size k. Taking a union bound over all $\binom{n}{k}^2$ choices of $(\mathcal{X}, \mathcal{Y})$, we find that

$$\mathbb{P}(W) \leq \binom{n}{k}^2 \frac{((\Delta n - \Delta k)!)^2}{(\Delta n)!(\Delta n - 2\Delta k)!}. \tag{7}$$

Using Stirling's approximation,

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left(-\frac{1}{12n+1}\right) \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left(-\frac{1}{12n}\right),$$

in (7), and ignoring the exponential factors (they can be bounded from above by $\exp(1/(cn))$ for some positive constant c, and hence be made arbitrarily close to 1), we thus obtain

$$\binom{n}{k}^2 \approx \frac{1}{2\pi\beta(1-\beta)n}\left(\frac{1}{\beta^{2\beta}(1-\beta)^{2(1-\beta)}}\right)^n$$

and

$$\frac{(\Delta n-\Delta k)!^2}{(\Delta n)!(\Delta n-2\Delta k)!} \approx \frac{1-\beta}{\sqrt{1-2\beta}}\left(\frac{(1-\beta)^{2\Delta(1-\beta)}}{(1-2\beta)^{\Delta(1-2\beta)}}\right)^n.$$

Hence, if

$$\frac{(1-\beta)^{2\Delta(1-\beta)}}{\beta^{2\beta}(1-\beta)^{2(1-\beta)}(1-2\beta)^{\Delta(1-2\beta)}} < 1,$$

then $\mathbb{P}(W) \to 0$ as $n \to \infty$. Thus, with positive probability G' is a Δ-regular graph with no bi-hole of size at least $k = \beta n$ (for n sufficiently large), as stated. □

Performing explicit computations in Lemma 3.4 for specific values of Δ yields the following bounds (see also Table 1).

Corollary 3.2. *There exists N_0 such that if $n \geq N_0$, then $f^*(n,3) < .4591n$, $f^*(n,4) < .4212n$, $f^*(n,5) < .3887n$, and $f^*(n,6) < .3621n$.*

In particular, one sees that $f(n,3) \leq f^*(n,3) < .4591n$ for sufficiently large n. Thus, combined with our earlier work, it follows that $.3411n < f(n,3) < .4591n$. It would be very interesting to improve either the lower or upper bound.

3.3 Bounding $f(n,\Delta)$ when Δ is large

In this section we address the behavior of $f(n,\Delta)$ for large Δ and prove Theorem 1.4. Before doing so, let us note the following simple result, which shows that Theorem 1.2 is tight (up to constants) when Δ is linear in n.

Proposition 3.2. *For any $\varepsilon \in (0,1)$ there is a constant $c = c(\varepsilon)$ such that for n sufficiently large $f(n,(1-\varepsilon)n) \geq c\log n$.*

Proof. Let ε be a positive constant, $\varepsilon < 1$. To show the lower bound on $f(n,(1-\varepsilon)n)$, consider a bipartite graph G with parts A, B of size n each such that $\deg(x) \leq (1-\varepsilon)n$ for every $x \in A$. Letting G' be the bipartite complement of G, we see that G' has at least εn^2 edges. The result then follows from the fact that for any $\varepsilon \in (0,1)$ and sufficiently large n, any $n \times n$ bipartite graph with εn^2 edges contains a $K_{t,t}$ where $t = c\log n$ for some constant $c = c(\varepsilon)$. This can be proved using the standard Kővári-Sós-Turán [22] double counting argument. □

Therefore, the behavior of $f(n,\Delta)$ is clear whenever Δ is linear, aside from more precise estimates of the constants involved. What happens when Δ is very large, more precisely, when $\Delta = n - o(n)$? This is partly addressed in Theorem 1.4, which we now prove.

Proof of Theorem 1.4. Recall that the classical Zarankiewicz number, $z(n,t)$, is the largest number of edges in an $n \times n$ bipartite graph that contains no copy of $K_{t,t}$. Assume first that $t \geq 4$.

The lower bound on $\Delta(t)$, follows from standard bounds on Zarankiewicz numbers. Indeed, $z(n,t) \le Cn^{2-1/t}$ for some constant $C = C(t)$, see for example [22]. Thus, any $n \times n$ bipartite graph on at least $Cn^{2-1/t}$ edges contains a copy of $K_{t,t}$, and so any $n \times n$ bipartite graph on at most $n^2 - Cn^{2-1/t}$ edges contains a bi-hole of size t. In particular, any $n \times n$ bipartite graph with maximum degree at most $n - Cn^{1-1/t}$ contains a bi-hole of size t. So the announced lower bound in Theorem 1.4 holds.

To determine the stated upper bound on $\Delta(t)$, we shall prove the existence of a $K_{t,t}$-free bipartite $n \times n$ graph with the additional constraint that the minimum degree of vertices (on one side) is large. For that, we shall alter the standard random construction used to prove lower bounds on Zarankiewicz numbers. For a graph F, we shall carefully control $X = X_F$, the total number of copies of $K_{t,t}$ in F, as well as $X(v) = X_F(v)$, the number of copies of $K_{t,t}$ containing a vertex v in F.

Let $N \coloneqq 2n$ and $p \coloneqq cN^{-2/(t+1)}$, for a constant c to be determined later. Consider the bipartite binomial random graph $G' \sim G(N,n,p)$ with parts A and B of sizes N and n, respectively. By Markov's inequality, $\mathbb{P}(X \ge 2\,\mathbb{E}[X]) \le 1/2$. Since $\deg(v)$, for $v \in A$, is distributed as $\mathrm{Bin}(n,p)$, Chernoff's inequality (6) from Lemma 3.2 with $\varepsilon \coloneqq 1/2$, implies that with high probability, every vertex $v \in A$ has degree at least $pn/2$. So with positive probability we have $X \le 2\,\mathbb{E}[X] = 2\binom{N}{t}\binom{n}{t}p^{t^2} \le 2\binom{N}{t}^2 p^{t^2}$ and $\deg(v) \ge pn/2$ for every $v \in A$.

Fix a bipartite graph G with these properties, i.e., G is a bipartite graph with parts A and B, the number $X = X_G$ of copies of $K_{t,t}$ satisfies $X \le 2\binom{N}{t}^2 p^{t^2}$ and $\deg(v) \ge pn/2$ for every $v \in A$. Observe that there are fewer than n vertices v in A with $X(v) > \frac{2t}{n}2\binom{N}{t}^2 p^{t^2}$. Indeed, otherwise $X \ge n\frac{4t}{n}\binom{N}{t}^2 p^{t^2}/t$, a contradiction.

Let $A' \subset A$ be a set of n vertices such that $X(v) \le \frac{2t}{n}2\binom{N}{t}^2 p^{t^2}$ for all $v \in A'$. Let H' be the subgraph of G induced by $A' \cup B$. Finally, let H be obtained from H' by removing an edge from each copy of $K_{t,t}$. Thus H has no copies of $K_{t,t}$. It remains to check that the degrees of vertices in A are sufficiently large. Indeed, for any $v \in A$,

$$\begin{aligned}
\deg_H(v) &\ge \deg_{H'}(v) - X(v) \\
&\ge np/2 - \frac{4t}{n}\binom{N}{t}^2 p^{t^2} \\
&\ge Np/4 - \frac{4t}{n}N^{2t}p^{t^2} \\
&\ge \frac{c}{4}N^{1-\frac{2}{t+1}} - \frac{4t}{n}N^{2t}p^{t^2} \\
&> (c/4 - 4tc^{t^2})n^{1-\frac{2}{t+1}} \\
&\ge \frac{1}{16}n^{1-\frac{2}{t+1}},
\end{aligned}$$

where the last inequality holds for $c = 1/2$. This concludes the proof of the general upper bound for $t \ge 4$.

Now, let $t = 2$ or $t = 3$. We have $z(n, 2) = (1 + o(1))n^{3/2}$ and $z(n, 3) = (1 + o(1))n^{5/3}$, see Kővári, Sós & Turán [22], Füredi [14], and Alon, Rónyai & Szabó [1] (we refer the interested reader to the survey by Füredi & Simonovits [15]). In fact, the extremal constructions are almost regular and therefore show that there are $n \times n$ bipartite graphs with no $K_{2,2}$ (no $K_{3,3}$) with minimum degree at least $(1 + o(1))n^{1/2}$ (at least $(1 + o(1))n^{2/3}$), respectively. This completes the proof. $\square$

4 Concluding remarks

We have made progress in determining the asymptotic behavior of $f(n, \Delta)$. However, we could not obtain better bounds for small Δ. The most glaring open problem is the case $\Delta = 3$.

Problem 4.1. *Determine the value of $f(n, 3)$ for n sufficiently large.*

Further, recall that Theorem 1.1 does not cover the full range of Δ that is covered in Theorem 1.2. In particular, Theorem 1.1 shows that $f(n, \Delta) = \Omega(\frac{\log \Delta}{\Delta} n)$ for $n \geq C\Delta \log \Delta$, while the upper bound in Theorem 1.2 holds for $n \geq C' \frac{\Delta}{\log \Delta}$. It would be interesting to address what happens in the range between these two estimates.

Problem 4.2. *Determine whether or not $f(n, \Delta) = \Theta(\frac{\log \Delta}{\Delta} n)$ whenever $c\frac{\Delta}{\log \Delta} \leq n \leq c'\Delta \log \Delta$, for suitable positive constants c and c'.*

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